Power Operations and Global Algebra

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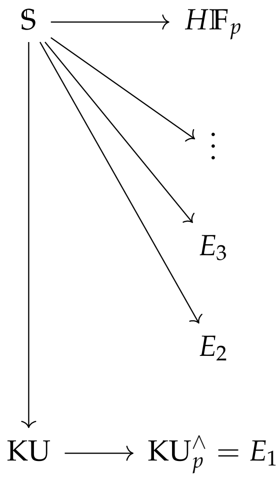
Abstract

These are my notes for an eCHT minicourse given by Nat Stapleton in May 2025. The content is is about global power functor structures on the complex representation ring, Burnside rings and Morava E-theory as well as a talk on partition functors yielding a sort of universal exponential relation.

Contents

1 Motivation

The chromatic story gives rise to LECTURE 1



each of which has a global equivariant refinement where the left objects are the usual global versions and the right side objects are given a Borel global equivariant structure. Thus, one can evaluate them on finite groups and this gives rise to objects in *global algebra*.

2 The Complex Representation Ring

2.1 Representations, Restrictions, Transfers

Starting in the height 1 story leads us into the theory of $\mathrm{RU}(G)$. Recall that this is the Grothendieck ring of the monoidal category of complex G-representations $\mathbf{Rep}_{\mathbb{C}}(G)$ with tensor product is given by $V \otimes_{\mathbb{C}} W$ endowed with the diagonal G-action.

Proposition 2.1. Every G-representation is a sum of irreducibles in a unique way.

Proof. This is a consequence of Schur's Lemma. □

So $\mathrm{RU}(G)$ is additively a free $\mathbb{Z}$-module with canonical basis given by the set of isomorphism classes of irreducible G-representations. In particular, the additive structure is not the most exciting thing to study on $\mathrm{RU}(G)$ and the story would end there without the multiplicative structure.

Example 2.2.

(i) The dimension function gives $\mathrm{RU}(e) \cong \mathbb{Z}$.

(ii) Consider $\mathrm{RU}(C_n)$ and let $\rho : C_n \circlearrowright \mathbb{C}$ via rotation by $\frac{2\pi i}{n}$. Thus, $\rho \otimes \rho : C_n \circlearrowright \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$ is given by rotation by $\frac{4\pi i}{n}$, and so on. In particular, $\rho^{\otimes n}$ is the trivial representation and all the powers $1, \rho^{\otimes 1}, \cdots, \rho^{\otimes n-1}$ are non-isomorphic, meaning that they form an additive basis whose multiplication we've just understood.

This implies $\mathrm{RU}(C_n) \cong \mathbb{Z}[x]/(x^n - 1)$ for $x = [\rho]$.

Moreover, since all irreducibles show up in the regular representation, it follows immediately by a dimension argument that

$$[\mathbb{C}[C_n]] = 1 + x + x^2 + \cdots + x^{n-1}.$$

For example, $\mathbb{C}[C_2] = \mathbb{C}(1+\tau) \oplus \mathbb{C}(1-\tau)$. More generally, let ζ be a primitive n-th root of unity, then $\mathbb{C}[C_n]$ decomposes into the subrepresentations spanned by

$$v_i = 1 + \zeta^i \rho + \zeta^{2i} \rho^2 + \cdots + \zeta^{(n-1)i} \rho^{n-1}.$$

One way to see this is to explicitly write out the matrix for ρ and computing its eigenvectors.

Construction 2.3. Let $f : H \to G$ be a group homomorphism.

(i) The map $\mathrm{Res}_f : \mathrm{RU}(G) \to \mathrm{RU}(H)$, $[G \circlearrowright V] \mapsto [H \to G \circlearrowright V]$ is a ring map.

(ii) The map $\mathrm{Tr}_f : \mathrm{RU}(H) \to \mathrm{RU}(G)$, $[H \circlearrowright V] \mapsto [G \circlearrowright \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V]$ is an additive map.

Note that transfers exist for any group homomorphism $f : H \to G$. This is a special feature of RU and is not in general available in global algebra, rather only transfers along injective group homomorphisms are accessible.

Example 2.4.

(i) Let $i : e \hookrightarrow G$. Then,

$$\mathrm{Res}_i : \mathrm{RU}(G) \to \mathrm{RU}(e) \cong \mathbb{Z},\ [V] \mapsto \dim V$$
$$\mathrm{Tr}_i : \mathbb{Z} \cong \mathrm{RU}(e) \to \mathrm{RU}(G),\ [\mathbb{C}] \mapsto [\mathbb{C}[G]].$$

(ii) For $f : H \twoheadrightarrow G$ we obtain $\mathrm{Tr}_f([V]) = [V/\ker f]$ using $G \cong H/\ker f$.

These interact in two important ways:

Fact 2.5.

(i) Double coset formula: Given $H, K \leq G$ we have

$$\mathrm{Res}^G_K \mathrm{Tr}^G_H = \sum_{KsH \in K\backslash G/H} \mathrm{Tr}^K_{K\cap sHs^{-1}} c_s \mathrm{Res}^H_{s^{-1}Ks\cap H}.$$

(ii) Frobenius Reciprocity: Let $x \in \mathrm{RU}(G)$ and $y \in \mathrm{RU}(H)$. Then, $\mathrm{Tr}_f(\mathrm{Res}_f(x)y) = x\mathrm{Tr}_f(y)$ for $f : H \to G$.

Corollary 2.6. Let $f : H \to G$. Then, $\mathrm{im}\,\mathrm{Tr}_f \subseteq \mathrm{RU}(G)$ is an ideal.

Proof. This follows from Frobenius reciprocity. □

Example 2.7. Let $n = kd$ and consider

$$C_k \xhookrightarrow{f} C_n \xrightarrow{g} C_n/C_k \cong C_d.$$

Let's compute some restrictions and transfers related to these maps.

(i) We obtain

$$\mathrm{Res}_f : \mathbb{Z}[x]/(x^n - 1) \cong \mathrm{RU}(C_n) \to \mathrm{RU}(C_k) \cong \mathbb{Z}[x]/(x^k - 1),\ x \mapsto x,$$

as the rotation by $\frac{2\pi i}{n}$ is restricted to rotation by $\frac{2\pi i d}{n} = \frac{2\pi i}{k}$ since $C_k \to C_n$, $\sigma_k \mapsto \sigma_n^d$ if σ denotes a generator. This map is in particular surjective.

(ii) We can describe

$$\mathrm{Tr}_f : \mathrm{RU}(C_k) \to \mathrm{RU}(C_n),\ [\mathbb{C}] \mapsto [\mathbb{C}[C_n] \otimes_{\mathbb{C}[C_k]} \mathbb{C}] = [\mathbb{C}[C_n/C_k]].$$

It's enough to describe this on $1 \in \mathrm{RU}(C_k)$ since Res_f is surjective by (i), so via Frobenius reciprocity $\mathrm{Tr}_f(\mathrm{Res}_f(x)y) = x\mathrm{Tr}_f(y)$ we can reach every input in Tr_f and immediately obtain a formula for it.

(iii) Similar to (i) we obtain

$$\mathrm{Res}_g : \mathrm{RU}(C_d) \to \mathrm{RU}(C_n),\ x \mapsto x^k.$$

In particular, $[\mathbb{C}[C_d]] \mapsto [\mathbb{C}[C_n/C_k]]$ which becomes

$$1 + x + x^2 + \cdots + x^{d-1} \mapsto 1 + x^k + x^{2k} + \cdots + x^{(d-1)k}.$$

This shows $[\mathbb{C}[C_n/C_k]] = 1 + x^k + x^{2k} + \cdots + x^{(d-1)k}$ in $\mathrm{RU}(C_n)$.

Example 2.8. We compute $\mathrm{Tr}^e_{C_2} : \mathrm{RU}(C_2) \to \mathbb{Z},\ 1 \mapsto 1,\ x \mapsto 0$ using 2.4(ii).

Fact 2.9 (Künneth Isomorphism)**.** The restriction of the projection maps induce a ring homomorphism $\boxtimes : \mathrm{RU}(G_1) \otimes \mathrm{RU}(G_2) \to \mathrm{RU}(G_1 \times G_2)$ which is an isomorphism.

Thus, we know understand $\mathrm{RU}(G)$ for every finite abelian group G! On the other hand, it is hard to describe $\mathrm{RU}(G)$ explicitly for finite groups G. The way to go is through character theory.

2.2 Character Theory

Let $\mathbb{Q}(\mu_\infty)$ denote $\mathbb{Q}$ with all roots of unities adjoined.

Construction 2.10. Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be a G-representation and consider $\mathrm{tr}(\rho(g))$. We make the following two observations.

(i) The eigenvalues of $\rho(g)$ must be roots of unities since g has finite order, thus $\mathrm{tr}(\rho(g))$ is a sum of roots of unity.[1]

(ii) The trace tr is invariant under conjugation.

Combining these yields the **character map**

$$\chi : \mathrm{RU}(G) \to \mathrm{Hom}_{\mathbf{Set}}(G/\mathrm{conj}, \mathbb{Q}(\mu_\infty)) = \mathrm{Cl}(G, \mathbb{Q}(\mu_\infty))$$

into the ring of $\mathbb{Q}(\mu_\infty)$-valued **class functions**.

Fact 2.11. The character map is an injective ring homomorphism and induces an isomorphism

$$\mathbb{Q}(\mu_\infty) \otimes \mathrm{RU}(G) \xrightarrow{\sim} \mathrm{Cl}(G, \mathbb{Q}(\mu_\infty)).$$

In fact, let $\widehat{\mathbb{Z}} = \lim_n \mathbb{Z}/n \subseteq \prod_n \mathbb{Z}/n$ denote the ring of profinite integers which shows up as

$$\mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \cong \widehat{\mathbb{Z}}^\times \cong \mathrm{Aut}(\widehat{\mathbb{Z}}).$$

In turns out that $G/\mathrm{conj} \cong \mathrm{Hom}_{\mathbf{Grp(Top)}}(\widehat{\mathbb{Z}}, G)/\mathrm{conj}$ and thus $\widehat{\mathbb{Z}}^\times \cong \mathrm{Aut}(\widehat{\mathbb{Z}})$ acts on it by precomposition. In particular, a conjugation action lets $\widehat{\mathbb{Z}}^\times$ act on $\mathrm{Cl}(G, \mathbb{Q}(\mu_\infty))$. It turns out that

$$\mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \cong \widehat{\mathbb{Z}}^\times \cong \mathrm{Aut}(\widehat{\mathbb{Z}})$$

is a $\widehat{\mathbb{Z}}^\times$-equivariant map. Taking fixed points thus yields:

Proposition 2.12. The character map induces an isomorphism $\chi : \mathbb{Q} \otimes \mathrm{RU}(G) \xrightarrow{\sim} \mathrm{Cl}(G, \mathbb{Q}(\mu_\infty))^{\widehat{\mathbb{Z}}^\times}$.

Example 2.13. Let $G = C_n$. Then,

LECTURE 2

$$\chi : \mathrm{RU}(C_n) \to \mathrm{Cl}(C_n, \mathbb{Q}(\mu_\infty)),\ x \mapsto (1, \mu_n, \mu_n^2, \cdots, \mu_n^{n-1}),$$

where μ_n is a primitive n-th root of unity (recall that x is represented by a one-dimensional representation!). So $\mathbb{Q}(\mu_\infty) \otimes \mathrm{RU}(C_n) \cong \prod_{C_n} \mathbb{Q}(\mu_\infty)$ by 2.11, as we can change the first coordinate 1 arbitrarily by an element of $\mathbb{Q}(\mu_\infty)$ and the rest of the tuple is determined by this C_n-action.

Example 2.14. Let $G = \Sigma_m$ and consider

$$\chi : \mathrm{RU}(\Sigma_m) \to \mathrm{Cl}(\Sigma_m, \mathbb{Q}(\mu_\infty))^{\widehat{\mathbb{Z}}^\times}.$$

Recall that an $[\sigma] \in \Sigma_m/\operatorname{conj}$ is determined by the length of the cycles in the cycle decomposition. The action $\widehat{\mathbb{Z}}^\times \circlearrowright \Sigma_m/\operatorname{conj}$ can be describedby sending $[\sigma] \in \Sigma_m/\operatorname{conj}$ to $[\sigma^\ell]$ where we really mean the image of ℓ under $\widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/|\sigma|)^\times$. Since $|\sigma| = \mathrm{lcm}(\text{length of cycles})$ we conclude that σ^ℓ has the same cycle decomposition as σ, as ℓ is relatively prime to the length of every cycle. In other words, the $\widehat{\mathbb{Z}}^\times$-action is trivial on $\Sigma_m/\operatorname{conj}$. In particular, $\widehat{\mathbb{Z}}^\times$ only acts on the target of $\mathrm{Cl}(\Sigma_m, \mathbb{Q}(\mu_\infty))$, so in particular, we get

$$\chi : \mathrm{RU}(\Sigma_m) \to \mathrm{Cl}(\Sigma_m, \mathbb{Q}(\mu_\infty))^{\widehat{\mathbb{Z}}^\times} \subseteq \mathrm{Cl}(\Sigma_m, \mathbb{Q})$$

using $\widehat{\mathbb{Z}}^\times = \mathrm{Gal}(\mathbb{Q}(\mu_\infty/\mathbb{Q}))$. On the other hand, those traces of these finite-order linear maps defining χ are given by suitably adding up some roots of unity which in particular lands in the ring of integers $\mathcal{O}_\mathbb{Q} \cong \mathbb{Z}$. So we obtain a factorization

$$\begin{array}{ccc} \mathrm{RU}(\Sigma_m) & \longrightarrow & \mathrm{Cl}(\Sigma_m, \mathbb{Q}) \\ & \searrow & \uparrow \\ & & \mathrm{Cl}(\Sigma_m, \mathbb{Z}) \end{array}$$

2.3 Relation to Restriction and Transfer

Let $\varphi : H \to G$, so we get a map $\varphi/\operatorname{conj} : H/\operatorname{conj} \to G/\operatorname{conj}$ which induces a map

$$\mathrm{Res}_{\varphi/\operatorname{conj}} : \mathrm{Cl}(G, \mathbb{Q}(\mu_\infty)) \to \mathrm{Cl}(H, \mathbb{Q}(\mu_\infty)).$$

Fact 2.15. There is a commutative square

$$\begin{array}{ccc} \mathrm{RU}(G) & \xrightarrow{\mathrm{Res}_\varphi} & \mathrm{RU}(H) \\ \downarrow & & \downarrow \\ \mathrm{Cl}(G, \mathbb{Q}(\mu_\infty)) & \xrightarrow[\mathrm{Res}_{\varphi/\operatorname{conj}}]{} & \mathrm{Cl}(H, \mathbb{Q}(\mu_\infty)) \end{array}$$

Construction 2.16.

(i) Let $H \leq G$. We define $\mathrm{Tr}_H^G : \mathrm{Cl}(H, \mathbb{Q}(\mu_\infty)) \to \mathrm{Cl}(G, \mathbb{Q}(\mu_\infty))$ by

$$\mathrm{Tr}_H^G(f)([g]) = \frac{1}{|H|} \sum_{\substack{\ell \in G \\ \ell g \ell^{-1} \in H}} f(\ell g \ell^{-1}) = \sum_{\substack{\ell H \in G/H \\ \ell g \ell^{-1} \in H}} f(\ell g \ell^{-1}).$$

(ii) If $\varphi : H \twoheadrightarrow G$, then we define $\mathrm{Tr}_\varphi(f)([g]) = \frac{1}{|\ker \varphi|} \sum_{h \in \varphi^{-1}(g)} f(h)$.

[1]A conceptual way to think about this is to put $\rho(g)$ in Jordan normal form.

2.4 Power Operations

Let $G \circlearrowright V$, then we obtain $G \wr \Sigma_m = (G^{\times m} \rtimes \Sigma_m) \circlearrowright V^{\otimes m}$.

Definition 2.17. Let $[V] \in \mathrm{RU}(G)$. We define $\mathbb{P}_m([V]) = [V^{\otimes m}] \in \mathrm{RU}(G \wr \Sigma_m)$.

Observation 2.18.

(i) Additivity fails: As vector spaces we have $(V \oplus W)^{\otimes m} = \bigoplus_{i+j=m} \binom{m}{i} V^{\otimes i} \otimes W^{\otimes j}$.

(ii) The failure of additivity is controlled by transfer maps: One can show

$$\mathbb{P}_m([V]+[W]) = \mathbb{P}_m([V]) + \sum_{\substack{i+j=m \\ i,j>0}} \mathrm{Tr}^{G\wr\Sigma_m}_{G\wr\Sigma_i \times G\wr\Sigma_j}\left(\mathbb{P}_i([V]) \boxtimes \mathbb{P}_j([W])\right).$$

Idea. Let $G = e$ and $\underline{m} = \underline{i} \cup \underline{j}$. Then, there is an inclusion

$$V^{\otimes \underline{i}} \otimes W^{\otimes \underline{j}} \hookrightarrow \bigoplus_{\substack{X \subseteq \underline{m} \\ |X|=i}} V^{\otimes X} \otimes W^{\otimes (\underline{m} \setminus X)}$$

where the right side is a Σ_m-representation (because we can swap around elements in $\underline{m}$) while the left side is a $(\Sigma_i \times \Sigma_j)$-representation. This is a $(\Sigma_i \times \Sigma_j)$-equivariant map. We can extend this to a Σ_m-equivariant map via the left adjoint $\mathbb{C}[\Sigma_m] \otimes_{\mathbb{C}[\Sigma_i \times \Sigma_j]} -$. You can check that this yields an isomorphism

$$\mathbb{C}[\Sigma_m] \otimes_{\mathbb{C}[\Sigma_i \times \Sigma_j]} \left(V^{\otimes \underline{i}} \otimes W^{\otimes \underline{j}}\right) \xrightarrow{\sim} \bigoplus_{\substack{X \subseteq \underline{m} \\ |X|=i}} V^{\otimes X} \otimes W^{\otimes (\underline{m} \setminus X)}.$$

□

Construction 2.19. We still need to extend $\mathbb{P}_m$ from G-representations to virtual G-representations.

(i) Let $\mathbb{P}_0(-[V]) = 1$.

(ii) Let $\mathbb{P}_1([V]) = -[V]$.

(iii) Let $\mathrm{Tr}_{i,j} = \mathrm{Tr}^{G\wr\Sigma_{i+j}}_{G\wr\Sigma_i \times G\wr\Sigma_j}$. We use induction to define $\mathbb{P}_m$ on $\mathrm{RU}(G)$, e.g.

$$0 = \mathbb{P}_2([V] + -[V]) = \mathbb{P}_2([V]) + \mathrm{Tr}_{1,1}([V] \boxtimes (-[V])) + \mathbb{P}_2(-[V]),$$

so $\mathbb{P}_2(-[V]) = \mathrm{Tr}_{1,1}([V] \boxtimes [V]) - \mathbb{P}_2([V])$.

Example 2.20. You can check

$$\mathbb{P}_2 : \mathbb{Z} \cong \mathrm{RU}(e) \to \mathrm{RU}(\Sigma_2) \cong \mathbb{Z}[x]/(x^2-1),\ k \mapsto \frac{k^2+k}{2} + \frac{k^2-k}{2}x.$$

Here are some properties of the $\mathbb{P}_m$'s.

Proposition 2.21.

(i) Special values: $\mathbb{P}_0(x) = 1,\ \mathbb{P}_1 = \mathrm{id}$.

(ii) $\mathbb{P}_m(x+y) =$ some big sum as above.

(iii) $\mathrm{Res}^{G\wr\Sigma_m}_{G\wr\Sigma_i \times G\wr\Sigma_j}(\mathbb{P}_m) = \mathbb{P}_i \boxtimes \mathbb{P}_j$.

Observation 2.22. About the failure of additivity, let

$$I_m = \mathrm{im}\left(\bigoplus_{\substack{i+j=m\\ i,j>0}} \mathrm{Tr}_{i,j}\right) \subseteq \mathrm{RU}(G \wr \Sigma_m)$$

be the ideal that controls additivity. So $\mathbb{P}_m/I_m : \mathrm{RU}(G) \to \mathrm{RU}(G \wr \Sigma_m)/I_m$ is additive, it is also called the **total additive power operation**.

Example 2.23. There is an isomorphism $\mathrm{RU}(\Sigma_m)/I_m \cong \mathbb{Z}$.[2] Then, it turns out that the composite

$$\chi_{[(1\ \cdots\ m)]} : \mathrm{RU}(\Sigma_m) \to \mathrm{RU}(\Sigma_m)/I_m \cong \mathbb{Z}$$

is the character map evaluated at the long cycle.

2.5 Interaction with the Character Map

Here is a new concept:

Definition 2.24.

(i) An **integer partition** of m, denoted $\lambda \vdash m$ is a function $\lambda : \mathbb{N}_{>0} \to \mathbb{N}$ such that $\sum_i \lambda_i i = m$.

(ii) An **integer partition of m decorated by G/conj** is a function

$$\lambda : \mathbb{N}_{>0} \times G/\operatorname{conj} \to \mathbb{N}, \sum_{i,[g]} \lambda_{i,[g]} i = m.$$

Denote the set of such decorated integer partitions by $\mathrm{Parts}(m, G/\operatorname{conj})$.

In other words, we remember by λ_i the number of times that i appears in an integer partition of m and each of these we decorate by a symbol $[g] \in G/\operatorname{conj}$. If we want to even remember this decoration, we take a finer function and let $\lambda_{i,[g]}$ remember that decoration.

Lemma 2.25. There is a canonical bijection $(G \wr \Sigma_m)/\operatorname{conj} \cong \mathrm{Parts}(m, G/\operatorname{conj})$.

Idea. Let $\sigma = (1\ \cdots\ m)$. Then, $(g_1, \cdots, g_m, \sigma) \sim (g_m \cdots g_1, e, \cdots, e, \sigma)$. □

Proposition 2.26. The function

$$\mathbb{P}^{\mathrm{cl}}_m : \mathrm{Cl}(G, \mathbb{Q}(\mu_\infty)) \to \mathrm{Cl}(G \wr \Sigma_m, \mathbb{Q}(\mu_\infty)),\ f \mapsto (\lambda \mapsto \prod_{i,[g]} f([g])^{\lambda_{i,[g]}})$$

makes the diagram

$$\begin{array}{ccc} \mathrm{RU}(G) & \xrightarrow{\mathbb{P}_m} & \mathrm{RU}(G \wr \Sigma_m) \\ \chi \downarrow & & \downarrow \\ \mathrm{Cl}(G, \mathbb{Q}(\mu_\infty)) & \xrightarrow[\mathbb{P}^{\mathrm{cl}}_m]{} & \mathrm{Cl}(G \wr \Sigma_m, \mathbb{Q}(\mu_\infty)) \end{array}$$

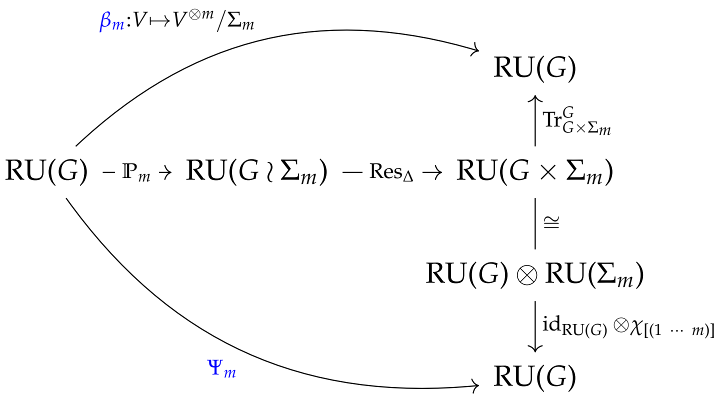
commute.

[2]Nat is not sure what the most beautiful proof of this is.

2.6 Symmetric Powers and Adams Operations from Power Operations

Here is a big diagram.

Construction 2.27. Start with the horizontal composite.



Once we have reached $\mathrm{RU}(G \times \Sigma_m)$, we have two ways of getting back to $\mathrm{RU}(G)$. The upper part follows because transfering along a surjection is given by modding out the kernel (2.4(ii)). The map Ψ_m is the **m-th Adams operation** which is additive since $\chi_{[(1\ \cdots\ m)]}$ factors through $\mathrm{RU}(\Sigma_m)/I_m$ by 2.23.

The fun thing is that there are formulas for everything. In terms of class functions (there is some abuse of notation going on here):

Proposition 2.28.

(i) $(\mathrm{Res}_\Delta \mathbb{P}_m)(f)([g], \tau \vdash m) = \prod_i f([g^i])^{\tau_i}$,

(ii) $\Psi_m(f)([g]) = f([g^m])$,

(iii) $\beta_m(f)([g]) = \frac{1}{m!} \sum_{\tau \vdash m} \frac{m!}{\prod_i (i^{\tau_i} \tau_i!)} \prod_i f([g^i])^{\tau_i} = \sum_{\tau \vdash m} \frac{1}{\prod_i \tau_i!} \prod_i \left(\frac{f([g^i])}{i}\right)^{\tau_i}$.

Corollary 2.29. There is an equality $\displaystyle\sum_{m \geq 0} \beta_m t^m = \exp\left(\sum_{k>0} \frac{\Psi_k}{k} t^k\right)$.

Proof. Check using the above formulas. □

The division by k should concern us homotopy theorists, it seems to live in $(\mathbb{Q} \otimes \mathrm{RU}(G))[\![t]\!]$. But it in fact lives in the divided power series ring $\mathrm{RU}(G)\langle\langle t \rangle\rangle \subseteq (\mathbb{Q} \otimes \mathrm{RU}(G))[\![t]\!]$.

3 Burnside Rings

Some of the discussion from last lecture prompts another gadget in global algebra, namely *global power functors*, which are global Green functors P together with maps $P(G) \to P(G \wr \Sigma_n)$. LECTURE 3

3.1 Definition and First Example

Today, we will try to speedrun a lot of the material that is analogous to the previous section.

Construction 3.1. The **Burnside ring** $A(G)$ is the Grothendieck ring of $(\mathbb{F}_G^{\mathrm{core}}, \amalg, \times)$.

The connection to (equivariant) homotopy theory is that it arises as $\pi_0^G(\mathbb{S}_G)$.

Observation 3.2. The building blocks for G-sets are transitive G-sets and $G/H \cong G/K$ if and only if H and K are conjugate. Thus, additively $A(G)$ is the free abelian group on the isomorphism classes of transitive G-sets, i.e. let Conj(G) = Sub(G)/ conj, then

$$A(G) \cong \bigoplus_{[H]\in \mathrm{Conj}(G)} \mathbb{Z}\{[G/H]\}.$$

So additively, it's pretty simple and we wish to understand it multiplicatively. You can actually do a little better than for RU, it's at least easier. To understand the multiplicative structure we need:

Observation 3.3. There is an isomorphism

$$G/H \times G/K \cong \coprod_{HgK\in H\backslash G/K} G/(H^g \cap K)$$

in $\mathbb{F}_G$.

Example 3.4. Consider $A(C_p)$. There are two isomorphism classes of transitives, namely $[C_p/C_p] = 1$ and $[C_p/e] = x$. So we need to understand x^2 to understand the multiplication of this ring. Note that the product of any G-set with a free G-set, we get a disjoint union of free G-sets. Or consider the double coset isomorphism from above (3.3) to see

$$C_p/e \times C_p/e \cong \coprod_{e\backslash C_p/e} C_p/e,$$

so $x^2 = px$. Thus, $A(C_p) \cong \mathbb{Z}[x]/(x^2 - px)$. In particular, $\operatorname{rk} A(C_p) = 2$ in contrast to $\operatorname{rk} \mathrm{RU}(C_p) = p$.

3.2 Linearization

Here is one relation of $A(G)$ and RU(G).

Construction 3.5. Consider the **linearization** map $L : A(G) \to \mathrm{RU}(G)$, $[G \circlearrowright X] \mapsto [G \circlearrowright \mathbb{C}\{x\}]$. This is a ring homomorphism.

Example 3.6. Can check for $G = C_p$ that it is injective.

Example 3.7. It is not injective or surjective: Consider $G = C_2 \times C_2$. The left side has rank 5 and the right side has rank 4. A fun exercise is to write down an element in the kernel of L.

3.3 Restrictions & Transfers & External Multiplication

Here is some more structure.

Construction 3.8. Let $\varphi : H \to G$.

(i) The map $\mathrm{Res}_\varphi : A(G) \to A(H)$, $[G \circlearrowright X] \mapsto \left[H \xrightarrow{\varphi} G \circlearrowright X\right]$ is a ring map.

(ii) The map $\mathrm{Tr}_\varphi : A(H) \to A(G)$, $[H \circlearrowright X] \mapsto [G \circlearrowright G \times_H X]$ is additive.

These satisfy the double coset formula and Frobenius reciprocity. This endows the Burnside ring with the structure of a global Green functor. In fact, it is the initial global Green functor. (This fact is a fun exercise.)

Example 3.9.

(i) Let $H \leq G$. Then, $\mathrm{Tr}_H^G([H/K]) = [G \times_H H/K] = [G/K]$. So Tr_H^G sends basis elements to basis elements. This is a big difference between $A(G)$ and $\mathrm{RU}(G)$ and is really helpful to show that A is the initial global Green functor.

(ii) Let $G \to e$. Then, $\mathrm{Tr}_G^e([X]) = [e \times_G X] = [X/G] = |X/G| \in A(e)$, i.e. this is counting the number of G-orbits in X.

Construction 3.10. There is an **external multiplication**

$$\boxtimes : A(G) \otimes A(H) \to A(G \times H),\ [G \circlearrowright X] \otimes [H \circlearrowright Y] \mapsto [(G \times H) \circlearrowright (X \times Y)].$$

This is not generally an isomorphism, for $G = H = C_2$ the left side has rank 4 while the right side has rank 5. It is an isomorphism more often than you think, for example if $\gcd(|G|, |H|) = 1$.

3.4 Character Theory

It turns out that these Burnside ring admit character maps that are extremely well behaved.

Definition 3.11. Let $\mathrm{Marks}(G, \mathbb{Z}) = \mathrm{Fun}(\mathrm{Conj}(G), \mathbb{Z})$ which people also call the **ghost ring** or the ring of **superclass functions**.

This terminology comes from the table of marks.

Construction 3.12. The **character map** is $\chi : A(G) \to \mathrm{Marks}(G, \mathbb{Z}),\ [G \circlearrowright X] \mapsto ([H] \mapsto |X^H|)$.

Fact 3.13. It is an injective ring homomorphism and rationally an isomorphism.

Thus, this is much easier than in the RU case, as the coefficient ring is just $\mathbb{Z}$ or $\mathbb{Q}$. There is no $\mathbb{Q}(\mu^\infty)$. Once again, you can ask for restrictions and transfers on Marks that are compatible with the ones on $A(G)$.

Construction 3.14.

(i) Restriction is induced by precomposition.

(ii) For $H \leq G$ have $\mathrm{Tr}_H^G(f)([K]) = \sum_{\substack{gH \in G/H \\ K^g \subseteq H}} f([K^g])$ similar to the RU story.

For surjections, the story is interesting. Over $\mathbb{Q}$ one gets an isomorphism and could just get a formula by inverting a matrix which people have done. But there is hope for a nice story. The hope is based on $\mathrm{Tr}_G^e(f)(e) = \frac{1}{|G|} \sum_{g \in G} f([\langle g \rangle])$ by Burnside's orbit counting lemma. It's very hard to see that this is the same formula as the one obtained for inverting the matrix. You can wonder about other transfers of this kind.

These are compatible with the formulas on class functions.

Construction 3.15. There is a map $L : \mathrm{Marks}(G, \mathbb{Z}) \to \mathrm{Cl}(G, \mathbb{Q}(\mu_\infty)),\ f \mapsto ([g] \mapsto f([\langle g \rangle]))$ compatible with the previous linearization map (3.5). This is compatible with restriction and transfer.

3.5 Power Operations

Now to power operations.

Definition 3.16. There are **power operations** given by

$$\mathbb{P}_m : A(G) \to A(G \wr \Sigma_m),\ [G \circlearrowright X] \mapsto [(G \wr \Sigma_m) \circlearrowright X^{\times m}].$$

Again, this is not additive but the failure is measured by transfers. It satisfies all the same formulas as in the previous section. So this seems very similar to the RU story. But that's actually where the stories begin to diverge: Something special happens!

Observation 3.17. Consider $\mathbb{P}_m([G/H]) = [(G/H)^{\times m}]$ but $G/H^{\times m} \cong (G \wr \Sigma_m)/(H \wr \Sigma_m)$ in $\mathbb{F}_{G\wr\Sigma_m}$. So $\mathbb{P}_m$ sends basis elements to basis element.

Example 3.18. Let $G = e$. Consider $\mathbb{P}_m : \mathbb{Z} \to A(\Sigma_m)$. Then,

$$\begin{aligned} 0 &\mapsto 0, \\ [e/e] = 1 &\mapsto [\Sigma_m/\Sigma_m] = 1 \\ 1+1 = 2 &\mapsto \sum_{i+j=m} \mathrm{Tr}^{\Sigma_m}_{\Sigma_i\times\Sigma_j}(\mathbb{P}_i(1) \boxtimes \mathbb{P}_j(1)) \\ &= \sum_{i+j=m} \mathrm{Tr}^{\Sigma_m}_{\Sigma_i\times\Sigma_j}([(\Sigma_i \times \Sigma_j)/(\Sigma_i \times \Sigma_j)]) \\ &= \sum_{i+j=m} [\Sigma_m/(\Sigma_i \times \Sigma_j)] \end{aligned}$$

Something interesting seems to happen since only special subgroups of Σ_m show up (namely those of the form $\Sigma_i \times \Sigma_j$). Continuing, we see that elements in the image of $\mathbb{P}_m$ are built out of Σ_m-sets of the form $[\Sigma_m/\Sigma_\lambda]$ where $\lambda \vdash m$, i.e. $\sum_i \lambda_i i = m$ and $\Sigma_\lambda = \prod_i \Sigma_i^{\times\lambda_i}$. These are called the *Young subgroups* of the symmetric groups.

This suggests that the image of the total power operation is smaller than you would expect it to be. This is very different from the RU case.

Definition 3.19. Let $\mathring{A}(G,m) \subseteq A(G \wr \Sigma_m)$ be the span of the smallest collection of basis elements (in the preferred basis) such that the span contains $\mathbb{P}_m(A(G))$.

Here are equivalent constructions of $\mathring{A}(G,m)$.

(i) A $G \wr \Sigma_m$-set X is called **submissive** if there is an embedding $X \hookrightarrow Y^{\times m}$ of $G \wr \Sigma_m$-sets for some G-set Y.

It turns out that $G \wr \Sigma_m$-sets are closed under $\amalg$, $\times$, for example $X \amalg X' \hookrightarrow (Y \amalg Y')^{\times m}$.

Then, $\mathring{A}(G,m)$ is the Grothendieck ring of isomorphism classes of submissive $G \wr \Sigma_m$-sets. (It's a good exercise to try to do this, it is not that hard.) In particular, this shows that $\mathring{A}(G,m)$ is a ring.

(ii) Let $\mathrm{Parts}(G,m)$ denote the set of Conj(G)-decorated integer partitions of m. In other words, an element is a function $\lambda : \mathrm{Conj}(G) \times \mathbb{N}_{>0} \to \mathbb{N}$ such that $\sum_{[H],i} \lambda_{[H],i} i = m$. Then, there is a map

$$\alpha : \mathrm{Parts}(G,m) \to \mathrm{Conj}(G \wr \Sigma_m),\ \lambda \mapsto \left[\prod_{[H],i} (H \wr \Sigma_i)^{\times\lambda_{[H],i}}\right]$$

If $G = e$, then this is already interesting: We are picking out Young subgroups using integer partitions.

It is a fact that α admits a preferred retract β. Let $\mathrm{Parks}(G,m) = \mathrm{Fun}(\mathrm{Parts}(G,m), \mathbb{Z})$. Then, there is a pullback square

$$\begin{array}{ccc} \mathring{A}(G,m) & \longrightarrow & A(G \wr \Sigma_m) \\ \downarrow & \lrcorner & \downarrow \chi \\ \mathrm{Parks}(G,m) & \xrightarrow[\beta^*]{} & \mathrm{Marks}(G \wr \Sigma_m, \mathbb{Z}) \end{array}$$

We win

$$\begin{array}{ccccc} A(G) & \xrightarrow{\mathbb{P}_m} & \mathring{A}(G,m) & \hookrightarrow & A(G \wr \Sigma_m) \\ \chi \downarrow & & \downarrow & & \downarrow \\ \mathrm{Marks}(G) & \xrightarrow[\mathbb{P}_m^{\mathrm{marks}}]{} & \mathrm{Parks}(G,m) & \xrightarrow[\beta^*]{} & \mathrm{Marks}(G \wr \Sigma_m) \end{array}$$

and now $\mathrm{Parks}(G,m)$ is quite combinatorial, so we can just write down

$$\mathbb{P}_m^{\mathrm{marks}} : f \mapsto \left(\lambda \mapsto \prod_{[H],i} f([H])^{\lambda_{[H],i}} \right).$$

The point is that giving to composite $\mathrm{Marks}(G) \to \mathrm{Marks}(G \wr \Sigma_m)$ is hard since the group $\mathrm{Marks}(G \wr \Sigma_m)$ is not very combinatorial. However, we can realize that $\mathbb{P}_m$ lands in a smaller ring which itself has its own character theory which is very combinatorial in nature. That allows us to give a power operation formula as desired on the level of marks!

We tried to focus on aspects of $A(G)$ that are different from $\mathrm{RU}(G)$ and this is an important difference!

4 Morava E-Theory

LECTURE 4

4.1 Prelude: p-adic K-theory

Let $\mathrm{KU}_p = \mathrm{KU}_p^\wedge$. There are maps

$$\mathrm{RU}(G) \longrightarrow \mathrm{KU}^0(BG) \longrightarrow \mathrm{KU}_p^0(BG).$$

and let us now make some comments about this:

- The first map is defined as follows: Start with $G \circlearrowright V$, in other words we have a G-vector bundle over $*$ and take the pullback

$$\begin{array}{ccc} EG \times V & \longrightarrow & V \\ \downarrow & \lrcorner & \downarrow \\ EG & \longrightarrow & * \end{array}$$

  and quotienting by G yields $EG \times_G V \to BG$. In more modern terms, we take the homotopy orbits $V \mapsto V_{hG}$.

- By the *Atiyah-Segal completion theorem* we have $\mathrm{KU}^0(BG) \cong \mathrm{RU}(G)^\wedge_{I_{\mathrm{aug}}}$ with augmentation ideal $I_{\mathrm{aug}} = \ker(\mathrm{RU}(G) \to \mathbb{Z})$. However, this is rather poorly behaved as a $\mathbb{Z}$-module.

- There is an isomorphism

$$\mathrm{KU}_p^0(BG) \cong \mathrm{RU}(G)^\wedge_{(p)+I_{\mathrm{aug}}} \cong \mathbb{Z}_p \otimes \mathrm{RU}(G)/J$$

  with $J = \ker(\mathrm{res} : \mathrm{RU}(G) \to \mathrm{RU}(S))$ where $S \leq G$ is any p-Sylow subgroups. Nat learned this from a paper of Strickland. Note that $\mathrm{RU}(G)/J \cong \mathrm{im}\,\mathrm{res}$ but $\mathrm{RU}(S)$ is a free abelian group, so $\mathrm{RU}(G)/J$ is one as well which we just base change to $\mathbb{Z}_p$. So the result is a free $\mathbb{Z}_p$-module!

It turns out that $\mathrm{rk}_{\mathbb{Z}_p} \mathrm{KU}_p^0(BG) = |G_p/\operatorname{conj}|$ where

$$\begin{aligned} G_p &= \text{set of } p\text{-power order elements in } G \\ &= \mathrm{Hom}_{\mathbf{Grp(Top)}}(\mathbb{Z}_p, G). \end{aligned}$$

Because G is a finite discrete group, a continuous map $\mathbb{Z}_p \to G$ must factor through some $\mathbb{Z}_p/p^k\mathbb{Z}_p \cong \mathbb{Z}/p^k$, hence the second equality. More detailedly, the kernel (i.e. the preimage of 0) must be open and thus contain some open neighbourhood of 0, so in particular some $p^k\mathbb{Z}_p$. Now G acts on $\mathrm{Hom}_{\mathbf{Grp(Top)}}(\mathbb{Z}_p, G)$ by conjugation.

Example 4.1.

(i) We have $\mathrm{KU}_p^0(B\mathbb{Z}/p^k) \cong \mathbb{Z}_p[x]/(x^{p^k} - 1)$ because $\mathbb{Z}/p^k$ is a p-group, so $J = 0$.

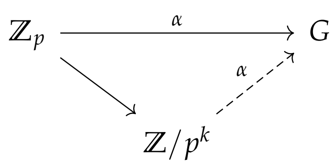
(ii) We have $\mathrm{KU}_p^0(BH) \cong \mathbb{Z}_p$ where $p \nmid |H|$ because then there are no p-power order elements in H. So from the perspective of p-adic K-theory, BH is a point.

There is a very nice character theory for p-adic K-theory due to Adams. See [Ada78] (or rather the second paper of that installment).

Construction 4.2. He constructs a character map

$$\chi : \mathrm{KU}_p^0(BG) \to \mathrm{Cl}_p(G, \mathbb{Q}_p(\mu_{p^\infty}) = \mathrm{Hom}_{\mathbf{Set}}(G_p/\operatorname{conj}, \mathbb{Q}_p(\mu_{p^\infty})).$$

Let $\alpha : \mathbb{Z}_p \to G$ picking out a map of p-power order, by adjunction we need to give a map $\mathrm{KU}_p^0(BG) \to \mathbb{Q}_p(\mu_{p^\infty})$ depending on α. Note that there is a factorization



so we get a map

$$\mathrm{KU}_p^0(BG) \xrightarrow{\alpha^*} \mathrm{KU}_p^0(B\mathbb{Z}/p^k) \cong \mathbb{Z}_p[x]/(x^{p^k} - 1) \twoheadrightarrow \mathbb{Z}_p[x]/\Phi_{p^k}(x) \hookrightarrow \mathbb{Q}_p(\mu_{p^\infty})$$

and in fact you can construct the character map for RU exactly like this as well. Note that the character map is a sort of restriction as signified as the restriction map α^*.

Adams proved that this is an injective ring homomorphism and that it is an isomorphism after base changing to $\mathbb{Q}_p(\mu_{p^\infty})$.

4.2 What is Morava E-Theory?

It's a challenge to talk about this in a few minutes. The theory is not hard but it requires some algebro-geometric language to carefully say what is going on. That's the challenge.

Let k be a perfect field of characteristic p and let Γ/k be a height n formal group law over k, i.e. an element $x +_\Gamma y \in k[\![x, y]\!]$ satisfying

(i) $x +_\Gamma 0 = x$,

(ii) $(x +_\Gamma y) +_\Gamma z = x +_\Gamma (y +_\Gamma z)$,

(iii) $x +_\Gamma y = y +_\Gamma x$.

Here, height n means $[p]_\Gamma(x) = x +_\Gamma \cdots +_\Gamma x = x^{p^n} + \cdots$.

Example 4.3. The formal group law $x +_{\widehat{\mathbb{G}}_m} y = x + y + xy$ over $\mathbb{F}_p$ is of height 1.

Fact 4.4 (Goerss-Hopkins-Miller)**.** This data determines a $K(n)$-local $\mathbb{E}_\infty$-ring $E = E(\Gamma, k)$ called **Morava E-theory** with even periodic homotopy groups

$$\pi_\bullet \cong W(k)[\![u_1, \cdots, u_{n-1}]\!][\beta^{\pm 1}]$$

with $|\beta| = 2$.[3]

Remark 4.5.

(i) If $k = \mathbb{F}_p$, then $W(\mathbb{F}_p) \cong \mathbb{Z}_p$.

(ii) The ring $\pi_0 E \cong W(k)[\![u_1, \cdots, u_{n-1}]\!]$ is called **Lubin-Tate ring** and is a complete local ring. Moreover, $W(k)$ is also a complete local ring with maximal ideal (p) and residue field k.

The Lubin-Tate ring carries the *universal deformation* of Γ/k.

Observation 4.6. Since E is a spectrum, the associated cohomology theory determines a global Mackey functor by plugging in BG for finite groups G. Since E is a homotopy commutative ring spectrum, the associated cohomology theory determines a global Green functor. Since E is an $\mathbb{E}_\infty$-ring spectrum, we even get a global power functor. Since E is $K(n)$-local, we get transfers along surjections.

(i) Power Operations: Consider $BG \to E$. Can take

$$B(G \wr \Sigma_m) \xrightarrow{\simeq} BG^{\times m}_{h\Sigma_m} \longrightarrow E^{\otimes m}_{h\Sigma_m} \xrightarrow{\mu_m} E$$

In other words, we obtain a map $\mathbb{P}_m : E^0(BG) \to E^0(BG \wr \Sigma_m)$.

(ii) $K(n)$-Local Transfers: Let $E_0^\wedge(BG) = \pi_0 L_{K(n)}(E \otimes BG)$. By $K(n)$-*local Tate vanishing*[4] there is a preferred isomorphism $E^0(BG) \cong E_0^\wedge(BG)$. So given $\varphi : H \to G$ we can define

$$\mathrm{Tr}_\varphi : E^0(BG) \cong E_0^\wedge(BH) \xrightarrow{E_0^\wedge(B\varphi)} E_0^\wedge(BG) \cong E^0(BG).$$

Since E is even periodic (and thus complex orientable), we get $E^0(BS^1) \cong E^0[\![x]\!]$ which is the ring of functions on a formal group. This isomorphism is not canonical and it is better to think of the right side as the ring of functions. We will write

$$\mathbb{G}_E = \operatorname{Spf} E^0(BS^1) : \mathbf{Alg}_{E^0}^{\text{compl,loc}} \to \mathbf{Ab}, \; (R, \mathfrak{m}_R) \mapsto (\mathfrak{m}_R, 0, +_{\mathbb{G}_E}).$$

where $\mathbf{Alg}_{E^0}^{\text{compl,loc}}$ means complete local E^0-algebras.

Example 4.7. Let $k = \mathbb{F}_p$ and $\Gamma = x + y + xy = x +_{\widehat{\mathbb{G}}_m} y$. In this case, $[p]_{\widehat{\mathbb{G}}_m}(x) = x^p + \cdots$, i.e. it has height 1 and it turns out that $E(\widehat{\mathbb{G}}_m, \mathbb{F}_p) \simeq \mathrm{KU}_p^\wedge$.

These Morava E-theories play the role of $\overline{\mathbb{Q}}$ in chromatic homotopy theory and is the main gadget we have to access the $K(n)$-local category.

Proposition 4.8. There is an isomorphism $\operatorname{Spf} E^0(B\mathbb{Z}/p^k) \cong \mathbb{G}_E[p^k] \cong \operatorname{Spf} E^0[\![x]\!]/[p^k]_{\mathbb{G}_E}(x)$.[5]

Proof Idea. This is a Gysin sequence argument and uses the fact that $[p^k]_{\mathbb{G}_E}(x)$ is not a zero divisor in $E^0[\![x]\!]$. □

The first isomorphism is canonical, the second one isn't.

[3] The isomorphism is non-canonical.

[4] This is a result from Greenlees-Sadofsky-Kuhn and also Strickland was involved in thinking about this.

[5] The notation means p^k-torsion. In the functor of points description, it outputs the p^k-torsion of $\mathfrak{m}_R$.

4.3 HKR Character Theory

The analogue of $\mathbb{Q}(\mu_\infty)$ in this story is a $\mathbb{Q} \otimes E^0$-algebra $C_0 = \mathbb{Q} \otimes D_\infty$ where D_∞ is the *Drinfeld ring at infinite level*. One can write

$$\mathbb{G}_E[p^k]^{\times n} \cong \underline{\mathrm{Hom}}_{\mathbf{Grp(Sch)}}(C_{p^k}^{\times n}, \mathbb{G}_E) \supseteq \underline{\mathrm{Level}}(C_{p^k}^{\times n}, \mathbb{G}_E)$$

where we write C_{p^k} instead of $\mathbb{Z}/p^k$ to think of this as its Pontryagin dual. Here, the idea is that Level caputures the injective maps $C_{p^k}^{\times n} \hookrightarrow \mathbb{G}_E$. Injective is not something you can say for the sake for functoriality but a good approximation for injective is a map $C_{p^k} \times n \to \mathbb{G}_E$ such that the image of any subgroup is one of the same size on the right side.

Example 4.9. One computes $\mathcal{O}_{\underline{\mathrm{Level}}(C_p, \mathbb{G}_E)} \cong E^0[\![x]\!]/\langle p \rangle_{\mathbb{G}_E}(x)$ where $\langle p \rangle_{\mathbb{G}_E}(x) = \frac{[p]_{\mathbb{G}_E}(x)}{x}$. In other words, we want a p-torsion point which is not 0. The not being 0 part is for the injectivity.

The ring of functions of $\underline{\mathrm{Level}}(C_{p^k}^{\times n}, \mathbb{G}_E)$ is much harder to describe. It was studied by Drinfeld and is extremely well behaved, e.g. it is a complete local domain, but writing it down in terms of generators and relations is super hard.

There is a natural map

$$\underline{\mathrm{Level}}(C_{p^k}^{\times n}, \mathbb{G}_E) \to \underline{\mathrm{Level}}(C_{p^{k-1}}^{\times n}, \mathbb{G}_E)$$

by restricting to the p^{k-1}-torsion. So this induces a map on rings of functions in the opposite direction.

Definition 4.10. We define $D_\infty = \mathrm{colim}_k\, \mathcal{O}_{\underline{\mathrm{Level}}(C_{p^k}^{\times n}, \mathbb{G}_E)} = \mathcal{O}_{\underline{\mathrm{Level}}(C_{p^\infty}^{\times n}, \mathbb{G}_E)}$.

Precomposition defines an action $\mathrm{GL}_n(\mathbb{Z}_p) \circlearrowright \mathcal{O}_{\underline{\mathrm{Level}}(C_{p^\infty}^{\times n}, \mathbb{G}_E)}$ and it turns out that $D_\infty^{\mathrm{GL}_n(\mathbb{Z}_p)} \cong E^0$ is the Lubin-Tate ring.

Fact 4.11. There is an isomorphism

$$\mathbb{Q} \otimes E^0(B(\mathbb{Z}/p^k)^{\times n})/I_{\mathrm{tr}} \cong \mathbb{Q} \otimes \mathcal{O}_{\underline{\mathrm{Level}}(C_{p^k}^{\times n}, \mathbb{G}_E)}$$

with $I_{\mathrm{tr}} = \mathrm{im}\left(\bigoplus_{A \subsetneq (\mathbb{Z}/p^k)^{\times n}} \mathrm{Tr}_A^{(\mathbb{Z}/p^k)^{\times n}}\right)$.

This gives a topological way of thinking about these level structures, at least after rationalization. More about this stuff can be read in [Sta20].

Let $G_p^{[n]} = \mathrm{Hom}_{\mathbf{Grp(Top)}}(\mathbb{Z}_p^{\times n}, G)/\,\mathrm{conj}$. Then, the HKR character map is a map

$$\chi : E^0(BG) \to \mathrm{Cl}_{n,p}(G, C_0) = \mathrm{Hom}_{\mathbf{Set}}(G_p^{[n]}, C_0)$$

which is constructed like Adams' character map, i.e. given $\alpha : \mathbb{Z}_p^{\times n} \to G$ there is a factorization

$$\begin{array}{ccc} \mathbb{Z}_p^{\times n} & \xrightarrow{\alpha} & G \\ & \searrow \quad \nearrow_{\alpha} & \\ & (\mathbb{Z}/p^k)^{\times n} & \end{array}$$

so we can take

$$E^0(BG) \xrightarrow{\alpha^*} E^0(B(\mathbb{Z}/p^k)^{\times n}) \longrightarrow \mathbb{Q} \otimes E^0(B(\mathbb{Z}/p^k)^{\times n})/I_{\mathrm{tr}} \hookrightarrow C_0$$

using 4.11 in the last map.

Theorem 4.12. The map χ is a ring map, is injective if $E^0(BG)$ is torsion-free, and $C_0 \otimes_{E^0} \chi$ is an isomorphism.

We know that $E^0(BG)$ is not always torsion-free but at least it is in the cases we care about.

Corollary 4.13. There is an isomorphism $\mathbb{Q} \otimes \chi : \mathbb{Q} \otimes E^0(BG) \xrightarrow{\sim} \mathrm{Cl}_{n,p}(G, C_0)^{\mathrm{GL}_n(\mathbb{Z}_p)}$.

Proof. Take $\mathrm{GL}_n(\mathbb{Z}_p)$-fixed points. □

Theorem 4.14 (Strickland)**.** Let $I_{\mathrm{tr}} = \mathrm{im}\left(\bigoplus_{\substack{i+j=m \\ i,j>0}} \mathrm{Tr}^{\Sigma_m}_{\Sigma_i \times \Sigma_j}\right) \subseteq E^0(B\Sigma_m)$. Then, $E^0(B\Sigma_m)/I_{\mathrm{tr}}$ is a free E^0-module and

$$\mathrm{Spf}\, E^0(B\Sigma_m)/I_{\mathrm{tr}} \cong \mathrm{Sub}_m(\mathbb{G}_E)$$

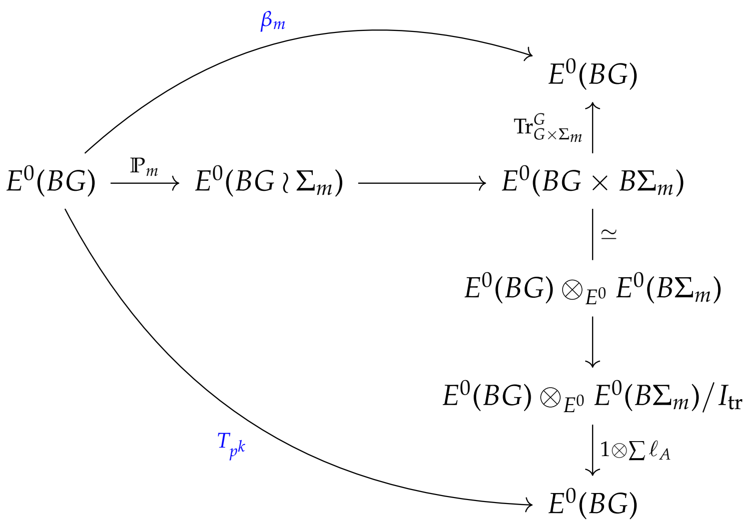
where the right side denotes subgroup schemes of order m.

We know that

$$E^0 \longrightarrow E^0(B\Sigma_m) \longrightarrow E^0(B\Sigma_m)/I_{\mathrm{tr}}$$

is a ring map and work of AHS gave an algebro-grometric interpretation. This is like the first ring map you can build out of the total power operation. So this allows us to use algebraic geometry to construct operations on Morava E-theory.

Construction 4.15. Let us redraw the big diagram that we already drew for RU (see 2.27).



where $1 \otimes \sum \ell_A$ only works for $m = p^k$. It was easier to get back down to $E^0(BG)$ in the RU-setting. So β_m is the **symmetric powers operator** and T_{p^k} is the **p^k-th Hecke operator**.

In Ganter's thesis:

Theorem 4.16 (Ganter)**.** We have $\sum_{m\geq 0} \beta_m t^m = \exp\left(\sum_{k\geq 0} \frac{T_{p^k}}{p^k} t^{p^k}\right)$.

So you can find a suitable logarithm. There are Adams operations on Morava E-theory but these Hecke operators take the position of the Adams operations in analogous relation with the symmetric powers.

In the final lecture we are going to try to understand where these exponential relations come from and we will try to produce a universal exponential relation.

5 Partition Functors & Universal Exponential Relations

This is joint work with Mehrle and Rose. This talk is a little different from the previous one in the sense that it has to do with current research. These are some ideas that came out of the Kentucky seminar three years ago. LECTURE 4

5.1 Motivation and Goals

Definition 5.1. A **global Mackey functor** is a pair of functors

$$M^* : \mathbf{Glo}^{\mathrm{op}} \to \mathbf{Ab},\ M_* : \mathbf{Glo}_{\mathrm{inj}} \to \mathbf{Ab}$$

with $M^*(G) = M_*(G)$ and which satisfy a double coset formula.

Here, **Glo** is a $(2,1)$-category of groups. As **Ab** is a 1-category, we obtain that conjugate maps go to the same map. In other words, inner automorphisms go to the identity maps.

Recall the exponential relations (2.29) on RU given by

$$\sum_{m\geq 0} \beta_m t^m = \exp\left(\sum_{k\geq 1} \frac{\Psi_k}{k} t^k\right)$$

which lives in $\mathrm{RU}(G)\langle\langle t\rangle\rangle \subseteq (\mathbb{Q}\otimes \mathrm{RU}(G))[\![t]\!]$ where elements on the left side are of the form $\sum_{i\geq 0} r_i \frac{t^i}{i!}$.

If R is a global power functor, then there is always the composite

$$\mathbb{P}_m/I_{\mathrm{tr}} : R(e) \xrightarrow{\mathbb{P}_m} R(\Sigma_m) \longrightarrow R(\Sigma_m)/I_{\mathrm{tr}}$$

which is additive. Our goal today is to describe an exponential relation of the form

$$\sum_{m\geq 0} \mathbb{P}_m t^m = \exp\left(\sum_{k\geq 1} \mathbb{P}_k/I_{\mathrm{tr}} t^k\right)$$

specializing to give the exponential relations we've described.

But if you look at this, it doesn't really seem to typecheck. So the first part of this talk is to try to find a home for an exponential relation like this.

5.2 Partition Functors

You should think about these things to being similar to global Mackey functor. Replace **Glo** by a $(2,1)$-category of partitions P.

Definition 5.2. Let P be a $(2,1)$-category given by:

- Objects: $\Lambda = (X, \sim_\Lambda)$ with a finite set X,
- Maps: Let $\Lambda = (X, \sim_\Lambda)$, $\Omega = (Y, \sim_\Omega)$. A map is a bijection $f : \Lambda \to \Omega$ such that $x \sim_\Lambda x' \implies f(x) \sim_\Omega f(x')$.
- 2-Maps: Let Σ_Ω denote the automorphisms of Ω that are identity mod $\sim_\Omega$. Now, let $f, g : \Lambda \to \Omega$, then a 2-morphism $\sigma : f \Rightarrow g$ is an element in Σ_Ω such that $\sigma f = g$.

In particular, there is at most one 2-morphism between any two 1-morphisms, so P is equivalent to a 1-category. However, this formulation is easier to compare to **Glo**. It's also a graded category and symmetric monoidal with respect to $\amalg$.

Example 5.3. Endow $\underline{m}$ with the indiscrete relation (i.e. identify every element). Then, $\Sigma_{\underline{m}} = \Sigma_m$ but $\Sigma_{\underline{i}\amalg\underline{j}} = \Sigma_i \times \Sigma_j$.

Definition 5.4. We say $\Lambda \leq \Omega$ if $X = Y$ and id_X is a map of partitions.

Definition 5.5. A **partition functor** M is a pair of functors

$$M^* : P^{\mathrm{op}} \to \mathbf{Ab}, \; M_* : P \to \mathbf{Ab}$$

such that $M^*(\Lambda) = M_*(\Lambda)$ and satisfying a double coset formula.

Example 5.6. Fix a group G. There are two functors

$$G \wr \Sigma_{(-)} : P \to \mathbf{Glo}, \; G \times \Sigma_{(-)} : P \to \mathbf{Glo}$$

of $(2,1)$-categories. Can restrict a global Mackey functor along these to get a partition functor.

5.3 The monad Div

This monad plays the role of integer-valued class functions on symmetric groups.

Construction 5.7. Let M be a partition functor and let

$$I_\Omega = \mathrm{im}\left(\bigoplus_{\Gamma<\Omega} (\Gamma < \Omega)\right) \leq M(\Omega).$$

If $\Omega \leq \Lambda$ and $\sigma \in \Sigma_\Lambda$, then $\sigma\Omega \leq \Lambda$ inducing and isomorphism $M(\Omega)/I_\Omega \xrightarrow{\sim} M(\sigma\Omega)/I_{\sigma\Omega}$. We define

$$\mathrm{Div}(M)(\Lambda) = \left(\bigoplus_{\Omega\leq\Lambda} M(\Omega)/I_\Omega\right)^{\Sigma_\Lambda}.$$

There is a summand $M(\Lambda)/I_\Lambda$ sitting inside.

Proposition 5.8. This $\mathrm{Div}(M)$ is a partition functor and the functor Div is a monad.

Proof Idea. A lot of bookkeeping but not so hard.

The restriction is given by projecting onto the respective summands. The trace is some sort of double coset formula. The most important map for today's purposes is the unit map of the monad. This unit map $\eta : M(\Lambda) \to \mathrm{Div}(M)(\Lambda)$ is given by

$$M(\Lambda) \longrightarrow M(\Omega) \longrightarrow M(\Omega)/I_\Omega$$

componentwise where the first map is restriction. Moreover, the monad multiplication map $\mathrm{Div}(\mathrm{Div}\, M) \to \mathrm{Div}\, M$ has to do with this distinguished summand sitting inside $\mathrm{Div}\, M$. □

Since P is symmetric monoidal, there are a lot of multiplicative structures you can try to put on partition functors, e.g. (lax) symmetric monoidal partition functors, partition rings, and so on.

Remark 5.9. Restricting a global Green functor to a partition functor (5.6) yields a partition ring.

Definition 5.10. A **partition power functor** is a partition ring R equipped with multiplicative operations $\mathbb{P}_m : R(\underline{1}) \to R(\underline{m})$ satisfying the usual identities.

The point of this is that global Green functors have a lot of information and a partition power functor is precisely the minimal information you need to talk about power operations on a single group by virtue of the restrictions (5.6).

Fact 5.11. The functor Div is a monad on all of these categories.

Example 5.12 (The initial partition ring)**.** Since the Burnside functor is the initial global Green functor, you might think that the initial partition ring is something like the Burnside ring of Σ_Λ. It turns out that it not quite correct.

Let $\mathring{A}(\Lambda) \subseteq A(\Sigma_\Lambda)$ be the subgroup generated by elements of the form $[\Sigma_\Lambda/\Sigma_\Gamma]$ for $\Gamma \leq \Lambda$, i.e. the Young subgroups of Σ_Λ. These assemble into a partition ring. This is the initial partition ring.

It's way smaller than $A(\Sigma_\Lambda)$ where you can mod out by any subgroup of Σ_Λ! Here is a surprise(?):

Fact 5.13. The composite

$$\mathring{A}(\Lambda) \hookrightarrow A(\Sigma_\Lambda) \xrightarrow{L} \mathrm{RU}(\Sigma_\Lambda)$$

is an isomorphism. In particular, $\mathrm{RU}(\Sigma_{(-)})$ is the initial partition ring.

Observation 5.14. One can check $\mathrm{RU}(\Sigma_\Lambda)/I_\Lambda \cong \mathbb{Z}$. So

$$\mathrm{Div}\left(\mathrm{RU}(\Sigma_{(-)})\right)(\underline{m}) = \left(\bigoplus_{\Omega\subseteq \underline{m}} \mathbb{Z}\right)^{\Sigma_m} \cong \bigoplus_{\lambda\vdash m} \mathbb{Z} = \mathrm{Cl}(\Sigma_m, \mathbb{Z}).$$

Moreover, $\eta : \mathrm{RU}(\Sigma_m) \to \mathrm{Div}(\mathrm{RU}(\Sigma_{(-)}))(\underline{m})$ is the character map.

This is probably the most important example and it shows the difference of partition rings to global Green functors, the initial objects are quite different!

5.4 Symmetric Functions in R

Let R be a lax symmetric monoidal partition functor.

Definition 5.15. Let $R[\Sigma] = \bigoplus_{m\geq 0} R(\underline{m})$.

This is a graded-commutative ring via

$$R(\underline{i}) \otimes R(\underline{j}) \xrightarrow{\boxtimes} R(\underline{i} \amalg \underline{j}) \xrightarrow{\mathrm{Tr}_{\underline{i}\amalg\underline{j}}^{\underline{i+j}}} R(\underline{i+j})$$

abusing the lax symmetric monoidal structure in the first map.

Definition 5.16. Let $R[\![\Sigma]\!] = \prod_{m\geq 0} R(\underline{m})$.

This is still a ring (but no long graded). We are interested in the map

$$R[\![\Sigma]\!] \to \mathrm{Div}(R)[\![\Sigma]\!] = R\langle\!\langle \Sigma \rangle\!\rangle.$$

If R is a partition power functor we can consider

$$\begin{array}{ccccc} R(\underline{1}) & \xrightarrow{\mathbb{P}_m} & R(\underline{m}) & & \\ & \searrow^{\mathbb{P}_m/I_{\underline{m}}} & \downarrow & \searrow^{\eta} & \\ & & R(\underline{m})/I_{\underline{m}} & \xhookrightarrow[u]{} & \mathrm{Div}(R)(\underline{m}) \end{array}$$

The left map is additive and the right triangle does not commute at all!

Theorem 5.17. Assume that R is a partition power functor. Then in $R\langle\langle\Sigma\rangle\rangle$ we have

$$\sum_{m\geq 0} \eta \mathbb{P}_m t^m = \exp\left(\sum_{k\geq 1} u\mathbb{P}_k / I_{\underline{k}} t^k\right).$$

Corollary 5.18. Let S be a lax symmetric monoidal Div-algebra and $R \to S$ be a map of lax symmetric monoidal partition functors. Then, there is a factorization

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & \nearrow & \\ \mathrm{Div}(R) & & \end{array}$$

and thus an exponential relation over S.

Nat was very excited about this when he first understood it but it turns out that it is not as powerful as hoped for because in practice it is pretty hard to construct maps $R \to S$. But you can prove a stronger theorem under stronger hypothesis.

Theorem 5.19. Let R be a symmetric monoidal partition ring and $R(\underline{m})/I_{\underline{m}}$ be a flat $R(\underline{0})$-module for all $m \in \mathbb{N}$. Then, $\mathrm{Div}(R)[\Sigma]$ is the divided power envelope of $R[\Sigma]$ (with ideal given by elements in grading > 0).

This is just a statement in algebra now. It is powerful enough to recover all the exponential relations out there (e.g. 2.29, 4.16) from 5.17 which thus should be viewed as a sort of universal exponential relation.

References

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