Six Functor Formalisms

QI ZHU

February 5, 2026

Abstract

These are my (live) TeX'd notes for the six functor formalisms seminar in Bonn, WiSe 2025/26. There is a more thorough abstract in the seminar program but it's pretty long, so summarized we are going to cover the following:

We start by setting up six functor formalisms following Heyer-Mann [HM24]. Then, we discuss an example on locally compact Hausdorff spaces via Verdier duality following Volpe [Vol21]. We end with an example from the theory of p-adic Lie groups resulting in the linearization hypothesis after Clausen [Cla25].

My notation and language is not always consistent with the speakers' choices. I also occassionally added some parts which were not included in the actual talks; such parts will always be indicated by a star like Lemma*.

Feel free to send me feedback. :-)

Contents

1 Spans and Six-Functor Formalisms (Thomas Blom)

Thomas changed the title of the talk and uses *span* instead of *correspondence* which has long had a meaning in category theory, namely a functor to [1]. TALK 1 16.10.2025

Kaif: *Also, span is another word that starts with* **Sp**.

1.1 What is a six functor formalism, morally?

Let $\mathscr{C}$ be a 'category of geometric objects', e.g. $\mathscr{C} = \mathbf{LCH}$. A *six functor formalism* on $\mathscr{C}$ consists of:

- an '∞-category of sheaves $D(X)$' for every $X \in \mathscr{C}$,
- $D(X)$ is closed symmetric monoidal, so there is $\otimes, \underline{\mathrm{Map}}(-,-), \mathbb{1}$,
- any $f : Y \to X$ induces a 'pullback' functor $f^* : \mathcal{D}(X) \to \mathcal{D}(Y)$ which has a right adjoint f_*, the *pushforward*,
- for 'any' $f : Y \to X$ there is an *exceptional pushforward* $f_! : D(Y) \to D(X)$ which has a right adjoint $f^!$, called *exceptional pullback*.

These satisfy:

- functoriality,
- pullback is strong symmetric monoidal, in particular $f^*(M \otimes N) \simeq f^*M \otimes f^*N$,
- proper base change: For a pullback square

$$\begin{tikzcd} X \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "f"] \\ Y \arrow[r, "g"'] & W \end{tikzcd}$$

  in $\mathscr{C}$ there is an equivalence $g^! f_* \simeq f'_*(g')^!$ or equivalently $f^* g_! \simeq g'_!(f')^*$ after passing to adjoints,
- projection formula: $M \otimes f_! N \simeq f_!(f^*M \otimes N)$.

Example 1.1. Let $\mathscr{C} = \mathbf{LCH}$. Consider $D(X) = \mathbf{Sh}(X, \mathcal{D}\mathbb{Z})$. A map $f : Y \to X$ then gives

- $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$,
- $(f^*\mathcal{F})(U) = L_{\mathbf{Sh}}\left(\mathrm{colim}_{V \supseteq f(U) \text{ open}} \mathcal{F}(V)\right)$,[1]
- $(f_!\mathcal{F})(U) = \operatorname*{colim}_{\substack{K \subseteq f^{-1}(U) \\ K \to U \text{ proper}}} \mathrm{fib}(\mathcal{F}(f^{-1}(U)) \to \mathcal{F}(f^{-1}(U \setminus K)))$[2],
- $f^!$ is kind of mysterious.

Suppose now for simplicity that $\mathscr{C}$ has a terminal object $*$. For $X \in \mathscr{C}$ we will almost always write $p : X \to *$.

[1] You want to evaluate on $f(U)$ but it's not open, so we approximate it by opens. Then, it need not be a sheaf, so we sheafify.

[2] If f is an open embedding, then this is extension by 0.

Definition 1.2. Let D be a six functor formalism on $\mathscr{C}$. For $X \in \mathscr{C}$ and $A \in D(X)$ we define

$$\Gamma(X, A) = p_* A \quad \text{and} \quad \Gamma_c(X, A) = p_! A.$$

Usually, $A = p^* \mathbb{1} = \mathbb{1}$.

Example 1.3. We have

$$H^\bullet(X, \mathbb{Z}) = p_* p^* \mathbb{Z} \quad \text{and} \quad H_c^\bullet(X, \mathbb{Z}) = p_! p^* \mathbb{Z}.$$

You can also do homology and get

$$H_\bullet(X, \mathbb{Z}) = p_! p^! \mathbb{Z} \quad \text{and} \quad H_\bullet^{\text{lf}}(X, \mathbb{Z}) = p_! p^! \mathbb{Z}.$$

Example 1.4 (Künneth)**.** Consider the pullback square

$$\begin{tikzcd}
X \times Y \arrow[r, "f_Y"] \arrow[d, "f_X"'] \arrow[dr, "p"] & Y \arrow[d, "p_Y"] \\
X \arrow[r, "p_X"'] & *
\end{tikzcd}$$

then

$$\begin{aligned}
\Gamma_c(X \times Y) &= p_! \mathbb{1} \\
&\simeq (p_Y)_! (f_A)_! f_X^* \mathbb{1} \\
&\simeq (p_Y)_! p_Y^* (p_X)_! \mathbb{1} \\
&\simeq (p_Y)_! (p_Y^* (p_X)_! \mathbb{1} \otimes \mathbb{1}) \\
&\simeq (p_X)_! \mathbb{1} \otimes (p_Y)_! \mathbb{1} \\
&\simeq \Gamma_c(X) \otimes \Gamma_c(Y)
\end{aligned}$$

Example 1.5 (Poincaré duality)**.** Let $\omega_X = p^! \mathbb{1}$. Then, $\Gamma(X, \omega_X) \simeq \underline{\mathrm{Map}}(\Gamma_c(X), \mathbb{1})$.

Proof. We perform a Yoneda argument, so

$$\begin{aligned}
\mathrm{Map}(M, \Gamma(X, \omega_X)) &\simeq \mathrm{Map}(M, p_* p^! \mathbb{1}) \\
&\simeq \mathrm{Map}(p_! p^* M, \mathbb{1}) \\
&\simeq \mathrm{Map}(M \otimes p_! \mathbb{1}, \mathbb{1}) \\
&\simeq \mathrm{Map}(M, \underline{\mathrm{Map}}(\Gamma_c(X), \mathbb{1})).
\end{aligned}$$

□

If X is orientable, then one can check $\omega_X \simeq \mathbb{1}[n]$, so this suggests Poincaré duality (but certainly this is not yet a proof of the Poincaré duality at this point because you need to prove this equivalence and show the existence of the 6FF and so on).

1.2 What is a six functor formalism, really?

The above 'definition' should hurt from the viewpoint of a homotopy theorist. Let's do it more coherently. Here is a 'pre-definition'.

Definition 1.6. Let $\mathscr{C}$ be an ∞-category with finite limits. Then, a **3FF** is a lax symmetric monoidal functor $D : (\mathbf{Span}(\mathscr{C}), \otimes) \to (\mathbf{Cat}_\infty, \times)$.

We take the cartesian monoidal structure on $\mathscr{C}$ but this does not induce the cartesian monoidal structure on $\mathbf{Span}(\mathscr{C})$.

Notation 1.7. We write

- $f^* = D(X \xleftarrow{f} Y = Y) : D(X) \to D(Y)$,
- $f_! = D(Y = Y \xrightarrow{f} X) : D(Y) \to D(X)$,
- There is a symmetric monoidal structure $(\otimes, \mathbb{1})$ on $D(X)$.[3]

Definition 1.8. A **6FF** is a 3FF such that $f^*, f_!$ admit right adjoints and $\otimes$ is closed.

At this point all exceptional pushforwards exist. The categorical fix is to simply mark those edges for which it should exist.

Definition 1.9. A **geometric setup** is a pair $(\mathscr{C}, \mathscr{E})$ where

(i) $\mathscr{E} \subset \mathscr{C}$ is a wide subcategory,

(ii) pullbacks of maps in $\mathscr{E}$ along any map exist in $\mathscr{C}$ and they lie in $\mathscr{E}$ again,

(iii) $\mathscr{E}$ has pullbacks and $\mathscr{E} \hookrightarrow \mathscr{C}$ preserves these.[4]

(iii') Equivalently: If $f, f \circ g \in \mathscr{E}$, then $g \in \mathscr{E}$.

Definition 1.10. The category **Span**$(\mathscr{C}, \mathscr{E})$ is the full subcategory of **Span**$(\mathscr{C})$ on those spans $\bullet \xleftarrow{f} \bullet \xrightarrow{g} \bullet$ such that $g \in \mathscr{E}$.

Remark 1.11. If we only start with a geometric setup, then we really assume only the existence of pullbacks along maps in $\mathscr{E}$ to be available in $\mathscr{C}$, so it need not have all pullbacks. In that sense, **Span**$(\mathscr{C})$ is ill-defined but this is not so bad, we can freely add in limits – e.g. we can define **Span**$(\mathscr{C}, \mathscr{E})$ as a full subcategory of **Span**$(\mathbf{PSh}(\mathscr{C}))$.

Now, we can redefine 3FF to:

Definition 1.12. Let $\mathscr{C}$ be an ∞-category with finite limits with a geometric setup $(\mathscr{C}, \mathscr{E})$. Then, a **3FF** is a lax symmetric monoidal functor $D : (\mathbf{Span}(\mathscr{C}, \mathscr{E}), \otimes) \to (\mathbf{Cat}_\infty, \times)$.

Proposition 1.13. If $\mathscr{C}$ has finite products, then $\times$ defines a symmetric monoidal structure on **Span**$(\mathscr{C}, \mathscr{E})$.

If $\mathscr{C}$ does not have finite products, then one obtains an ∞-operad **Span**$(\mathscr{C}, \mathscr{E})$. Heyer-Mann spend a lot of effort on this [HM24, Proposition 2.3.3] but according to Thomas one could also just enlarge **Span**$(\mathscr{C}, \mathscr{E})$ to a category with finite products and then take the suitable suboperad.

1.3 Sanity Check

Let's unpack our abstract definition and recover all those desired properties of 6FFs.

(i) The composite

$$(\mathscr{C}^{\mathrm{op}})^{\amalg} \longrightarrow \mathbf{Span}(\mathscr{C}, \mathscr{E}) \longrightarrow \mathbf{Cat}_\infty^\times$$

[3] In $\mathscr{C}^\times$ every objects admits a preferred cocommutative coalgebra structure via the diagonal and by the backwards functoriality via $(-)^*$ it becomes a commutative algebra in **Span**$(\mathscr{C})$. So the lax symmetric monoidality of D sends this to a commutative algebra in **Cat**.

[4] In part (ii) the pullback need not be a pullback in $\mathscr{E}$ because the unique maps coming from the universal property are in $\mathscr{C}$ and need not be in $\mathscr{E}$.

is lax symmetric monoidal.[5] By [Lur17, Theorem 2.4.3.18] this is equivalent to a functor $\mathscr{C}^{\mathrm{op}} \to \mathbf{CMon}(\mathbf{Cat}_\infty)$. This gives $\otimes$ and strong symmetric monoidality on f^*.

(ii) There is an equivalence

$$D(X \xleftarrow{f} Y = Y) \circ D(Z \xleftarrow{g} X = X) \simeq D(Z \xleftarrow{gf} Y = Y)$$

which gives functoriality on f^*. Similarly $f_!$.

(iii) There is an equivalence

$$D(X \xleftarrow{f} Y \xrightarrow{g} Z) \simeq D(Y = Y \xrightarrow{g} Z) \circ D(X \xleftarrow{f} Y = Y) \simeq g_! f^*.$$

With

$$\begin{tikzcd}
& & X \arrow[dl, "f'"'] \arrow[dr, "g'"] & & \\
& Y \arrow[dl, equal] \arrow[dr, "g"'] & & W \arrow[dl, "f"] \arrow[dr, equal] & \\
Y & & Z & & W
\end{tikzcd}$$

we obtain $g'_!(f')^* \simeq f^* g_!$, i.e. proper base change.

(iv) The projection formula is a bit trickier.

$$\begin{tikzcd}
& & Y \arrow[dl, "{(f,\mathrm{id}_Y)}"'] \arrow[dr, "f"] & & \\
& X \times Y \arrow[dl, equal] \arrow[dr, "\mathrm{id}_X \times f"'] & & X \arrow[dl, "\Delta_X"] \arrow[dr, equal] & \\
X \times Y & & X \times X & & X
\end{tikzcd}$$

Thus,

$$\Delta_X^* \circ (\mathrm{id} \times f)_! \simeq f_!(f, \mathrm{id})^* \simeq f_! \circ \Delta_Y^* \circ (f \times \mathrm{id})^*.$$

Using symmetric monoidality of pullbacks we deduce $(-) \otimes f_!(-) \simeq f_!(f^*(-) \otimes (-))$.

1.4 Constructing a 6FF

Often, one can construct some pullback/pushforwards, etc. by hand – see e.g. our example for **LCH** – and so it is desirable that one could glue these together into a 6FF.

Definition 1.14. Let $(\mathscr{C}, \mathscr{E})$ be a geometric setup. A **suitable decomposition** of $\mathscr{E}$ is a pair $I, P \subset \mathscr{E}$ of wide subcategories such that:

(i) $P \circ I = \mathscr{E}$,

(ii) $(\mathscr{C}, I), (\mathscr{C}, P)$ are geometric setups,

(iii) Any $f \in I \cap P$ is n-truncated for some $n \geq -2$.

Condition (iii) is rather technical but at least in the example **LCH** is not relevant since this is an honest $(1,1)$-category.

[5] The first functor is strong symmetric monoidal.

Example 1.15. Let $\mathscr{C} = \mathbf{LCH}$. Any $f : X \to Y$ is a composite

$$X \xrightarrow{i} \overline{X} \xrightarrow{p} Y$$

of an open embedding and a proper map. Indeed, take $\overline{X}$ as the closure of $\Gamma(f) \subseteq X^+ \times Y$.

Theorem 1.16. Let $(\mathscr{C}, \mathscr{E})$ be a geometric setup admitting finite products and $I, P \subset \mathscr{E}$ be a suitable decomposition. Let $D : \mathscr{C}^{\mathrm{op}} \to \mathbf{CMon}(\mathbf{Cat}_\infty)$ be a functor. Suppose:

(i) For $j : U \to X$ in I the functor j^* admits a left adjoint $j_!$ such that the following are satisfied:

(a) Base change: Given a pullback

$$\begin{tikzcd} U' \arrow[r, "i"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "f"] \\ U \arrow[r, "j"'] & X \end{tikzcd}$$

the Beck-Chevalley map $i_! g^* \Rightarrow f^* j_!$ is an equivalence.[6]

(b) Projection: The preferred map $j_!(j^*(-) \otimes (-)) \Rightarrow (-) \otimes j_!(-)$ is an equivalence.

(ii) For $p : Y \to X$ in P the functor p^* admits a right adjoint p_* such that the following are satisfied:

(a) Base change: Given a pullback

$$\begin{tikzcd} Y' \arrow[r, "q"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "f"] \\ Y \arrow[r, "p"'] & X \end{tikzcd}$$

the Beck-Chevalley map $f^* p_* \Rightarrow q_* g^*$ is an equivalence.

(b) Projection: The preferred map $(-) \otimes p_*(-) \Rightarrow p_*(p^*(-) \otimes (-))$ is an equivalence.

(iii) For a pullback square

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "p \in P"] \\ U \arrow[r, "j \in I"'] & X \end{tikzcd}$$

the Beck-Chevalley map $j_! q_* \Rightarrow p_* i_!$ is an equivalence.

Then, D extends to a 3-functor formalism $D : \mathbf{Span}(\mathscr{C}, \mathscr{E}) \to \mathbf{Cat}_\infty$.

One observation is that for this to have any chance to be true, we will have $j^* \simeq j^!$.

There are some extended ways to look at this for which we also obtain uniqueness then [CLL25, DK25]. These solve some really fundamental problems because Liu-Zheng's work is really hard to understand, as the combinatorics going on in their papers is too involved.

Example 1.17. We get a 3FF on $\mathbf{LCH}$ given by $X \mapsto \mathbf{Sh}(X, \mathcal{D}\mathbb{Z})$.

You can still get a 6FF which can be done using Verdier duality. Thomas is not aware of a different way than this to obtain the 6FF on $\mathbf{LCH}$.

[6]Note that this is a property!

1.5 Extending 6FF

This is the last part that we forced Thomas to speak about.

Definition 1.18. Let D be a 3FF on $(\mathscr{C}, \mathscr{E})$ and $\mathscr{C}$ be a site. Then, D is **sheafy** if $D^* : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ is a sheaf.

Proposition 1.19. Let D be a sheafy 3FF on $(\mathscr{C}, \mathscr{E})$ and $\mathscr{C}$ be subcanonical. Let $\mathscr{E}' \subseteq \mathbf{Sh}(\mathscr{C})$ be those maps $f : T \to R$ such that for all $X \in \mathscr{C}$ and all maps $X \to R$ the map $T \times_R X \to X$ lies in $\mathscr{E}$. Then,

(i) $(\mathscr{C}, \mathscr{E}) \to (\mathbf{Sh}(\mathscr{C}), \mathscr{E}')$ is a map of geometric setups,

(ii) D extends to a sheafy 3-functor formalism D' on $(\mathbf{Sh}(\mathscr{C}), \mathscr{E}')$,[7]

(iii) if D is a presentable 6FF, then so is D'.

Remark 1.20. The condition on $\mathscr{E}'$ is essentially that it consists of those maps which locally lie in $\mathscr{E}$. That's one of the main points for the above results. These decompositions into open immersions and proper maps may not be possible but only possible locally.

In practice, $\mathscr{E}'$ is often chosen much larger and indeed, there are ways of enlargening $\mathscr{E}'$ which often is quite specific to the 3FF we consider.

Remark* 1.21. There is another extension result allowing 'stacky' maps [HM24, Theorem 3.4.11] which was not mentioned in this talk. It might feature in future talks.

2 Kernel Categories 1 (Jonah Epstein)

2.1 Recollection on Enriched and $(\infty, 2)$-Categories

TALK 2
23.10.2025

There are several ways to set this up; we follow Gepner-Haugseng [GH15].

Recall that a colored operad/multicategory $\mathscr{M}$ consists of objects and for $X_1, \cdots, X_n, Y \in \mathscr{M}$ a set of multimorphisms $\mathscr{M}(X_1, \cdots, X_n; Y)$ together with identity, composition and associativity assumptions.

Example 2.1. Let S be a set. Then, there is a multicategory $\mathscr{O}_S$ defined as follows:

(i) Objects: $S \times S$,

(ii) Maps: We have

$$\mathscr{O}_S\left((X_0, Y_1), (X_1, Y_2), \cdots, (X_{n-1}, Y_n); (X_0, Y_n)\right) = \begin{cases} * & Y_i = X_i \quad \text{for all } i \\ \varnothing & \text{else.} \end{cases}$$

This multicategory is supposed to encode composition.

Definition 2.2. An **enriched category** with objects S over a monoidal category $\mathscr{V}$ is an $\mathscr{O}_S$-algebra in $\mathscr{V}$.

So this is a map $\mathscr{O}_S \to \mathscr{V}$ and the intuition is that on objects we have $(X, Y) \mapsto \mathrm{Hom}^{\mathscr{V}}(X, Y)$ and on multimorphisms

$$(((X, Y), (Y, Z)) \to (X, Z)) \mapsto \left(\mathrm{Hom}^{\mathscr{V}}(X, Y) \otimes \mathrm{Hom}^{\mathscr{V}}(Y, Z) \to \mathrm{Hom}^{\mathscr{V}}(X, Z)\right).$$

This now generalizes to the ∞-world. If we only want a set of objects, then we can take $\mathscr{O}_S$ and directly use the ∞-version of the above definition [GH15, Definition 2.2.17]. For spaces, Gepner-Haugseng define a generalized ∞-operad Δ_S^{op} [GH15, after Remark 2.4.4].

[7] Recall that a contravariant functor starting from an ∞-topos is a *sheaf* if it is limit-preserving.

Definition* 2.3. Let $S \in \mathcal{S}$. Consider $\Delta^{\mathrm{op}} \to \mathcal{S}$, $[n] \mapsto S^{\times n}$. It can be checked to satisfy the Rezk-Segal conditions [GH15, after Remark 2.4.4], so it unstraightens to a double ∞-category[8] $\Delta_S^{\mathrm{op}} \to \Delta^{\mathrm{op}}$.

So we want to consider the following definition:

Definition 2.4. Let $S \in \mathcal{S}$ and $\mathscr{V}$ be a monoidal ∞-category. A **$\mathscr{V}$-enriched ∞-category** with space of objects S is an Δ_S^{op}-algebra in $\mathscr{V}$.

To define the ∞-category of enriched ∞-categories, we want one for all possible $S \in \mathcal{S}$.

There exists a cartesian fibration $\mathbf{Alg}(\mathscr{V}) \to \mathbf{Op}_\infty^{\mathrm{ns,gen}}$ whose fiber over $\mathscr{O}$ is $\mathbf{Alg}_{\mathscr{O}}(\mathscr{V})$.

Definition 2.5.

(i) The pullback

$$\begin{tikzcd}
\mathbf{Alg}_{\mathrm{cat}}(\mathscr{V}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Alg}(\mathscr{V}) \arrow[d] \\
\mathcal{S} \arrow[r, "\Delta_-^{\mathrm{op}}"'] & \mathbf{Op}_\infty^{\mathrm{ns,gen}}
\end{tikzcd}$$

is the **∞-category of categorical algebras** in $\mathscr{V}$.

(ii) We let $\mathbf{Enr}_{\mathscr{V}} \subseteq \mathbf{Alg}_{\mathrm{cat}}(\mathscr{V})$ be the reflective subcategory where fully faithful and essentially surjective functors are inverted. This is the ∞-category of **$\mathscr{V}$-∞-categories**.

For (ii) we really want to impose some condition forcing $S \simeq \mathscr{C}^{\mathrm{core}}$ for a $\mathscr{V}$-enriched category $\mathscr{C}$. That's why we don't take $\mathbf{Alg}_{\mathrm{cat}}(\mathscr{V})$ but rather $\mathbf{Enr}_{\mathscr{V}}$.

We can transfer enrichments.

Construction 2.6. Let $\alpha : \mathscr{V} \to \mathscr{W}$ be lax monoidal. Then, there is a functor

$$\tau_\alpha : \mathbf{Enr}_{\mathscr{V}} \to \mathbf{Enr}_{\mathscr{W}},\ X \mapsto X,\ \mathrm{Map}^{\mathscr{V}}(X, Y) \mapsto \alpha\left(\mathrm{Map}^{\mathscr{V}}(X, Y)\right).$$

Remark* 2.7. If $\mathscr{V}$ is a presentably monoidal ∞-category, then $\mathrm{Map}_{\mathscr{V}}(\mathbb{1}_{\mathscr{V}}, -) : \mathscr{V} \to \mathcal{S}$ is lax monoidal [GH15, Example 4.3.20], so we can transfer enrichments to obtain the underlying ∞-category in this case. Does some variant work if $\mathscr{V}$ is not presentable?

Remark 2.8. If $\mathscr{V}$ is closed monoidal, then, $\mathscr{V}$ is enriched over itself with mapping objects $\mathrm{Map}^{\mathscr{V}}(X, Y) = \underline{\mathrm{Map}}_{\mathscr{V}}(X, Y)$.

Comment.* It seems like Gepner-Haugseng need to pass to a different model to show this [GH15, Corollary 7.4.10]. In Lurie's model of enriched ∞-categories this is also discussed in [HM24, Example C.1.12]. □

Example 2.9.

(i) $\mathbf{Cat}_\infty$ is self-enriched.

(ii) $\mathbf{Span}(\mathscr{C})$ is self-enriched with $\underline{\mathrm{Map}}_{\mathbf{Span}(\mathscr{C})}(X, Y) = X \times Y$.

Definition 2.10. We write $\mathbf{Cat}_{(\infty,2)} = \mathbf{Enr}_{\mathbf{Cat}_\infty}$ as the ∞-category of $(\infty, 2)$-categories.

Definition 2.11. A map $f : Y \to X$ is an $(\infty, 2)$-category $\mathscr{C}$ is **left adjoint** if there exists a map $g : X \to Y$ and 2-morphisms $\eta : \mathrm{id}_Y \Rightarrow gr$ and $\varepsilon : fg \Rightarrow \mathrm{id}_X$ called **(co-)unit** such that the diagrams

$$\begin{tikzcd}
f \arrow[r, Rightarrow, "f\eta"] \arrow[dr, equal] & fgf \arrow[d, Rightarrow, "\varepsilon f"] \\
& f
\end{tikzcd}
\qquad
\begin{tikzcd}
g \arrow[r, Rightarrow, "\eta g"] \arrow[dr, equal] & gfg \arrow[d, Rightarrow, "g\varepsilon"] \\
& g
\end{tikzcd}$$

commute.

[8] I.e. a generalized non-symmetric ∞-operad $\Delta_S^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ which is a coCartesian fibration.

2.2 Category of Kernels

Throughout the entire talk, assume that $(\mathscr{C}, \mathscr{E})$ is a geometric setup with finite products.

Definition 2.12. Let $D : \mathbf{Span}(\mathscr{C}, \mathscr{E}) \to \mathbf{Cat}_\infty$ be a 3FF.

(i) If $\mathscr{E} = \text{all}$, let $\mathscr{K}_D = \tau_D(\mathbf{Span}(\mathscr{C})) \in \mathbf{Cat}_{(\infty,2)}$ be the **$(\infty, 2)$-category of kernels**.

(ii) Let $S \in \mathscr{C}$ and put $\mathscr{C}_{\mathscr{E}} \subset \mathscr{C}$ be the subcategory spanned by E. Then, $(\mathscr{C}_{\mathscr{E}/S}, E)$ is a geometric setup and there is a map of geometric setups $(\mathscr{C}_{\mathscr{E}/S}, \text{all}) \to (\mathscr{C}, \mathscr{E})$ which thus induces a 3FF
$$D_S : \mathbf{Span}\left(\mathscr{C}_{\mathscr{E}/S}\right) \to \mathbf{Span}(\mathscr{C}, \mathscr{E}) \to \mathbf{Cat}_\infty$$
which hence allows us to define
$$\mathscr{K}_{D,S} = \tau_{D_S}\left(\mathbf{Span}(\mathscr{C}_{\mathscr{E}/S})\right) \in \mathbf{Cat}_{(\infty,2)},$$
the **$(\infty, 2)$-category of kernels**.

Concretely, $\mathscr{K}_{D,S}$ is the following $(\infty, 2)$-category:

- Objects: Maps $X \to S$ in $\mathscr{E}$,
- Maps: $\mathrm{Fun}_S(Y, X) = D(X \times_S Y)$,
- Composition: Let $N \in D(X \times_S Y), M \in D(Y \times_S Z)$. Consider $X \times_S Y \times_S Z$ along with all the possible projections. Then, $M \circ N = (\pi_{13})_!(\pi_{12}^* M \otimes \pi_{23}^* N) \in D(X \times_S Z)$.

Note the self-symmetry. It suggests the following.

Remark 2.13 ([HM24, Proposition 4.1.4])**.** There is an equivalence $\mathscr{K}_{D,S}^{\mathrm{op}} \simeq \mathscr{K}_{D,S}$.

Proposition 2.14 ([HM24, Proposition 4.1.5])**.** The functor D_S from 2.12 splits as
$$\mathbf{Span}(\mathscr{C}_{\mathscr{E}/S}) \xrightarrow{\Phi_{D,S}} \mathscr{K}_{D,S} \xrightarrow{\Psi_{D,S}} \mathbf{Cat}_{(\infty,2)}$$
where the functors are given as follows:

- The functor $\Phi_{D,S}$ is id on objects and on morphisms we start with $f = [Y \leftarrow Z \to X]$ which induces a map $f' : Z \to X \times_S Y$ and we put $\Phi_{D,S}(f) = f'_!(\mathbb{1}) \in D(X \times_S Y)$.
- Put $\Psi_{D,S}(X) = D(X)$ on objects and on maps let $M \in \mathrm{Fun}_S(Y, X) = D(X \times_S Y)$, then we put
$$\Psi_{D,S}(M) = (\pi_1)_! \left(M \otimes \pi_2^*(-)\right) : D(Y) \to D(X).$$

This M is often called *kernel* of the Fourier-Mukai transform $(\pi_1)_!(M \otimes \pi_2^*(-))$.

Remark* 2.15. Kevin Lin's answer in `https://mathoverflow.net/questions/9834` is a great way to motivate this terminology. Recall that a classical Fourier transform is a function $g(y) = \int f(x) e^{2\pi i x y}\, dx$. Here, M resembles $f(x)$ and $e^{2\pi i x y}$ resembles $\pi_2^*(-)$. Integration along the fiber comes from the pushforward. The term $f(x)$ is typically called *integral kernel* and kernel here does not have the meaning from algebra but rather stands for the central object in this integration – in German *Kern* (der Sache). See `https://mathoverflow.net/questions/24098`.

Remark 2.16 ([HM24, Section 4.2])**.** This construction is functorial in D and S.

2.3 Descent

Recall that for a sieve $\mathscr{U} \subseteq \mathscr{C}_{/U}$ a functor $\mathcal{F} : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ *satisfies descent* if $\mathcal{F}(U) \xrightarrow{\simeq} \lim_{V \in \mathscr{U}^{\mathrm{op}}} \mathcal{F}(V)$.

Definition 2.17. A functor $\mathcal{F} : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ satisfies **universal descent** along $\mathscr{U}$ if it has descent along all pullbacks

$$\begin{tikzcd} f^*\mathscr{U} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{U} \arrow[d] \\ \mathscr{C}_{/U'} \arrow[r] & \mathscr{C}_{/U} \end{tikzcd}$$

on $\mathscr{U}$.

If $D : \mathbf{Span}(\mathscr{C}, \mathscr{E}) \to \mathbf{Cat}_\infty$ is a 3FF, then we write $D^* : \mathscr{C}^{\mathrm{op}} \to \mathbf{Span}(\mathscr{C}, \mathscr{E}) \to \mathbf{Cat}_\infty$.

Proposition 2.18 ([HM24, Proposition 4.3.1, 4.3.3])**.**

(i) Let D be a 3FF and $\mathscr{U}$ be a sieve for $X \in \mathscr{C}$. Suppose that D^* satisfies universal descent along $\mathscr{U}$. Then,
$$\mathscr{K}_{D,X} \to \lim_{U \in \mathscr{U}^{\mathrm{op}}} \mathscr{K}_{D,U}$$
is fully faithful.

(ii) Let $\mathscr{U}$ be a sieve on $X \in \mathscr{C}$ and suppose $\mathscr{E} = $ all. Suppose that D^* satisfies universal descent along $\mathscr{U}$. Then,
$$\operatorname*{colim}_{U \in \mathscr{U}} U \xrightarrow{\simeq} X$$
in $\mathscr{K}_D$.

2.4 Suave/Prim Objects and Morphisms

We obtain interesting interactions between $*$ and $!$.

Definition 2.19. Let D be a 3FF on $(\mathscr{C}, \mathscr{E})$. Fix a map $f : X \to S$ in $\mathscr{E}$ and $P \in D(X)$.

(i) We say that P is **f-suave** if it is a left adjoint morphism in $\mathscr{K}_{D,S}$. We write $\mathrm{SD}_f(P) \in D(X)$ for the right adjoint, called **f-suave dual** of P.

(ii) We say that P is **f-prim** if it is a right adjoint morphism in $\mathscr{K}_{D,S}$. We write $\mathrm{PD}_f(P) \in D(X)$ for the left adjoint, called **f-prim dual** of P.

Remark 2.20. Let $\mathrm{id}_X : X \to X$. Then, $P \in D(X)$ is (id_X-)suave if and only if it is prim if and only if it is dualizable.

*Proof**. Each of those three conditions corresponds to the existence of $Q \in \mathrm{Fun}_X(X, X) \simeq D(X)$ with commuting diagrams

$$\begin{tikzcd} P \arrow[r, Rightarrow] \arrow[dr, equal] & P \otimes Q \otimes P \arrow[d, Rightarrow] \\ & P \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[r, Rightarrow] \arrow[dr, equal] & Q \otimes P \otimes Q \arrow[d, Rightarrow] \\ & Q \end{tikzcd}$$

so we win. □

Lemma 2.21 ([HM24, Lemma 4.4.5])**.** Let D be a 6FF. Let $(f : X \to S) \in \mathscr{E}$ and $P \in D(X)$. Then, P is f-suave if and only if the natural map

$$\pi_1^* \underline{\mathrm{Map}}_{D(X)}(P, f^! \mathbb{1}) \otimes \pi_2^* P \to \underline{\mathrm{Map}}_{D(X \times_S X)}(\pi_1^* P, \pi_2^! P)$$

becomes an equivalence after applying $\mathrm{Map}_{D(X)}(\mathbb{1}, \Delta^!(-))$. Then, $\mathrm{SD}_f(P) \simeq \underline{\mathrm{Map}}_{D(X)}(P, f^! \mathbb{1})$.

Here, $\underline{\mathrm{Map}}$ is one of the six functors. There is a similar criterion for primness [HM24, Lemma 4.4.6]. In practice it's often not so hard that these two objects are equivalent but rather that it comes from this natural map.

Heyer-Mann [HM24, Section 4.4] show a bunch of additional stuff about these such as locality on source and target. These are maybe meant to be introduced when we really need to apply them.

Lemma 2.22 ([HM24, Lemma 4.4.18]).

(i) Suppose that $(\Delta_f)_!\mathbb{1} \in D(X \times_S X)$ is compact, then f-suave objects in $D(X)$ are compact.

(ii) Suppose that $\mathbb{1} \in D(S)$ is compact, then f-prim objects in $D(X)$ are compact.

Remark* 2.23. You can also deduce some general duality-type statement and relation between suave and prim duals [HM24, Lemma 4.4.17, 4.4.19]. We see a special case of it in 2.26.

2.5 Suave & Prim Maps

We have just discussed suave and prim objects. Now we discuss suave and prim maps.

Definition 2.24. Let D be a 3FF. Let $f : Y \to X$ be a map in $\mathscr{E}$.

(i) Then, f is **D-suave** if $\mathbb{1} \in D(Y)$ is f-suave. We call $\omega_f = \mathrm{SD}_f(\mathbb{1}) \in D(Y)$ the **dualizing complex**.

(ii) Then f is **D-prim** if $\mathbb{1} \in D(Y)$ is f-prim. We call $\delta_f = \mathrm{PD}_f(\mathbb{1}) \in D(Y)$ the **codualizing complex**.

(iii) A D-suave map f is **D-smooth** if ω_f is invertible.

The dualizing complex is relatively common in geometry but as of now this is not really the case for the codualizing complex.

Lemma 2.25 ([HM24, Lemma 4.5.4, 4.5.5]). Let D be a 6FF and let $\pi_1, \pi_2 : Y \times_X Y \to Y$.

(i) Then, $f : Y \to X$ is D-suave if and only if $\pi_1^* f^! \mathbb{1}_{D(X)} \to \pi_2^! \mathbb{1}_{D(Y)}$ is an equivalence. In this case, $\omega_f \simeq f^! \mathbb{1}_{D(X)}$.

(ii) Then, $f : Y \to X$ is D-prim if and only if $f_!(\pi_2)_* \Delta_! \mathbb{1}_{D(Y)} \to f_* \mathbb{1}_{D(Y)}$ is an equivalence. In this case, $\delta_f \simeq (\pi_2)_* \Delta_! \mathbb{1}_{D(Y)}$.

Now, finally the interaction of $*$ and !; they are related by a twist given suaveness/primness conditions.

Proposition 2.26 ([HM24, Corollary 4.5.11]). Let D be a 6FF.

(i) If f is D-suave, then $\omega_f \otimes f^* \simeq f^!$ and $f^* \simeq \underline{\mathrm{Map}}_{D(Y)}(\omega_f, f^!)$.

(ii) If f is D-prim, then $f_!(\delta_f \otimes -) \simeq f_*$ and $f_! \simeq f_* \underline{\mathrm{Map}}_{D(Y)}(\delta_f, -)$.

Theorem 2.27 (General base change, [HM24, Lemma 4.5.13]). Let

$$\begin{tikzcd}
Y' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "f"] \\
X' \arrow[r, "g"'] & X
\end{tikzcd}$$

be a pullback diagram in $\mathscr{C}_{\mathscr{E}}$.

(i) If g is D-suave, then the natural maps

$$g^* f_* \overset{\simeq}{\Rightarrow} f'_* g'^*, f'_! g'^! \overset{\simeq}{\Rightarrow} g^* f_!, f'^* g^! \overset{\simeq}{\Rightarrow} g'^! f^*, g'^* f^* \overset{\simeq}{\Rightarrow} f'^! g^*$$

are equivalences.

(ii) If g is D-prim, then the natural maps

$$f^* g_* \overset{\simeq}{\Rightarrow} g'_* f^*, g'_! f'^! \overset{\simeq}{\Rightarrow} f^! g_!, g_! f'_* \overset{\simeq}{\Rightarrow} f_* g'_!, f_! g'_* \overset{\simeq}{\Rightarrow} g_* f'_!$$

are equivalences.

Proposition 2.28 ([HM24, Corollary 4.5.18])**.** Let $(f : X \to S) \in \mathscr{E}$.

(i) If f is D-suave, then every dualizable object $P \in D(X)$ is f-suave and $\mathrm{SD}_f(P) \simeq P^\vee \otimes \omega_f$.

(ii) If Δ_f is D-suave, then every f-suave object $P \in D(X)$ is dualizable and in this case, $\mathrm{SD}_f(P) \simeq P^\vee \otimes \omega_{\Delta_f}^{-1}$.

There are some more results in [HM24, Section 4.5] and it's best to have them introduced when we actually need them. All results of this talk are essentially proved by pure abstract nonsense.

3 Category of Kernels 2 (Maria Stroe)

3.1 Étale & Proper Maps

TALK 3
06.11.2025

Recall (2.26): Let D be a 6FF on $(\mathscr{C}, \mathscr{E})$ with $(f : Y \to X) \in \mathscr{E}$.

(i) If f is D-suave, then $\omega_f \otimes f^* = \mathrm{SD}_f(\mathbb{1}) \otimes f^* \simeq f^! : D(X) \to D(Y)$.

(ii) If f is D-prim, then $f_!(\delta_f \otimes -) \simeq f_* : D(Y) \to D(X)$.

We wish to discuss notions that trivialize these twists (3.3) – that's the point of étale and proper maps.

Definition 3.1. Let D be a 3FF on $(\mathscr{C}, \mathscr{E})$ and let $(f : Y \to X) \in \mathscr{E}$ be a truncated map.

(i) We say that f is ***D*-étale** if it is D-suave and Δ_f is D-étale or an equivalence.

(ii) We say that f is ***D*-proper** if it is D-prim and Δ_f is D-proper or an equivalence.

We can make this inductive definition by the inductive nature of truncatedness: f is n-truncated if Δ_f is $(n-1)$-truncated and (-2)-truncated if it is an equivalence.

Remark* 3.2. Suppose that f is n-truncated. Then, f is D-étale if $f, \Delta_f, \Delta_{\Delta_f}, \cdots$ are all D-suave (eventually, we get an equivalence).

The next result shows that twists are trivialized in the étale/proper setting.

Proposition 3.3. Let D be a 6FF on $(\mathscr{C}, \mathscr{E})$ and $(f : Y \to X) \in \mathscr{E}$ such that Δ_f is D-étale. Then, there exists a preferred natural transformation $f^! \Rightarrow f^*$ of functors $D(X) \to D(Y)$ such that TFAE:

(i) f is D-étale.

(ii) $f^! \mathbb{1}_X \to f^* \mathbb{1}_X$ is an equivalence in $D(Y)$.

(iii) $f^! \Rightarrow f^*$ is an equivalence of functors $D(X) \to \mathcal{D}(Y)$

Proof. Induction on truncatedness n of f. By induction hypothesis suppose that $\Delta_f^! \simeq \Delta_f^*$.

Let's discuss the base step, so say that $g : X \to Y$ is an equivalence. By base change on

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, equal] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "g^{-1}"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

we get $g^! \simeq g^*$.

Then, we construct

$$f^! \simeq \Delta_f^* \pi_2^* f^! \simeq \Delta_f^! \pi_2^* f^! \Rightarrow \Delta_f^! \pi_1^! f^* \simeq f^*$$

with

$$\begin{tikzcd} Y \arrow[dr, "\Delta_f"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, equal] & & \\ & Y \times_X Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "f"] \\ & Y \arrow[r, "f"'] & X \end{tikzcd}$$

Here, we use functoriality and an explicit natural transformation $\pi_2^* f^! \Rightarrow \pi_1^! f^*$ which we now construct. For this, we use the two ingredients:

- Base change: $\pi_{1!}\pi_2^* \overset{\simeq}{\Rightarrow} f^* f_!$,
- $f_! \dashv f^!, \pi_{1!} \dashv \pi_1^!$.

So we can write

$$\pi_2^* f^! \overset{\eta_{\pi_1}}{\Longrightarrow} \pi_1^! \pi_{1!} \pi_2^* f^! \overset{\simeq}{\Rightarrow} \pi_1^! f^* f_! f^! \overset{\varepsilon_f}{\Rightarrow} \pi_1^! f^*.$$

Now let's start the proof.

(ii) $\implies$ (i): We want to show that f is D-suave. Consider the map

$$\pi_2^* f^! \mathbb{1}_X \to \pi_1^! f^* \mathbb{1}_X \simeq \pi_1^! \mathbb{1}_Y.$$

By (ii) it is an equivalence after applying $\Delta_f^!$, so we are done by [HM24, Lemma 4.5.4].

(i) $\implies$ (iii): From suave base change (2.27) we get that $\pi_2^* f^! \Rightarrow \pi_1^! f^*$ is an equivalence by suave base change and applying $\Delta_f^!$ to this implies $f^! \overset{\simeq}{\Rightarrow} f^*$.

□

There is a similar result for properness by replacing $(-)^!, (-)^*$ by $(-)_!, (-)_*$ [HM24, Lemma 4.6.4(ii)].

Lemma 3.4. Let D be a 3FF on $(\mathscr{C}, \mathscr{E})$.

(i) Then, D-étale maps are stable under base change and composition, and if $f, g \in \mathscr{E}$ and fg, Δ_f are D-étale, then g is D-étale.

(ii) The D-étaleness of a truncated map is D^*-local on the target. If D is compatible with small colimits, then we can check D-étaleness of f on a universal D^*-cover of D-étale maps in $\mathscr{C}_{\mathscr{E}}$ of the source.

(ii') An $f \in \mathscr{E}$ is D-proper if it is so on a universal D-proper D^*-cover on the source such that $f_!$ commutes with $\mathscr{U}^{\mathrm{op}}$-indexed limits.

Proof.

(i) Let's only do base change. Suppose that $f : Y \to X$ is D-étale and that $f' : Y' \to X'$ is a base change of f. Induction on n. Since f is n-truncated, also f' is n-truncated.

- f' is D-suave: D-suaveness is preserved under base change.
- $\Delta_{f'}$ is D-étale: Note that $\Delta_{f'}$ is $(n-1)$-truncated and a pullback of Δ_f, so by induction hypothesis $\Delta_{f'}$ is D-étale.

(ii) Assume that f is n-truncated and locally D-étale on some universal D^*-cover $\mathscr{U}$ of the target. Then, f is D-suave since D-suaveness is stable under base change. We're left to show that Δ_f is D-étale. Consider the diagram

$$\begin{tikzcd} V \arrow[r, "\Delta_{f_U}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V \times_U V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d] \\ Y \arrow[r, "\Delta_f"'] & Y \times_X Y \arrow[r, "\pi"'] & X \end{tikzcd}$$

where $\pi = f \circ \pi_1 = f \circ \pi_2$. By assumption, Δ_{f_U} is D-étale, so Δ_f is locally D-étale on the universal D^*-cover $\pi^*\mathscr{U}$.

$\square$

3.2 Descendability and Exceptional Descent

Goal: Suave and prim maps are good sources of $*$-covers and !-covers. To begin, let's recall some descent statements.

Definition 3.5 (Descent data)**.** Let $\mathscr{C}, \mathscr{V} \in \mathbf{Cat}_\infty$ and $\mathscr{U} \subseteq \mathscr{C}_{/U}$ be a sieve and $\mathcal{F} : \mathscr{C}^{\mathrm{op}} \to \mathscr{V}$. Then, we write

$$\mathrm{Desc}(\mathscr{U}, \mathcal{F}) = \lim_{V \in \mathscr{U}^{\mathrm{op}}} \mathcal{F}(V).$$

We say that $\mathcal{F}$ **descends along** $\mathscr{U}$ if $\mathcal{F}(U) \xrightarrow{\simeq} \mathrm{Desc}(\mathscr{U}, \mathcal{F})$.

Definition 3.6. Let D be a 6FF on $(\mathscr{C}, \mathscr{E})$.

(i) We say that $\mathscr{U} \subseteq (\mathscr{C}_{\mathscr{E}})_{/U}$ is a $D^!$**-cover** if it is generated by a small family $\{U_i \to U\}_i$ and $D^!$ descends along $\mathscr{U}$.

(ii) We say that $\mathscr{U}$ is a **universal** $D^!$**-cover** if for every $V \to U$ in $\mathscr{C}$ the family $\{U_i \times_U V \to V\}$ generates a $D^!$-cover.

(iii) A map $f : Y \to X$ is a **(universal)** $D^!$**-cover** if the sieve generated by f is a (universal) $D^!$-cover.

Lemma 3.7. Let D be a 6FF on $(\mathscr{C}, \mathscr{E})$ and $(f : Y \to X) \in \mathscr{E}$ be such that $D(X)$ has all countable limits and colimits. If f is D-suave and $f^* : D(X) \to D(Y)$ is conservative, then f is a universal !-cover and $*$-cover.[9]

[9] We need limits for ! and colimits for $*$.

Proof. One can view $D^!$-descent as

$$D^!(X) \xrightarrow{\simeq} \lim_{[n]\in\Delta} D^!(Y_n)$$

with $Y_n = Y^{\times_X(n+1)}$. We use Lurie's Beck-Chevalley condition [Lur17, Corollary 4.7.5.3]:

(1) The ∞-category $D(X)$ admits geometric realizations of $f^!$-split simplicial objects and these geometric realizations are preserved by $f^!$.

(2) Let $\alpha : [m] \to [n]$, then

$$\begin{tikzcd} D^!(Y_m) \arrow[r, "d^{0!}_{(m)}"] \arrow[d, "\alpha^!"'] \arrow[dr, phantom, "\lrcorner", very near start] & D^!(Y_{m+1}) \arrow[d, "\alpha^!"] \\ D^!(Y_n) \arrow[r, "d^{0!}_{(n)}"'] & D^!(Y_{n+1}) \end{tikzcd}$$

is left adjointable where $d^0 : Y_{m+1} \simeq Y_m \times_X Y \to Y_m \times_X X$.

Here is why:

(1) By assumption, $D(X)$ has all countable colimits. By D-suaveness, $f^!(-) \simeq \omega_f \otimes f^*(-)$ and f^* is a left adjoint, so it preserves colimits. Thus, $f^!$ preserves colimits.

(2) By suave base change $d^0_{(n)!}\alpha'^! \overset{\simeq}{\Rightarrow} \alpha^! d^0_{(m)!}$ (see 2.27).

By (1) & (2) we get that $D^!(X) \to \lim_{n\in\Delta} D^!(Y_n)$ has a fully faithful left adjoint. Moreover, $f^* \simeq \underline{\mathrm{Map}}_{D(Y)}(\omega_f, f^!)$, so $f^!$ (see 2.26) is also conservative. $\square$

There is a similar criterion for prim maps but it is slightly more involved. To discuss this, we will take a detour first.

3.3 Mathew's Notion of Descent

Let us take a quick detour to Mathew's descendability notion [Mat16].

Definition 3.8. Let $(\mathscr{C}, \otimes, \mathbb{1})$ be a symmetric monoidal stable ∞-category and $A \in \mathbf{CAlg}(\mathscr{C})$. We write $\langle A \rangle \subseteq \mathscr{C}$ for the thick $\otimes$-ideal generated by A.

(i) We say that A is **descendable** if $\mathbb{1}_{\mathscr{C}} \in \langle A \rangle$.

(ii) Suppose furthermore that $\mathscr{C}$ is presentable and that $\otimes$ commutes with all colimits in both variables. Then, $\mathscr{C}$ is called a **stable homotopy theory**.

(iii) Let $\mathscr{C}$ be a stable homotopy theory. Then, $A \to B$ **admits descent** if B is descendable in $\mathbf{CAlg}(\mathbf{Mod}_A(\mathscr{C}))$.

In particular, A being descendable means that $\mathbb{1} \to A$ admits descent.

Example 3.9.

(i) Let $R \in \mathbf{CRing}$, then $\mathcal{D}(R)$ is a stable homotopy theory.

(ii) Let X be a scheme or a prestack. Then, $\mathbf{QCoh}_X$ is a stable homotopy theory.

Definition 3.10. Let $\mathscr{C}$ be an ∞-category with finite colimits.

(i) A filtered diagram $F : I \to \mathscr{C}$ is **ind-constant** if it lies in the essential image of $\mathscr{C} \to \mathrm{Ind}(\mathscr{C})$.

(ii) A simplicial object $M_\bullet : \Delta^{\mathrm{op}} \to \mathscr{C}$ is **ind-constant** if $\mathbb{Z}_{\geq 0} \to \mathscr{C}$, $n \mapsto \lim_{m \in \Delta_{\leq n}} M_n$ is ind-constant.

There is a dual notion for *pro-constant cofiltered diagrams* and *pro-constant cosimplicial objects*.

Definition 3.11. Let $A \in \mathbf{CAlg}(\mathscr{C})$. We write

$$\mathrm{CB}^\bullet(A) = (A \rightrightarrows A \cdots)$$

for the **cobar resolution**.

Proposition 3.12 ([Mat16, Proposition 3.20])**.** An object $A \in \mathbf{CAlg}(\mathscr{C})$ is descendable if and only if $\mathrm{CB}^\bullet(A)$ defines a constant pro-object on $\{\mathrm{Tot}_n\, \mathrm{CB}^\bullet(A)\}_n$ which converges to $\mathbb{1}$ and $(\mathbb{1}_{\mathscr{C}})_n \to \{\mathrm{Tot}_n\, \mathrm{CB}^\bullet(A)\}_n$ is a pro-isomorphism.

Proposition 3.13 ([Mat16, Proposition 3.22])**.** Let $\mathscr{C}$ be a stable homotopy theory and A be descendable. Then, the adjunction

$$\mathscr{C} \underset{U}{\overset{A\otimes -}{\rightleftarrows}} \mathbf{Mod}_A(\mathscr{C})$$

is comonadic. In particular,

$$\mathscr{C} \xrightarrow{\simeq} \mathrm{Tot}\,(\mathbf{Mod}_A(\mathscr{C}) \rightrightarrows \mathbf{Mod}_{A\otimes A}(\mathscr{C}) \cdots)\,.$$

3.4 Back to Six Functor Formalisms

One can state Mathew's notions slightly more generally, namely in a stable monoidal ∞-category $\mathscr{C}$ instead of in $\mathbf{CAlg}(\mathscr{C})$.

Lemma 3.14 ([HM24, Lemma 4.7.4])**.** Let D be a stable 6FF and $(f : Y \to X) \in \mathscr{E}$ such that $D(X)$ has all countable colimits and limits. Suppose that f is D-prim and that $f_*\mathbb{1} \in D(X)$ is descendable.

(i) Then, f is a universal D^*-cover and $D^!$-cover.

(ii) Every $f^!$-split simplicial object in $D(X)$ is ind-constant and every f^*-split cosimplicial object in $D(X)$ is pro-constant.

Proof Idea. We want to show that $f_! f^! \in \mathrm{Fun}(D(X), D(X))$ is descendable (but not in the algebra category). This can be reduced into descendability of $f_*\mathbb{1}$. □

4 Six-Functor Formalism on Condensed Anima (Gabriel Ong)

We will work light today, i.e. set $\kappa = \aleph_1$. Some set-theoretic technicalities go away then. TALK 4 13.11.2025

4.1 Condensed Math

We want topological algebra to behave better categorically.

Example 4.1. The category **TopAb** is not an abelian category, but **Cond**(**Ab**) is.

We try to rebuild ordinary algebra in condensed land. Instead of sets, abelian groups, rings, ... consider condensed sets, condensed abelian groups, condensed rings and so on. But really you should look at *analytic rings*. These date back to old ideas like Johnstone's observation that **CHaus** $\to$ **Set** is monadic.

Definition 4.2.

(i) Let **ProFin** be the full subcategory of the 1-category of topological spaces spanned by sequential limits of finite sets.[10]

(ii) It becomes a site with covers the finitely jointly surjective families.

Definition 4.3. A **condensed anima** X is a hypersheaf of anima on **ProFin**. This gives rise to an ∞-category **Cond(An)**.

Definition 4.4.

(i) A **surjection** of condensed anima is an effective epimorphism.

(ii) A **quasicompact** condensed anima is an object if all covers[11] admits a finite subcover.

(iii) A condensed anima X is **quasiseparated** if for $Y, Z \to X$ with Y, Z qc also $Y \times_X Z$ is qc.

For $X \in \mathbf{Top}$ consider $\mathrm{Hom}_{\mathbf{Top}}(-, X) : \mathbf{ProFin}^{\mathrm{op}} \to \mathbf{Set}$. This is a condensed set.

Proposition 4.5.

(i) This construction gives a fully faithful embedding

$$\text{(metrizable compactly generated spaces)} \hookrightarrow \mathbf{Cond}(\mathbf{An}).$$

(ii) This restricts to an equivalence

$$\text{(metrizable compact Hausdorff spaces)} \simeq \mathbf{Cond}(\mathbf{Set})^{\mathrm{qcqs}}.$$

We shall briefly remark that metrizable spaces are already compactly generated.

4.2 Six Functor Formalism for Λ-Sheaves

Recall that singular cohomology $H^i(X; \mathbb{Z})$ can be written as sheaf cohomology $H^i(X; \underline{\mathbb{Z}})$ for nice enough spaces.

Definition 4.6. Let S be a finite set and Λ be a ring. We denote by $\Lambda(S) = \prod_{x \in S} \Lambda$ the set of Λ-valued continuous functions.

Proposition 4.7 ([HM24, Construction 3.5.16, Lemma 3.5.12])**.**

(i) The functor $\mathbf{Fin}^{\mathrm{op}} \to \mathbf{CAlg}(\mathbf{Pr}^L_{\mathrm{st}})$, $S \mapsto \mathbf{Mod}_{\Lambda(S)}$ extends to a 6FF on **ProFin** given by $\lim_i S_i \mapsto \mathrm{colim}_i\, \mathbf{Mod}_{\Lambda(S_i)}$.

(ii) This 6FF satisfies hyperdescent.

Proof.

(i) Use monadicity and construction from suitable decompositions with $E = P =$ (all) and $I =$ (equiv).

(ii) We do this for static Λ. More generally, run Lurie's faithfully flat descent [Lur18, Appendix D]. It's enough to show that for every surjection $S' \twoheadrightarrow S$ between profinite sets that $\Lambda(S) \to \Lambda(S')$ is faithfully flat. Pass to presentations and check this termwise.

[10] These are light profinite sets!
[11] These are defined by surjections from (i).

□

There are two ways to extend to stacks.

(i) New !-able maps to those locally !-able in the site.

(ii) More generally, take E' to be bigger than this locally !-able class.

Here is an example from scheme theory.

Example 4.8. Let $\mathscr{C} = \mathbf{AffSch}$. Then, $\mathbb{P}^1_{\mathbb{Z}}$ is a stack but pullback over $\operatorname{Spec} k \to \operatorname{Spec} \mathbb{Z}$ gives $\mathbb{P}^1_k$ but $\mathbb{P}^1_k \to \operatorname{Spec} k$ is not an affine map.

This suggests that (i) is often not so useful.[12]

Theorem 4.9 ([HM24, Theorem 3.4.11])**.** Let $D_0 : \mathbf{Span}(\mathscr{C}, \mathscr{E}) \to \mathbf{Pr}^L$ be a 6FF with (hyper)descent for (hyper)subcanonical $\mathscr{C}$. Then, there exists some (minimal choice of) $\mathscr{E}'$ which is

- $*$-local on the target,
- !-local on the source or target,
- tame, i.e. every map $f : Y \to X$ in $\mathscr{E}'$ with $X \in \mathscr{C}$ is !-locally on the source in $\mathscr{E}$.

such that D_0 extends uniquely to $D : \mathbf{Span}(\mathscr{X}, \mathscr{E}') \to \mathbf{Cat}_\infty^\times$.

Applied to $D(-, \Lambda) : \mathbf{Span}(\mathbf{ProFin}) \to \mathbf{Cat}_\infty$ we deduce:

Theorem 4.10. There is a collection of maps $\mathscr{E}'$ in $\mathbf{Cond}(\mathbf{An})$ uniquely extending the 6FF on $\mathbf{ProFin}$ to $D : \mathbf{Span}(\mathbf{Cond}(\mathbf{An}), \mathscr{E}') \to \mathbf{Cat}_\infty^\times$ where:

(i) *-local: An $f : X \to Y$ lies in $\mathscr{E}'$ if and only if for every representable S there exists $S \to Y$ we have $X \times_Y S \to S$ is in $\mathscr{E}'$.

(ii) !-local: Membership in $\mathscr{E}'$ can be checked on after composition or pullback with a universal !-cover.

(iii) Tame: Maps $f : X \to S$ are in $\mathscr{E}'$ if and only if there exists a !-cover of X and the composition of f with any map in the cover is in $\mathscr{E}$.

Definition 4.11. The **6FF for Λ-sheaves on $\mathbf{Cond}(\mathbf{An})$** is the one from the theorem (**4.10**). The maps in $\mathscr{E}'$ are called **Λ-fine**.

Definition 4.12. Let Λ be a ring and $f : X \to *$ for $X \in \mathbf{Cond}(\mathbf{An})$.

(i) The **Λ-valued cohomology** is $\Gamma(X, \Lambda) = f_* \mathbb{1}_X \in \mathbf{Mod}_\Lambda$.

(ii) The **compactly supported Λ-valued cohomology** is $\Gamma_c(X, \Lambda) = f_! \mathbb{1}_X \in \mathbf{Mod}_\Lambda$ for Λ-fine f.

Similarly, cohomology of sheaves.

[12] However, one could first try to enlarge $\mathscr{C}$ and then apply (i).

4.3 Poincaré Duality

As usual in the stacky world, an open/closed immersion of condensed anima is a map which after pulling back along a representable is an open/closed subset.

Lemma 4.13. Let $\Lambda \in \mathbf{CRing}$.

(i) Every open immersion of condensed anima is étale.

(ii) Every map from a profinite set to a qs condensed set is proper. In particular, every closed immersion of condensed anima is proper.

Proof.

(i) By locality reduce to profinite sets $f : X \to Y$. Reduce further to f being an inclusion of clopens. Any open is a finite disjoint union of clopens in the light setting. The functors have an explicit description here, f_* is the restriction, check explicitly with unit and counit.

(ii) Check after pulling back to a profinite set. Consider $f : S \to X$ and $T \to X$. Show that $S \times_X T \to T$ is proper. Consider the subset $S \times_X T \subseteq X \times T$. The source and target are profinite and by conditions from last time we can deduce that this is proper.

□

Proposition 4.14. Let X be a metrizable compact Hausdorff space and

$$U \xrightarrow{j \text{ open}} X \xleftarrow{i \text{ closed}} Z$$

with $X = U \sqcup Z$ in **Set**. Then,

$$j_! \mathbb{1}_U \longrightarrow \mathbb{1}_X \longrightarrow i_* \mathbb{1}_Z$$

is a fiber sequence in $D(X, \Lambda)$.

Proof. Base change shows $j^* i_* \mathbb{1}_Z \simeq 0$. So $j^* \operatorname{fib}(\mathbb{1}_X \to i_* \mathbb{1}_Z) \simeq j^* \mathbb{1}_X \simeq \mathbb{1}_U$. Adjunction gives a map

$$j_! \mathbb{1}_U \to \operatorname{fib}(\mathbb{1}_X \to i_* \mathbb{1}_Z).$$

Since this was a disjoint decomposition, (j^*, i^*) is a conservative family of functors [HM24, Lemma 4.8.5]. So we are done. □

Lemma 4.15. Let $f : [0, 1] \to *$ be the projection.

(i) Then, f is proper.

(ii) Let X be a metrizable locally compact Hausdorff space and let $P : X \times [0, 1] \to X$. Then, P^* is fully faithful.

Proof.

(i) Consider $g : \{0, 1\}^{\mathbb{N}} \twoheadrightarrow [0, 1]$. Then, g is proper and $f \circ g : \{0, 1\}^{\mathbb{N}} \to *$ is proper. By cancellation we need to show that $g_* \mathbb{1}_{\{0,1\}^{\mathbb{N}}}$ is descendable.

For $n > 0$ let C_n be the disjoint union of 2^n closed intervals, e.g. $C_1 = [0, 1/2] \amalg [1/2, 1]$, with $\{0, 1\}^{\mathbb{N}} \cong \lim_n C_n$. Write $h_n : C_n \to [0, 1]$. Then, $g_* \mathbb{1}_{\{0,1\}^{\mathbb{N}}} \simeq \operatorname{colim}_n h_{n*} \mathbb{1}_{C_n}$, so it's enough to show that $h_{n*} \mathbb{1}_{C_n}$ is descendable. The pushforward of $\mathbb{1}$ on this closed cover is descendable in this setting.

□

Theorem 4.16. Let $f : X \to *$ be the projection from a manifold X. Then, ω_X is locally equivalent to $\Lambda[-n]$ where n is the local dimension.

Proof. By locality, it's enough to consider $X = \mathbb{R}^n$. Can further reduce to $X = \mathbb{R}$. Consider $j : \mathbb{R} \xrightarrow{\sim} (0, 1) \hookrightarrow [0, 1]$. We have $f_* j_! \mathbb{1}_{\mathbb{R}} \simeq \Gamma_c(\mathbb{R}, \Lambda)$ is the fiber of $\Gamma([0, 1], \Lambda) \to \Gamma(\partial[0, 1], \Lambda)$ which is $\Lambda[-1]$. Thus, $\omega_X \simeq \Lambda[-n]$ by locality. □

Corollary 4.17. Let X be a manifold and $\Lambda \in \mathbf{CRing}$. Then, $\Gamma(X, \omega_X) \simeq \Gamma_c(X, \Lambda)^\vee$.

Proof. We have $\Gamma(X, \omega_X) \simeq f_* f^! \mathbb{1} \simeq f_* \underline{\mathrm{Map}}_X(\mathbb{1}_X, f^! \mathbb{1}_*) \simeq \mathrm{Map}_*(f_! \mathbb{1}_X, \mathbb{1}_*) \simeq \Gamma_c(X, \Lambda)^\vee$. □

5 Applications to (Smooth) Representation Theory (Qi Zhu)

Surely, you know representation theory. Now, you just have to be smooth about it! TALK 5 20.11.2025

5.1 Smooth Representation Theory through Condensed Anima

5.1.1 Condensed Anima & Classifying Stacks

Recall from last talk that **Cond(An)** $= \mathbf{Sh}^{\mathrm{hyp}}(\mathbf{ProFin})$ and the six functor formalism of condensed anima.

Recollection 5.1. Let $\Lambda \in \mathbf{CAlg}$, then the universal property of Ind yields a diagram

$$\begin{tikzcd}
* \arrow[r, "\Lambda"] \arrow[d, hook, "\langle 1 \rangle"'] & \mathbf{CAlg} \\
\mathbb{F}^{\mathrm{op}} \arrow[ur, dashed] \arrow[d, hook] & \\
\mathbf{ProFin}^{\mathrm{op}} \arrow[uur, dotted, "(S_i)_i \mapsto \mathrm{colim}_i \prod_{x \in S_i} \Lambda"'] &
\end{tikzcd}$$

and postcomposing with $\mathbf{Mod}_{(-)}$ gives $D(-, \Lambda) : \mathbf{ProFin}^{\mathrm{op}} \to \mathbf{Cat}_\infty$. This can be extended to $\mathbf{Cond}(\mathbf{An})^{\mathrm{op}}$ and then to a six functor formalism $D(-, \Lambda) : \mathbf{Span}(\mathbf{Cond}(\mathbf{An}), \Lambda\text{-fine}) \to \mathbf{Cat}_\infty$ [HM24, Construction 3.5.16].

The ∞-topos $\mathbf{Cond}(\mathbf{An})$ contains **Top** but also a homotopical direction – in particular, it allows us to form classifying stacks of topological groups. We will use this observation to study smooth representation theory of locally profinite groups.

Definition 5.2. A **locally profinite group** is a Hausdorff, locally compact, totally disconnected topological group.

So compact locally profinite groups are precisely the profinite groups.

Example 5.3. This includes profinite groups like Galois groups of (infinite) field extensions $\mathrm{Gal}(L/K)$ or the Morava stabilizer group $\mathbb{G}$, but also discrete groups, $\mathbb{Q}_p$ and p-adic Lie groups such as $\mathrm{GL}_n(\mathbb{Q}_p)$.

Let G be a locally profinite group, then it is in particular a group object in $\mathbf{Cond}(\mathbf{An})$. If it acts on some $X \in \mathbf{Cond}(\mathbf{An})$, then we can form the stacky quotient

$$X /\!\!/ G = \operatorname*{colim}_{[n] \in \Delta^{\mathrm{op}}} G^{\times n} \times X \in \mathbf{Cond}(\mathbf{An}).$$

We will in particular care about the classifying stacks $* /\!\!/ G$. Indeed, it gives information about representation theory as follows!

5.1.2 Representation Theory

Let's define smooth representation theory!

Definition 5.4. Let G be a locally profinite group, $\Lambda \in \mathbf{CRing}$ and V be a continuous G-representation. It is **smooth** if $\mathrm{Stab}_G(v) \subseteq G$ is open for every $v \in V$. We write $\mathbf{Rep}_\Lambda(G)^\heartsuit$ for the 1-category of smooth G-representations and

$$\mathbf{Rep}_\Lambda(G) = \mathcal{D}\left(\mathbf{Rep}_\Lambda(G)^\heartsuit\right) \quad \text{and} \quad \widehat{\mathbf{Rep}}_\Lambda(G) = \widehat{\mathbf{Rep}_\Lambda(G)}$$

for its unbounded derived category and the left t-completion thereof.

Theorem 5.5 ([HM24, Proposition 5.1.12])**.** Let $\Lambda \in \mathbf{CRing}$ and G be a locally profinite group. There, there is a natural t-exact equivalence $D(* /\!\!/ G, \Lambda) \simeq \widehat{\mathbf{Rep}}_\Lambda(G)$.

Proof Idea. The proof strategy is by *derived descent from abelian descent*.

1. One first develops some general abstract nonsense to discuss the question for which $X \in \mathbf{Cond}(\mathbf{An})$ the ∞-category $D(X, \Lambda)$ is the (left t-completion[13] of the) derived category of its heart. This turns out to be true for $* /\!\!/ G$, so $D(* /\!\!/ G, \Lambda) \simeq \widehat{\mathcal{D}}\left(D(* /\!\!/ G, \Lambda)^\heartsuit\right)$ [HM24, Example 5.1.2, Proposition 5.1.8].

   Thus, it suffices to prove $D(* /\!\!/ G, \Lambda)^\heartsuit \simeq D(* /\!\!/ G, \Lambda)^\heartsuit$. In other words, it suffices to study the relevant abelian descent data to obtain derived descent.

2. To perform abelian descent one notices $D(G^n, \Lambda) \simeq \mathcal{D}\left(\mathbf{Mod}^\heartsuit_{\Lambda_c(G^n)}\right)$ where we denote by $\Lambda_c(G^n) \subseteq \Lambda_c(G^n)$ locally constant functions $G^n \to \Lambda$ with compact support [HM24, Lemma 5.1.9].[14] Writing out the descent diagram for $* /\!\!/ G$ and noting that we are working in 1-categories, we obtain that $D(* /\!\!/ G, \Lambda)^\heartsuit$ is the limit of

$$\mathbf{Mod}^\heartsuit_\Lambda \rightrightarrows \mathbf{Mod}^\heartsuit_{\Lambda_c(G)} \underset{\pi_1^*}{\overset{\pi_2^*}{\underset{- m^* \to}{\rightrightarrows}}} \mathbf{Mod}^\heartsuit_{\Lambda_c(G\times G)},$$

   i.e. abelian descent.

   At this point, writing out an equivalence $\mathbf{Rep}_\Lambda(G)^\heartsuit \to D(* /\!\!/ G, \Lambda)^\heartsuit$ is a 1-categorical problem which can be handled by hand [HM24, Proposition 5.1.12].

□

Remark 5.6 ([HM24, Corollary 5.1.14, Remark 5.1.15])**.** Let $\varphi : H \to G$ be a map of locally profinite groups. This induces an adjunction

$$D(* /\!\!/ G, \Lambda) \underset{f_*}{\overset{f^*}{\rightleftarrows}} D(* /\!\!/ H, \Lambda)$$

which can be described in terms of smooth representations.

(i) The pullback f^* is the derived functor of taking a G-representation to its underlying H-representation. It is called **restriction/inflation** depending on whether f is injective or surjective.

[13] This part is automatic [HM24, Lemma 3.5.14].

[14] This result is stated for disjoint unions of profinite sets. To apply it to the locally profinite G we note that by van Dantzig's theorem there exists a compact open subgroup $K \leq G$, so we obtain a disjoint union decomposition $G = \bigsqcup_{[g]\in G/K} gK$.

(ii) If φ is the inclusion of a closed subgroup, then f_* is the right derived functor of smooth induction $R\operatorname{Ind}_H^G$. It φ is a topological quotient map with kernel U, then f_* is the right derived functor of taking U-fixed points $R(-)^U$, also denoted $(-)^U$.

(iii) The symmetric monoidal structure corresponds to the underlying tensor product of Λ-modules with diagonal G-action.

5.2 Six Functors in Representation Theory

We have already described some of the six operations. Now, we shall also describe the !-functor and discuss some of the six functor formulaic features.

5.2.1 !

Let $\Lambda \in \mathbf{CRing}$ and G be a locally profinite group, then natural maps such as $* /\!\!/ G \to *$ need not be Λ-fine, but we want shriekability to study six functor phenomena like being suave/prim. We fix this by posing mild conditions.

Definition 5.7. Let $\Lambda \in \mathbf{CRing}$.

(i) Let G be a profinite group. We call

$$\operatorname{cd}_\Lambda G = \sup\left\{n : H^n(G,V) \neq 0 \text{ for some } V \in \mathbf{Rep}_\Lambda(G)^\heartsuit\right\} \in \mathbb{N} \cup \{\infty\}$$

the **Λ-cohomological dimension** of G.

(ii) We say that a locally profinite group G has **locally finite Λ-cohomological dimension** if there exists an open profinite subgroup $K \leq G$ such that $\operatorname{cd}_\Lambda K < \infty$.

Many p-adic Lie groups satisfy this condition [HM24, Example 5.2.2].

Lemma 5.8. Let $\Lambda \in \mathbf{CRing}$.

(i) Let G be a locally profinite group and $H \leq K \leq G$ be compact subgroups with open K and (closed) H. The map $* /\!\!/ K \to * /\!\!/ G$ is Λ-étale and $* /\!\!/ H \to * /\!\!/ K$ is Λ-proper.

(ii) Let G be a profinite group with $\operatorname{cd}_\Lambda G < \infty$. Then, $* /\!\!/ G \to *$ is Λ-proper.

(iii) Let $H \to G$ be a map of locally profinite groups with locally finite Λ-cohomological dimension. Then, $* /\!\!/ H \to * /\!\!/ G$ is Λ-fine.

Proof.

(i) First note that $* \to * /\!\!/ G$ is a $*$-cover since $* \to * /\!\!/ G$ is an effective epimorphism[15] and D is sheafy. Thus, we need to check that the pullback[16] $G/K \to *$ is Λ-étale [HM24, Lemma 4.6.3(ii)].[17] This can be checked on open covers [HM24, Corollary 4.8.4(i)] but G/K is discrete, so it reduces to $* \to *$ being Λ-étale.

Similarly, for Λ-properness, we need to check that $K/H \to *$ is Λ-proper. This is true because K/H is a profinite set [HM24, Lemma 4.8.2(ii)].

(ii) We apply backwards 2-out-of-3 [HM24, Corollary 4.7.5] on

[15] This means that it is equivalent to its Čech nerve, which can be checked by hand.

[16] To compute the pullback we use the delooping $\Omega(* /\!\!/ G) \simeq G$, some pullback pastings and the LES associated to fiber sequences [NSS15, Definition 2.26].

[17] This pullback is truncated, so in particular, the map $* /\!\!/ K \to * /\!\!/ G$ is truncated.

$$* \xrightarrow{g} * /\!\!/ G \xrightarrow{f} *$$

so we need to show that g is Λ-prim, $fg = \mathrm{id}_*$, that f is truncated, Λ-proper and that $g_*\mathbb{1} \in D(* /\!\!/ G, \Lambda)$ is descendable. The first part follows from (i), the second part is clear. Truncatedness follows from $\Omega B \simeq \mathrm{id}$ [Lur09, Lemma 7.2.2.1]. and that $g_*\mathbb{1}$ is descendable requires the finite cohomological dimension [HM24, Proposition 5.2.5].

(iii) Since the shriekable maps are right cancellative (by definition of geometric setups), it suffices to check that $* /\!\!/ G \to *$ (and $* /\!\!/ H \to *$) is Λ-fine. This can be checked after restriction to $* /\!\!/ K$ for some compact open subgroup $K \leq G$ with $\mathrm{cd}_\Lambda K < \infty$.

Indeed, such $K \leq G$ exists by locally finite Λ-cohomological dimension and (i) shows that $* /\!\!/ K \to * /\!\!/ G$ is Λ-suave. It is furthermore $*$-conservative since this is just the restriction of a representation. Thus, the map is a universal !-cover and Λ-fine maps can be tested !-locally on the source. This then follows from (ii).

□

In particular, those maps $* /\!\!/ H \to * /\!\!/ G$ are shriekable, so we should describe the shrieks.

Construction 5.9. Let $\Lambda \in \mathbf{CRing}$ and let $H \leq G$ be a closed subgroup of a locally profinite group.

(i) For $V \in \mathbf{Rep}_\Lambda(H)^\heartsuit$ we set $\text{c-Ind}_H^G(V)$ as the set of elements $f : G \to V$ such that

   (a) f is locally constant,
   (b) $f(hg) = hf(g)$ for all $h \in H, g \in G$,
   (c) the image of $\operatorname{supp} f$ in $H \setminus G$ is compact.

   It becomes a smooth G-representation via the right translation action on the domain.

(ii) The functor c-Ind_H^G is exact, so we denote its derived functor by

$$\text{c-Ind}_H^G : \widehat{\mathbf{Rep}}_\Lambda(H) \to \widehat{\mathbf{Rep}}_\Lambda(G).$$

This is the **compact induction functor**.

Proposition 5.10 ([HM24, Lemma 5.4.2, Proposition 5.4.4])**.** Let $\Lambda \in \mathbf{CRing}$ and $H \leq G$ be a closed subgroup in a locally profinite group with locally finite Λ-cohomological dimension.

(i) Then, $f_! : D(* /\!\!/ H, \Lambda) \to D(* /\!\!/ G, \Lambda)$ is t-exact.

(ii) The diagram

$$\begin{tikzcd} \widehat{\mathbf{Rep}}_\Lambda(H) \arrow[r, "\text{c-Ind}_H^G"] \arrow[d, "\simeq"'] & \widehat{\mathbf{Rep}}_\Lambda(G) \arrow[d, "\simeq"] \\ D(* /\!\!/ H, \Lambda) \arrow[r, "f_!"'] & D(* /\!\!/ G, \Lambda) \end{tikzcd}$$

commutes.

Remark 5.11. In fact, $\widehat{\mathbf{Rep}} \simeq \mathbf{Rep}$ in this setting [HM24, Proposition 5.3.10].

5.2.2 Suave & Prim in Representation Theory

Let us describe suave and prim objects and hence recover notions of duality.

Definition 5.12. Let $\Lambda \in \mathbf{CRing}$ and let G be a locally profinite group with $f : * /\!\!/ G \to *$.

(i) Let $V \in D(* /\!\!/ G, \Lambda)$. We write $V^G = \Gamma(* /\!\!/ G, V) = f_* V$ for the **derived invariants** of V.

(ii) Suppose that G has locally finite Λ-cohomological dimension. An object $V \in D(* /\!\!/ G, \Lambda)$ is called **admissible** if $V^K \in \mathbf{Mod}_\Lambda$ is dualizable for all compact open $K \leq G$ with $\operatorname{cd}_\Lambda K < \infty$.

(iii) Suppose that G is a profinite group with $d = \operatorname{cd}_\Lambda G < \infty$. We say that it is **$\Lambda$-Poincaré** (of dimension d) if $f_* : D(* /\!\!/ G, \Lambda) \to \mathbf{Mod}_\Lambda$ preserves dualizable objects.

(iv) A locally profinite group is **locally Λ-Poincaré** (of dimension d) if it admits an open profinite subgroup which is Λ-Poincaré (of dimension d).

Lemma 5.13 ([HM24, Lemma 5.3.11])**.** Let $\Lambda \in \mathbf{CRing}$ and G be a locally profinite group with $i_K : K \hookrightarrow G$ a compact open subgroup with $\operatorname{cd}_\Lambda K < \infty$. Let $V \in D(* /\!\!/ G, \Lambda)$. The following are equivalent:

(i) V is dualizable,

(ii) $i_K^* V$ is dualizable in $D(* /\!\!/ K, \Lambda)$,

(iii) the underlying Λ-module of V is dualizable.

Proof.

(i) $\implies$ (iii): The implication (i) $\implies$ (iii) is because $D(* /\!\!/ G, \Lambda) \to \mathbf{Mod}_\Lambda$ is symmetric monoidal.

(ii) $\implies$ (i): Let $V^\vee = \underline{\operatorname{Map}}_{D(* /\!\!/ G, \Lambda)}(V, \mathbb{1})$. It suffices to check that $V \otimes V^\vee \to \underline{\operatorname{Map}}_G(V, V)$ is a G-equivariant equivalence. To do so, consider the following commutative diagram:

$$\begin{tikzcd}
i_K^*(V \otimes V^\vee) \arrow[rr] \arrow[d, "\simeq"'] & & i_K^* \underline{\operatorname{Map}}_G(V, V) \arrow[d, "\simeq"] \\
i_K^* V \otimes i_K^*(V^\vee) \arrow[r, "\simeq"'] & i_K^* V \otimes (i_K^* V)^\vee \arrow[r, "\simeq"'] & \underline{\operatorname{Map}}_K(i_K^* V, i_K^* V)
\end{tikzcd}$$

The left map is an equivalence since i_K^* is symmetric monoidal. The lower right map is an equivalence by assumption (ii). For the remaining equivalences, we consider the projection formula

$$i_{K!} \circ (i_K^* V \otimes -) \overset{\simeq}{\Rightarrow} V \otimes i_{K!}(-)$$

whose two-fold right adjoints form an equivalence

$$i_K^* \underline{\operatorname{Map}}_G(V, -) \overset{\simeq}{\Rightarrow} \underline{\operatorname{Map}}_K(i_K^* V, i_K^* -).$$

This explains the bottom left and the right equivalence. In particular, the top arrow must be an equivalence. We conclude with conservativity of i_K^*.[18]

[18] On models, we are just forgetting an action but the map being an equivalence can be tested underlying.

(iii) $\implies$ (ii): Since $f_K : * /\!/ K \to *$ is Λ-proper (5.8(ii)) we conclude that the f_K-prim and dualizable objects in $D(* /\!/ K, \Lambda)$ agree [HM24, Lemma 4.6.3(iii)]. So (iii) means that q^*V is prim where $q : * \to * /\!/ K$ and we need to show that V is f_K-prim. But q is Λ-prim (5.8(i)) and $q_*\mathbb{1}$ is descendable [HM24, Proposition 5.2.5].[19] So V is prim [HM24, Corollary 4.7.5].

□

In special settings there are more checkable conditions for admissibility [HM24, Remark 5.3.13]. Another finiteness condition is compactness which will thus naturally show up in our arguments below. Let us briefly state it here.

Lemma 5.14 ([HM24, Corollary 5.3.4])**.** Consider $\Lambda \in \mathbf{CRing}$ and a profinite group G with $\mathrm{cd}_\Lambda G < \infty$. Then, $\mathbb{1} \in D(* /\!/ G, \Lambda)$ is compact.

Proof. By 5.8(ii) the map $f : * /\!/ G \to *$ is Λ-proper, so we can compute

$$\mathrm{RHom}_{D(*/\!/G,\Lambda)}(\mathbb{1}, -) \simeq f_*\underline{\mathrm{Map}}_G(\mathbb{1}, -) \simeq f_* \simeq f_!$$

which commutes with colimits as a left adjoint. Here, the first equivalence follows by passing to left adjoints. Now we can pass to the underlying spectrum and then apply Ω^∞ to obtain the underlying space and both of these passages commute with filtered colimits. □

Proposition 5.15 ([HM24, Proposition 5.3.14, 5.3.19])**.** Let $\Lambda \in \mathbf{CRing}$ and G be a profinite group with locally finite Λ-cohomological dimension. Let $V \in D(* /\!/ G, \Lambda)$ and $f : * /\!/ G \to *$.

(i) The object V is f-prim if and only if it is compact.

  (a) In this case, $\mathrm{PD}_f(V) \simeq \underline{\mathrm{Map}}_G(V, \Lambda_c(G))$.

  (b) If $K \leq G$ is a compact open subgroup with $\mathrm{cd}_\Lambda K < \infty$ and $V \in D(* /\!/ K, \Lambda)$ is dualizable, then $\mathrm{PD}_f(\text{c-Ind}_K^G V) \simeq \text{c-Ind}_K^G V^\vee$.

(ii) The object V is f-suave if and only if it is admissible. In this case, $\mathrm{SD}_f(V) \simeq \underline{\mathrm{Map}}_G(V, f^!\mathbb{1})$.

(iii) The map $* /\!/ G \to *$ is Λ-suave if and only if G is locally Λ-Poincaré.

Proof. Let's start by recalling a classical result from smooth representation theory that we will use.

Lemma [HM24, Lemma 5.3.7]**.** For $V \in D(* /\!/ G, \Lambda)$ we have $\mathrm{colim}_{\substack{K \leq G \text{ open} \\ \mathrm{cd}_\Lambda K < \infty}} V^K \simeq V$.

The fun thing is that you can also recover this result via a 6FF argument [HM24, Lemma 5.3.7].

(i) Note that $\Lambda \in \mathbf{Mod}_\Lambda$ is compact. This implies that every f-prim object is compact in $D(* /\!/ G, \Lambda)$ [HM24, Lemma 4.4.18(ii)]. So onto the converse.

Claim. Let

$$\mathcal{G} = \{i_{K!}\mathbb{1} : i_K : * /\!/ K \to * /\!/ G,\ K \leq G \text{ compact open with } \mathrm{cd}_\Lambda K < \infty\}.$$

Then, $\mathcal{G}$ consists of compact and f-prim objects and generates $D(* /\!/ G, \Lambda)$.

[19] This uses $\mathrm{cd}_\Lambda K < \infty$.

Proof. We have seen that $f_K : * /\!\!/ K \to *$ is Λ-prim (5.8(ii)), i.e. $\mathbb{1} \in D(* /\!\!/ K, \Lambda)$ is f_K-prim. Moreover, i_K is Λ-suave (5.8(i)), so $i_{K!}\mathbb{1}$ is f-prim [HM24, Lemma 4.4.9(ii)].

Furthermore, $\mathbb{1}$ is compact by 5.14. Since $i_{K!} \dashv i_K^! \simeq i_K^* \dashv i_{K*}$ by Λ-étaleness of i_K (see 5.8(i)), it admits a right adjoint who commutes with (filtered) colimits and hence preserves compact objects. So $i_{K!}\mathbb{1}$ is compact.

To see that $\mathcal{G}$ is generating we observe

$$P^K = f_{K*} i_K^* P \simeq f_* i_{K*} \underline{\mathrm{Map}}_K(\mathbb{1}, i_K^* P) \simeq f_* \underline{\mathrm{Map}}_K(i_{K!}\mathbb{1}, P) \simeq \mathrm{RHom}_{D(*/\!\!/ G,\Lambda)}(i_{K!}\mathbb{1}, P)$$

where the third equality is general 6FF nonsense [HM24, Proposition 3.2.2]. By the result discussed in the beginning of the proof, we conclude. □

Denote by $\langle \mathcal{G} \rangle \subseteq D(* /\!\!/ G, \Lambda)$ the full subcategory generated by $\mathcal{G}$ under (co-)fibers and retracts. Since primness is closed under these operations [HM24, Corollary 4.4.13] we get $\langle \mathcal{G} \rangle \subseteq \mathrm{Prim}(* /\!\!/ G)$. On the other hand, $\mathrm{Ind}(\langle \mathcal{G} \rangle) \simeq D(* /\!\!/ G, \Lambda)$ since $\mathcal{G}$ consists of compact generators [Lur09, Proposition 5.3.5.11]. Passing to compact objects yields $\langle \mathcal{G} \rangle \simeq D(* /\!\!/ G, \Lambda)^\omega$.

(a) This follows from the general prim dual formula [HM24, Lemma 4.4.6] while using the c-Ind to understand $\Delta_!$ from that formula.

(b) The map $f_K : * /\!\!/ K \to *$ is Λ-proper (5.8(ii)). So, the dualizables agree with the f_K-prims in $D(* /\!\!/ K, \Lambda)$ [HM24, Lemma 4.6.3(iii)] which in particular means that f_K-prim duality is the usual duality. Moreover, $h_! = \text{c-Ind}_K^G$ commutes with prim duality [HM24, Lemma 4.4.9]. So

$$\mathrm{PD}_f(\text{c-Ind}_K^G V) \simeq \text{c-Ind}_K^G \left(\mathrm{PD}_{f_K}(V)\right) \simeq \text{c-Ind}_K^G V^\vee$$

as desired.

(ii) We use

Lemma [HM24, Corollary 4.4.15]**.** Let D be a 6FF on some geometric setup $(\mathscr{C}, \mathscr{E})$ and $f : X \to S$ be a map in $\mathscr{E}$. Let $(Q_i)_{i\in I}$ be a family of objects in $D(X)$. Assume that the Q_i are f-prim and $D(X \times_S X)$ is generated by $\pi_1^* Q_i \otimes \pi_2^* Q_j$. Then, $P \in D(X)$ is f-suave if and only if $f_* \underline{\mathrm{Map}}(Q_i, P)$ is dualizable for all Q_i.

We take the family $(Q_i)_{i\in I} = \mathcal{G}$ from (i). We have seen there that its consists of Λ-prim objects and moreover,

$$\pi_1^* i_{K!}\mathbb{1} \otimes \pi_2^* i_{K'!}\mathbb{1} \simeq i_{(K\times K')!}\mathbb{1}$$

generates $D(* /\!\!/ (G \times G), \Lambda)$ by the same argument as in (i). We have also seen in the proof of (i) that $f_* \underline{\mathrm{Map}}(i_{K!}\mathbb{1}, V) \simeq V^K$, so the only if part of the statement translates to admissibility. The suave dual formula is an instance of the general formula [HM24, Lemma 4.4.5].

(iii) Suppose first that G is locally Λ-Poincaré. Let $H \leq G$ be a compact open Λ-Poincaré subgroup. As in the proof of 5.8(iii) we see that $* /\!\!/ H \to * /\!\!/ G$ is a universal !-cover, so it suffices to show that $* /\!\!/ H \to *$ is Λ-suave [HM24, Lemma 4.5.8(i)]. So WLOG G is Λ-Poincaré.

We need to show that $\mathbb{1} \in D(* /\!\!/ G, \Lambda)$ is Λ-suave, i.e. admissible by (ii). In other words, we need that $V^K = f_{K*}\mathbb{1}$ is dualizable for every compact open $K \leq G$ with $\mathrm{cd}_\Lambda K < \infty$. For this, we note $f_{K*}\mathbb{1} \simeq f_* i_{K*}\mathbb{1}$ and f_* preserves dualizables because G is Λ-Poincaré. On the other hand, i_K is Λ-proper (5.8(i)), so $i_{K*}\mathbb{1} \simeq i_{K!}\mathbb{1}$. Now $\mathbb{1}$ is compact by 5.14 and $i_{K!}$ preserves compacts as demonstrated in the proof of (i). On the other hand, compacts and dualizables agree (5.16).

Conversely, suppose that $* /\!\!/ G \to *$ is Λ-suave. Since G has locally finite Λ-cohomological dimension, it has a compact open subgroup K with $\mathrm{cd}_\Lambda K < \infty$. Moreover, being Λ-suave is the same as admissibility by (ii), so $* /\!\!/ K \to *$ is still Λ-suave. So WLOG G is profinite with $\mathrm{cd}_\Lambda G < \infty$. Since $\mathbb{1} \in D(* /\!\!/ G, \Lambda)$ is f-suave, i.e. admissible, the object $f_* i_{K*}\mathbb{1} \simeq f_{K*}\mathbb{1}$ is dualizable in $\mathbf{Mod}_\Lambda$ for every compact open $K \leq G$. On the other hand, $i_{K*}\mathbb{1} \simeq i_{K!}\mathbb{1}$ generate the dualizables in $D(* /\!\!/ G, \Lambda)$ under (co-)fibers and retractions as demonstrated in (i). So f_* preserves dualizables.

□

This prim duality is also called *Bernstein–Zelevinsky duality* and it is an example of a statement that is really terrible to prove by writing down formulas but follows formally from six functor nonsense! Just from the formulas, it's not clear that this formula for the prim duality is interesting and it's hard to get this explicit prim duality formula on compact inductions by only playing around with the formulas. With 6FF nonsense it's not that bad!

Corollary 5.16. Let $\Lambda \in \mathbf{CRing}$ and G be a profinite group with $\mathrm{cd}_\Lambda G < \infty$.

(i) Then, $D(* /\!\!/ G, \Lambda)$ is compactly generated.

(ii) An object is compact if and only if it is dualizable.

Proof.

(i) We have seen this in the proof of 5.15(i), it is compactly generated by what we called $\mathcal{G}$.

(ii) By 5.15(i) the compact objects agree with the f-prim objects where $f : * /\!\!/ G \to *$. So we need to show that f-primality agrees with dualizability. But $* /\!\!/ G \to *$ is Λ-proper (5.8(ii)) and in this setting we are done [HM24, Lemma 4.6.3(iii)].

□

Example 5.17 ([HM24, Example 5.3.21, 5.3.22])**.** Let p be a prime.

(i) Let Λ be a $\mathbb{Z}[1/p]$-algebra and G be locally pro-p. Then, G is locally Λ-Poincaré.

(ii) Let Λ be a $\mathbb{Z}/p^n$-algebra and G be a p-adic Lie group. Then, G is locally Λ-Poincaré.

In each case one can give explicit descriptions of the dualizing complex and so suave duality (5.15(iii)) recovers Poincaré duality in these settings. This is not really a new proof of Poincaré duality because it relies on results from classical representation theory which are close to Poincaré duality.

5.3 What the Hecke?

What the heck is a Hecke algebra?

They show up in various areas of mathematics. Frankly, I know neither of the motivations but https://www.math.columbia.edu/~martinez/Notes/introtohecke.pdf seems useful.

Definition 5.18. Let Λ be a field with $\mathrm{char}\,\Lambda = p > 0$ and $K \leq G$ be a compact open subgroup of a locally profinite group with $V \in \mathbf{Rep}_\Lambda(K)^\heartsuit$. Then, $\mathcal{H}(G, K, V) = \mathrm{End}_G(\text{c-Ind}_K^G V)$ is the associated **Hecke algebra**.

Fact 5.19 ([HM24, Remark 5.5.1]).

(i) There is an isomorphism

$$\mathcal{H}(G,K,V) \xrightarrow{\sim} \{f : G \to \mathrm{End}_\Lambda(V) : f \text{ is } K\text{-}K\text{-linear}, \operatorname{supp} f \text{ compact}\}.$$

(ii) Under this identification there is an involutive anti-isomorphism of algebras

$$\iota : \mathcal{H}(G,K,V) \xrightarrow{\sim} \mathcal{H}(G,K,V^*),\ \iota(T)(g) = (T(g^{-1}))^*.$$

There are more refined derived versions of this construction by taking derived endomorphisms instead of the underived version [HM24, Remark 5.5.1].

Definition 5.20. Let $\Lambda \in \mathbf{CRing}$ and G be a locally profinite group with a compact open subgroup $K \leq G$ with $\mathrm{cd}_\Lambda K < \infty$.

(i) We denote by $\mathcal{H}_K$ the $\mathbf{Mod}_\Lambda$-enriched ∞-category whose objects are the dualizable objects in $\mathbf{Rep}_\Lambda(K)$ and whose mapping objects are

$$\mathcal{H}_K(V,W) = \mathrm{RHom}_G(\text{c-Ind}_K^G V, \text{c-Ind}_K^G W) \in \mathbf{Mod}_\Lambda.$$

(ii) We denote by $\mathcal{H}_K^\bullet = \mathcal{H}_K(\mathbb{1},\mathbb{1}) \in \mathbf{Alg}_{\mathbb{E}_1}(\mathbf{Mod}_\Lambda)$ and **derived Hecke algebra** of weight $\mathbb{1}$.

Theorem 5.21 ([HM24, Proposition 5.5.4, 5.5.6]). Let $\Lambda \in \mathbf{CRing}$ and G be a locally profinite group with a compact open subgroup $K \leq G$ with $\mathrm{cd}_\Lambda K < \infty$.

(i) Prim duality PD on $\mathrm{Prim}(* /\!\!/ G)$ induces an involutive equivalence

$$\mathcal{H}_K^{\mathrm{op}} \xrightarrow{\simeq} \mathcal{H}_K,\ V \mapsto V^\vee = \mathrm{RHom}_\Lambda(V,\Lambda)$$

of $\mathbf{Mod}_\Lambda$-enriched ∞-categories.

(ii) Let Λ be a field with $\mathrm{char}\,\Lambda = p > 0$ and G be a p-adic Lie group with a p-torsionfree compact open subgroup $I \leq G$. Then, (i) induces an anti-involution $\mathrm{Inv} : (\mathcal{H}_I^\bullet)^{\mathrm{op}} \xrightarrow{\simeq} \mathcal{H}_I^\bullet$ which coincides with Schneider–Sorensen's anti-involution $\mathrm{Inv}_{\mathrm{SS}}$ [HM24, Remark 5.5.1].

It seems like previously this was only defined for fields of positive characteristic Λ and you need to work a little to write down these maps. Prim duality immediately yields a map and works for all $\Lambda \in \mathbf{CRing}$.

A fruitful plan of developing new mathematics seems to be: Find/Take any six functor formalism and try to specialize all of the general abstract 6FF notions that we have learned to the example. Anyhow, the next goal of the seminar will be to carry out this plan on the category of topological spaces.

6 Sheaves and (A Bit of) Cosheaves (Christian Kremer)

Christian began by backing out from dissing me. TALK 6 27.11.2025

I wanted to make fun of Qi and how he was scared about representation theory with Peter May's quote that everyone should do some computations... and then talk about topos theory for 90 minutes.

6.1 Cocomplete Categories

Fix two universes $V \in W$. Let **Cocont** be the ∞-category of V-cocomplete W-small ∞-categories with V-cocontinuous functors. We get mapping functor ∞-categories $\mathrm{Fun}_!(\mathscr{C}, \mathscr{D})$, so **Cocont** is an $(\infty, 2)$-category. We obtain a bilinear functor $\mathscr{C} \times \mathscr{D} \to \mathscr{C} \otimes \mathscr{D}$ and it is initial with respect to this property:

$$\begin{tikzcd} \mathscr{C} \times \mathscr{D} \arrow[r] \arrow[dr] & \mathscr{C} \otimes \mathscr{D} \arrow[d, dashed] \\ & \mathscr{E} \end{tikzcd}$$

In symbols, $\mathrm{Fun}_{!\times!}(\mathscr{C} \times \mathscr{D}, \mathscr{E}) \simeq \mathrm{Fun}_!(\mathscr{C} \otimes \mathscr{D}, \mathscr{E})$.

Fact 6.1. This yields a symmetric monoidal ∞-category $(\mathbf{Cocont}, \otimes)$ with unit $\mathbb{1} = \mathbf{An}$.

The way you construct the Lurie tensor product on $\mathbf{Pr}^L$ is that you first construct a tensor product on **Cocont** which you then show to restrict to $\mathbf{Pr}^L$. In other words, $\mathbf{Pr}^L \to \mathbf{Cocont}$ is symmetric monoidal.

Example 6.2. There is an equivalence

$$\begin{aligned} \mathrm{Fun}_!(\mathbf{PSh}(I) \otimes \mathbf{PSh}(J), \mathscr{E}) &\simeq \mathrm{Fun}_{!\times!}(\mathbf{PSh}(I) \times \mathbf{PSh}(J), \mathscr{E}) \\ &\simeq \mathrm{Fun}(I \times J, \mathscr{E}) \\ &\simeq \mathrm{Fun}_!(\mathbf{PSh}(I \times J), \mathscr{E}). \end{aligned}$$

6.2 Sheaves

Let X be a topological space with an open cover $X = \bigcup_i U_i$. We write

$$R\{U_i\}_i = \{V \subseteq X : \exists i : V \subseteq U_i\}$$

for the generated sieve.

Definition 6.3.

(i) Let $\mathscr{C}$ be complete and $\mathcal{F} \in \mathbf{PSh}(\mathbf{Open}(X), \mathscr{C})$. Then, $\mathcal{F}$ is a **sheaf** if for each open $V \subseteq X$ and open cover $V = \bigcup_i U_i$ the comparison map

$$\mathcal{F}(V) \to \lim_{W \in R\{U_i\}_i} \mathcal{F}(W)$$

is an equivalence.

(ii) Dually, let $\mathscr{C}$ be cocomplete. Then, $\mathcal{F} \in \mathrm{Fun}(\mathbf{Open}(X), \mathscr{C})$ is a **cosheaf** if for each open $V \subseteq X$ and open cover $V = \bigcup_i U_i$ the comparison map

$$\operatorname*{colim}_{W \in R\{U_i\}_i} \mathcal{F}(W) \to \mathcal{F}(V)$$

is an equivalence.

The word 'dually' should be taken literally: $\mathbf{CoSh}(X, \mathscr{C}) \simeq \mathbf{Sh}(X, \mathscr{C}^{\mathrm{op}})^{\mathrm{op}}$. Here,

$$\mathbf{CoSh}(X, \mathscr{C}) \subseteq \mathrm{Fun}(\mathbf{Open}(X), \mathscr{C}) \quad \text{and} \quad \mathbf{Sh}(X, \mathscr{C}^{\mathrm{op}})^{\mathrm{op}} \subseteq \mathrm{Fun}(\mathbf{Open}(X)^{\mathrm{op}}, \mathscr{C}^{\mathrm{op}})^{\mathrm{op}}$$

explaining Lucas' joke that sheaves are (co-)cosheaves at the beginning of the talk.

Remark 6.4.

(i) If $\mathscr{C}$ is presentable, then $\mathbf{Sh}(X, \mathscr{C}) \simeq \mathbf{Sh}(X) \otimes \mathscr{C}$. More precisely,

$$\mathbf{Sh}(X, \mathscr{C}) \simeq \mathrm{Fun}_*(\mathbf{Sh}(X)^{\mathrm{op}}, \mathscr{C}) \simeq \mathbf{Sh}(X) \otimes \mathscr{C}$$

where the first equivalence is an elementary check and the second equivalence needs presentability.

(ii) There is an equivalence $\mathbf{Sh}(X \times Y) \simeq \mathbf{Sh}(X) \otimes \mathbf{Sh}(Y)$. The idea is that

$$\{U \times V : U \subseteq X, V \subseteq Y\}$$

is a basis for $X \times Y$. Here,

$$\mathbf{Sh}(X \times Y) \subseteq \mathbf{PSh}(\mathbf{Open}(X) \times \mathbf{Open}(Y))$$
$$\mathbf{Sh}(X) \otimes \mathbf{Sh}(Y) \subseteq \mathbf{PSh}(\mathbf{Open}(X)) \otimes \mathbf{PSh}(\mathbf{Open}(Y)).$$

and one checks that the subcategories are the same.

6.3 Topoi & Local Contractibility

Definition 6.5. An **∞-topos** is a presentable ∞-category $\mathscr{X}$ satisfying the two equivalent definitions below:

(i) There exists a small ∞-category J and an adjunction

$$\mathbf{PSh}(J) \underset{\longleftarrow}{\overset{\text{lex}}{\longrightarrow}} \mathscr{X},$$

i.e. $\mathscr{X}$ is a left-exact localization of a presheaf category.

(ii) Descent: The slice functor $\mathscr{X}_{/(-)} : \mathscr{X}^{\mathrm{op}} \to \mathbf{CAT}_\infty$ preserves limits.[20]

A **geometric morphism** of ∞-topoi $f : \mathscr{X} \to \mathscr{Y}$ is a functor $f_* : \mathscr{X} \to \mathscr{Y}$ which admits a left exact left adjoint f^*. We write $\mathbf{Top}^R$ for the ∞-category of ∞-topoi.

An example of geometric morphisms appeared in the definition of ∞-topoi!

Example 6.6. Consider geometric morphisms $\mathbf{An} \to \mathscr{X}$. There is a unique left adjoint taking $*_{\mathbf{An}}$ to $*_{\mathscr{X}}$. One checks that it really is left-exact and its left adjoint is denoted by $\Gamma : \mathscr{X} \to \mathbf{An}$.

Example 6.7. The ∞-category $\mathbf{Sh}(X)$ is an ∞-topos. Indeed, $\mathbf{Sh}(X) \hookrightarrow \mathbf{PSh}(X)$ is an accessible limit-preserving functor and it has a left adjoint, essentially by constructing $\mathbf{Sh}(X)$ through localizations. The left adjoint turns out to be left-exact.

Idea. Let $\mathcal{F} \in \mathbf{PSh}(X)$ and consider

$$\mathcal{F}^\dagger : \mathbf{Open}(X)^{\mathrm{op}} \to \mathbf{An},\ U \mapsto \operatorname*{colim}_{\{U_i\}_i \to U \text{ cover}} \lim_{V \in R\{U_i\}_i} \mathcal{F}(V),$$

trying to naively force the sheaf condition. This is usually not a sheaf. Let $\mathcal{G}$ be a sheaf, then $\mathrm{Map}(\mathcal{F}^\dagger, \mathscr{G}) \simeq \mathrm{Map}(\mathcal{F}, \mathcal{G})$. Moreover, there is always the trivial cover $\{U\} \to U$ in the colimit index, so this induces a map $\mathcal{F} \to \mathcal{F}^\dagger$. This is enough to get a sheaf in 1-topoi land but in ∞-land you have to do it more often:

$$\mathcal{F} \longrightarrow \mathcal{F}^\dagger \longrightarrow (\mathcal{F}^\dagger)^\dagger \longrightarrow ((\mathcal{F}^\dagger)^\dagger)^\dagger \longrightarrow \cdots.$$

You wanna take a colimit over some κ-iterate of this (with κ depending on the space). This is a sheaf then. So this constructs the left adjoint. What did we do? We took some limits and some filtered colimits. Both of these constructions commute with finite limits and that's why the left adjoint is left-exact. □

[20] The functoriality is by pullbacks.

6.4 Local Contractibility

Definition 6.8. A geometric morphism $f : \mathscr{X} \to \mathscr{Y}$ is **essential** if $f^* : \mathscr{Y} \to \mathscr{X}$ has a left adjoint $f_\sharp$.

For $U \in \mathscr{Y}$ there is a geometric morphism $f_{/U} : \mathscr{X}_{/f^*U} \to \mathscr{Y}_{/U}$ with left adjoint

$$(W \to U) \mapsto (f^*W \to f^*U).$$

The right adjoint is given by $(V \to f^*U) \mapsto (f_\sharp V \to f_\sharp f^*U \to U)$.

Lemma 6.9. Suppose that $f : \mathscr{X} \to \mathscr{Y}$ is essential and let $U \in \mathscr{Y}$. Then, $f_{/U} : \mathscr{X}_{/f^*U} \to \mathscr{Y}_{/U}$ is essential.

Proof. By the universal property of mapping spaces of slice categories, we compute

$$\begin{aligned}\mathrm{Map}_{\mathscr{Y}_{/U}}(f_\sharp(V) \to U, W \to U) &\simeq \mathrm{Map}(f_\sharp(V), W) \times_{\mathrm{Map}(f_\sharp(V),U)} \{f_\sharp(V) \to U\}\\ &\simeq \mathrm{Map}(V, f^*W) \times_{\mathrm{Map}(V,f^*U)} \{V \to f^*U\}\\ &\simeq \mathrm{Map}_{\mathscr{X}_{/f^*U}}(V \to f^*U, f^*W \to f^*U),\end{aligned}$$

showing the adjunction. □

Definition 6.10. A geometric morphism f is **locally contractible** if it is essential and for every $U \to V$ in $\mathscr{Y}$ the square

$$\begin{tikzcd} \mathscr{X}_{/f^*U} \arrow[d, shift right, "f_\sharp"'] & \mathscr{X}_{/f^*V} \arrow[l, "q^*"'] \arrow[d, shift right, "f_\sharp"'] \\ \mathscr{Y}_{/U} \arrow[u, shift right, "f^*_{/U}"'] & \mathscr{Y}_{/V} \arrow[l, "q^*"] \arrow[u, shift right, "f^*_{/V}"'] \end{tikzcd}$$

commutes (given by the Beck–Chevalley map).

Explicitly: Given $U \to V$ in $\mathscr{Y}$ and $W \to f^*V$ in $\mathscr{X}$ we can write down a comparison map

$$f_\sharp(f^*U \times_{f^*V} W) \to U \times_V f_\sharp W.$$

By definition, f is locally contractible if all such comparison maps are equivalences.

Lemma 6.11. The geometric morphism $\mathscr{X} \to \mathbf{An}$ is locally contractible if and only if it is essential.

Proof. Fix $W \in \mathscr{X}$. Consider

$$p_\#(p^* - \times_{p^*-} W) \Rightarrow - \times_- p_\sharp W$$

as a natural transformation of functor $\mathbf{An}^{[1]} \to \mathbf{An}$. This is checked by hand for $\emptyset \to *$ and $* \to *$ and both sides commute with colimits. □

Definition 6.12. An ∞-topos is **locally contractible** if $p : \mathscr{X} \to \mathbf{An}$ is.

By 6.11 we equivalently need to check that p^* preserves limits. Consider the composition

$$\mathbf{An} \xrightarrow{p^*} \mathscr{X} \xrightarrow{p_*} \mathbf{An}.$$

This functor p_*p^* is called the **shape** of $\mathscr{X}$. This is left-exact and because we did nothing crazy, it is also accessible, i.e. $p_*p^* \in \mathrm{Fun}^{\mathrm{lex,acc}}(\mathbf{An}, \mathbf{An})$. Consider

$$\begin{tikzcd} & \mathscr{X} \arrow[dl, "p"'] \arrow[dd, "f"] \\ \mathbf{An} & \\ & \mathscr{Y} \arrow[ul, "q"] \end{tikzcd}$$

Then, we can write down $q_*q^* \Rightarrow q_*f_*f^*q^* \simeq p_*p^*$. This is the functoriality of a functor

$$\mathbf{Top}^R \to \mathrm{Fun}^{\mathrm{lex,acc}}(\mathbf{An}, \mathbf{An})^{\mathrm{op}} \simeq \mathrm{Pro}(\mathbf{An}),\ \mathscr{Y} \mapsto q_*q^*.$$

We have $\mathbf{An} \hookrightarrow \mathrm{Pro}(\mathbf{An})$ but also

$$\mathbf{An} \simeq \mathrm{Fun}_{\mathrm{cocont}}(\mathbf{An}, \mathbf{An}) \simeq \mathrm{Fun}^{\mathrm{cont,acc}}(\mathbf{An}, \mathbf{An})^{\mathrm{op}} \hookrightarrow \mathrm{Fun}^{\mathrm{lex,acc}}(\mathbf{An}, \mathbf{An})^{\mathrm{op}}.$$

Definition 6.13. An ∞-topos $\mathscr{X}$ has **constant shape** if $\mathrm{sh}(\mathscr{X}) = p_*p^* \in \mathrm{Pro}(\mathbf{An})$ lies in the subcategory $\mathbf{An} \subseteq \mathrm{Pro}(\mathbf{An})$.

Remark 6.14. Note that constant shape does not imply locally contractible. But consider:

$$\begin{tikzcd} & \mathscr{X}_{/U} \arrow[r, "q^*"] & \mathbf{An} \\ \mathbf{An} \arrow[ur, "q^*"] \arrow[dr, "p^*"'] & & \\ & \mathscr{X} \arrow[uu, "j_U^*"'] & \end{tikzcd}$$

If $\mathscr{X}_{/U}$ has constant shape for all $U \in \mathscr{X}$, then p^* preserves limits, so p is locally contractible. Indeed, we want to show that $p^* \lim_i X_i \to \lim_i p^* X_i$ is an equivalence. On the other hand, $q_*j_U^*$ is a set of conservative functors and this is limit-preserving. So it suffices to check this after applying $q_*j_U^*$ which reduces it to the assumption.

Theorem 6.15 (Properties of locally contractible morphisms)**.**

(i) They are closed under compositions.

(ii) If $\mathscr{X} \to \mathscr{Y}$ is locally contractible, then so is $\mathscr{X} \times \mathscr{Z} \to \mathscr{Y} \times \mathscr{Z}$.[21]

(iii) Let $\mathscr{X}$ be locally contractible. Consider a retract $\mathscr{X}' \hookrightarrow \mathscr{X}$. Then, $\mathscr{X}'$ is locally contractible.

(iv) The map $\mathscr{X}_{/U} \to \mathscr{X}$ is locally contractible for all $U \in \mathscr{X}$.

(v) The map $\mathbf{Sh}(\mathbb{R}) \to \mathbf{An}$ is locally contractible.

Part (ii) is not super elementary to prove.

Corollary 6.16. Let X be an Euclidean Neighborhood retract. Then, $\mathbf{Sh}(X)$ is locally contractible.

[21] The product is taken in $\mathbf{Top}^R$ which is the same as the Lurie tensor product $\otimes$ in $\mathbf{Pr}^L$.

6.5 Base Change

Consider a diagram

$$\begin{tikzcd} \mathscr{X}' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{X} \arrow[d, "g"] \\ \mathscr{Y}' \arrow[r, "f"'] & \mathscr{Y} \end{tikzcd}$$

with essential g and g'.

Question 6.17. Is $f^* g_\sharp \simeq g'_\sharp f'^*$?

Proposition 6.18. Let g, g' be locally contractible. Then yes.

Let's discuss the following cases.

- Case 1: Consider

$$\begin{tikzcd} \mathscr{X} \times \mathscr{Y} \arrow[r] \arrow[d, "q_*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{X} \arrow[d, "p_*"] \\ \mathscr{Y} \arrow[r] & \mathbf{An} \end{tikzcd}$$

  and we can pass to left adjoints, so this is $(\mathbf{An} \to \mathscr{Y}) \otimes (\mathbf{An} \to \mathscr{X})$. We have $p^* \otimes \mathscr{Y} \simeq q^*$, so $p_\sharp \otimes \mathscr{Y} \dashv q^*$, so $p_\sharp \otimes \mathscr{Y} \simeq q_\sharp$.

- Case 2: Consider

$$\begin{tikzcd} \mathscr{X} \times \mathscr{Y}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{X} \times \mathscr{Y} \arrow[d, "\mathrm{proj}_{\mathscr{Y}}"] \\ \mathscr{Y}' \arrow[r] & \mathscr{Y} \end{tikzcd}$$

  and you can use essentially the same argument.

Definition 6.19. A map of topological spaces $g : X \to Y$ is a **shape submersion** if X has a basis of the form

$$\begin{tikzcd} U \times V \arrow[r, hook] \arrow[d] & X \arrow[d] \\ V \arrow[r, hook] & Y \end{tikzcd}$$

Lemma 6.20 ([Vol21, Lemma 3.24, 3.25])**.** Shape submersions of nice topological spaces induce essential geometric morphisms. Consider a pullback

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "g"] \\ Y' \arrow[r, "f"'] & Y \end{tikzcd}$$

with shape submersions g, g'. Then, there is base change $f^* g_\sharp \simeq g'_\sharp f'^*$.

7 Verdier Duality (Anupam Datta)

Recap: Let $f : X \to Y$ be a map of topological spaces. Then, we saw an adjunction

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$$\mathbf{Sh}(X) \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathbf{Sh}(Y)$$

with left-exact f^*.

7.1 Localization Sequences

Theorem 7.1. Let X be a topological space and $i : Z \hookrightarrow X$ be a closed subspace. Moreover, let $j : U = X \setminus Z \hookrightarrow X$. These induce maps

$$\mathbf{Sh}(Z) \xrightarrow{i_*} \mathbf{Sh}(X) \xleftarrow{j_*} \mathbf{Sh}(U)$$

which witnesses $\mathbf{Sh}(X)$ as a recollement of $\mathbf{Sh}(U)$ and $\mathbf{Sh}(Z)$.

Definition 7.2. Let $\mathscr{C}$ be an ∞-category which admits finite limits and let $\mathscr{C}_0, \mathscr{C}_1 \subseteq \mathscr{C}$ be full subcategories. We will say that $\mathscr{C}$ is a **recollement** of $\mathscr{C}_0$ and $\mathscr{C}_1$ if:

(i) $\mathscr{C}_0, \mathscr{C}_1$ are stable under equivalences,

(ii) The inclusion functors $\mathscr{C}_0 \hookrightarrow \mathscr{C}$ and $\mathscr{C}_1 \hookrightarrow \mathscr{C}$ admit left-exact left adjoints L_0 and L_1 respectively.

(iii) The functor L_1 carries every object of $\mathscr{C}_0$ to the final object of $\mathscr{C}$.

(iv) If α is a map in $\mathscr{C}$ such that $L_0\alpha$ and $L_1\alpha$ are equivalences, then α is an equivalence.

Example 7.3. Set $\mathscr{C} = \mathbf{Sh}(X), \mathscr{C}_0 = i_*\mathbf{Sh}(Z) \simeq \mathbf{Sh}(Z), \mathscr{C}_1 = j_*\mathbf{Sh}(U) \simeq \mathbf{Sh}(U)$.

Proof. The equivalences come from checking that the counits $i^*i_* \Rightarrow \mathrm{id}$ and $j^*j_* \Rightarrow \mathrm{id}$ are equivalences. □

Lemma 7.4. Let $i : Z \hookrightarrow X$. Then, $i_* : \mathbf{Sh}(Z) \to \mathbf{Sh}(X)$ commutes with contractible colimits.

Proof Sketch. One first checks

$$\operatorname{im} i_* \simeq \{F \in \mathbf{Sh}(X) : j^*F \simeq *\}.$$

So let $F_j \in \mathbf{Sh}(Z)$ over a contractible indexing category J. Since $i_* : \mathbf{Sh}(Z) \to \mathbf{Sh}(X)$ is fully faithful, it suffices to show that $\operatorname{colim}_J i_*F_j$ already lies in $\operatorname{im} i_*$. To do so, we compute

$$j^*\left(\operatorname*{colim}_I F_i\right) \simeq \operatorname*{colim}_I j^*F_i \simeq |I|,$$

which is contractible. □

Corollary 7.5. Let $\mathscr{C}$ be pointed and presentable. Then, there is an adjunction

$$\mathbf{Sh}(Z, \mathscr{C}) \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathbf{Sh}(X, \mathscr{C}).$$

We are in the situation

$$\begin{tikzcd} \mathbf{Sh}(Z,\mathscr{C}) \arrow[r, "i_*^{\mathscr{C}}" description] & \mathbf{Sh}(X,\mathscr{C}) \arrow[l, bend right, "i^*_{\mathscr{C}}"'] \arrow[l, bend left, "i^!_{\mathscr{C}}"] \arrow[r, "j^*_{\mathscr{C}}" description] & \mathbf{Sh}(U,\mathscr{C}) \arrow[l, bend right, "j_\sharp^{\mathscr{C}}"'] \arrow[l, bend left, "j_*^{\mathscr{C}}"] \end{tikzcd}$$

where $i^!_{\mathscr{C}}$ exists if $\mathscr{C}$ is pointed and $j^{\mathscr{C}}_\sharp$ comes from the shape submersions being essential (6.20) from last talk.

Proposition 7.6. Let $\mathscr{C}$ be pointed and presentable and $F \in \mathbf{Sh}(X, \mathscr{C})$. We have a cofiber sequence

$$j^{\mathscr{C}}_\sharp j^*_{\mathscr{C}} F \longrightarrow F \longrightarrow i^{\mathscr{C}}_* i^*_{\mathscr{C}} F.$$

Dually, $i^{\mathscr{C}}_* i^!_{\mathscr{C}} F \to F \to j^{\mathscr{C}}_* j^*_{\mathscr{C}} F$ is a fiber sequence.

Proof Sketch. The dual statement follows from the universal properties of pushout and pullback. Reduce to $\mathscr{C} = \mathcal{S}_*$. We write $\alpha : \mathbf{Sh}(X, \mathcal{S}_*) \to \mathbf{Sh}(X, \mathcal{S})$ for the forgetful functor. Then, we claim that there is a pushout square

$$\begin{tikzcd}
j_\sharp(よ(U)) \arrow[r] \arrow[d] & よ(X) \arrow[d] \\
j_\sharp j^* \alpha(F) \arrow[r] & \alpha(j^{\mathcal{S}_*}_\sharp j^*_{\mathcal{S}_*} F)
\end{tikzcd}$$

We know that α reflects pushouts and we have an equivalence $\alpha i^{\mathcal{S}_*}_* i^*_{\mathcal{S}_*} \simeq i_* i^* \alpha$, so it suffices to show. It suffices to show that

$$\begin{tikzcd}
\alpha(j^{\mathcal{S}_*}_\sharp j^*_{\mathcal{S}_*} F) \arrow[r] \arrow[d] & \alpha(F) \arrow[d] \\
よ(X) \arrow[r] & i_* i^* \alpha(F)
\end{tikzcd}$$

is a pushout. You can check this with some pushout pasting arguments. □

Proof of 7.1. Check part (iv) of the definition. Use the above cofiber sequences and 2-out-of-3 for equivalences. □

7.2 Pullbacks with Stable Bicomplete Coefficients

Let X be locally compact and Hausdorff. Let $\mathscr{C}$ be stable and bicomplete.

Definition 7.7. Let $\mathcal{K}(X)$ be the poset of compact subsets of X.

(i) Then, $\mathbf{Sh}(\mathcal{K}(X), \mathscr{C}) = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{K}(X)^{\mathrm{op}}, \mathscr{C})$.

(ii) An $F \in \mathbf{Sh}(\mathcal{K}(X), \mathscr{C})$ is a **$\mathcal{K}$-sheaf** if the preferred map

$$\operatorname*{colim}_{K \Subset K'} \Gamma(K', F) \to \Gamma(K, F)$$

is an equivalence.[22] Let $\mathbf{Sh}_{\mathcal{K}}(X, \mathscr{C})$ denote the ∞-category of $\mathcal{K}$-sheaves.

Fact 7.8 ([Lur09, Theorem 7.3.4.9])**.** There is an equivalence $\mathbf{Sh}_{\mathcal{K}}(X, \mathscr{C}) \simeq \mathbf{Sh}(X, \mathscr{C})$.

Remark 7.9. This comes form the functors

$$\Theta_{\mathscr{C}} : \mathrm{Fun}(\mathbf{Open}(X)^{\mathrm{op}}, \mathscr{C}) \to \mathrm{Fun}(\mathcal{K}(X)^{\mathrm{op}}, \mathscr{C}),\ F \mapsto \left(K \mapsto \operatorname*{colim}_{K \subseteq U} \Gamma(U, F)\right)$$

and

$$\Psi_{\mathscr{C}} : \mathrm{Fun}(\mathcal{K}(X)^{\mathrm{op}}, \mathscr{C}) \to \mathrm{Fun}(\mathbf{Open}(X)^{\mathrm{op}}, \mathscr{C}),\ G \mapsto \left(U \mapsto \lim_{K \subseteq U} \Gamma(K, G)\right).$$

[22] The symbol $K \Subset K'$ means that there exists an open U such that $K \subseteq U \subseteq K'$.

Let $\mathbf{CoSh}_{\mathcal{K}}(X, \mathscr{C}) = \mathbf{Sh}_{\mathcal{K}}(X, \mathscr{C}^{\mathrm{op}})^{\mathrm{op}}$.

Corollary 7.10. There is an equivalence $\mathbf{CoSh}(X, \mathscr{C}) \simeq \mathbf{CoSh}_{\mathcal{K}}(X, \mathscr{C})$.

Proof. This comes from

$$\mathbf{CoSh}(X, \mathscr{C}) \simeq \mathbf{Sh}(X, \mathscr{C}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathbf{Sh}_{\mathcal{K}}(X, \mathscr{C}^{\mathrm{op}})^{\mathrm{op}} = \mathbf{CoSh}_{\mathcal{K}}(X, \mathscr{C})$$

using 7.8. □

Definition 7.11. Let $F \in \mathbf{Sh}(X, \mathscr{C})$ with $U \in \mathbf{Open}(X)$ and $K \subseteq X$ be closed.

(i) If $K \subseteq U$, let the **sections of F supported at K** as $\Gamma_K(U, F) = \mathrm{fib}(\Gamma(U, F) \to \Gamma(U \setminus K, F))$.

(ii) The **compactly supported sections of F over U** are $\Gamma_c(U, F) = \operatorname{colim}_{\substack{K \subseteq U \\ \text{compact}}} \Gamma_K(U, F)$.

Lemma 7.12. Consider $K \subseteq U \subseteq \overline{U} \subseteq K' \subseteq X$ where U is open and the other subsets are compact. Then, we have a fiber sequence

$$\Gamma_{K' \setminus U}(X, F) \longrightarrow \Gamma_{K'}(X, F) \longrightarrow \Gamma(U, F).$$

Remark 7.13. We have a functor

$$\mathrm{css}_{\mathscr{C}} : \mathrm{Fun}(\mathbf{Open}(X)^{\mathrm{op}}, \mathscr{C}) \to \mathrm{Fun}(\mathcal{K}(X), \mathscr{C}),\ F \mapsto (K \mapsto \Gamma_K(X, F)).$$

The functor

$$\mathrm{Fun}(\mathbf{Open}(X)^{\mathrm{op}}, \mathscr{C}) \to \mathrm{Fun}(\mathbf{Open}(X), \mathscr{C}),\ F \mapsto (U \mapsto \Gamma_c(U, F))$$

is the composition $\Psi^{\mathrm{op}}_{\mathscr{C}^{\mathrm{op}}} \circ \mathrm{css}_{\mathscr{C}}$.

Theorem 7.14 (Verdier duality)**.** The functor above restricts to an equivalence

$$\mathbb{D}_{\mathscr{C}} : \mathbf{Sh}(X, \mathscr{C}) \to \mathbf{CoSh}(X, \mathscr{C}).$$

Proof. You can check that if F is a sheaf, then $\mathbb{D}_{\mathscr{C}}$ is a cosheaf. We show $\mathbb{D}^{\mathrm{op}}_{\mathscr{C}^{\mathrm{op}}}$ is an inverse to $\mathbb{D}_{\mathscr{C}}$. Let us just show that it is a left inverse. We claim that

$$\Gamma_c(X \setminus K, F) \longrightarrow \Gamma_c(X, F) \longrightarrow \Gamma(K, F)$$

is a fiber sequence for a compact $K \subseteq X$. In the setting of the previous lemma we had

$$\Gamma_{K' \setminus U}(X, F) \longrightarrow \Gamma_{K'}(X, F) \longrightarrow \Gamma(U, F)$$

and take a colimit over all $K' \supseteq U$ to get a colimit

$$\operatorname{colim}_{K' \supseteq U} \Gamma_{K' \setminus U}(F) \longrightarrow \Gamma_c(X, F) \longrightarrow \Gamma(U, F).$$

Note that

$$\operatorname*{colim}_{K' \supseteq U} \Gamma_{K' \setminus U}(X, F) \simeq \operatorname*{colim}_{K' : K' \cap U = \varnothing} \Gamma_{K'}(X, F)$$

via $K' \leftrightarrow K' \setminus U$. Have a fiber sequence

$$\operatorname{colim}_{\{K' : K' \cap U = \varnothing\}} \Gamma_{K'}(X, F) \longrightarrow \Gamma_c(X, F) \longrightarrow \Gamma(U, F).$$

Take a colimit over $P = \{U \in \mathbf{Open}(X) : \overline{U} \in \mathcal{K}(X),\ U \supseteq K\}$. This was the main ingredient of the proof. □

Corollary 7.15. Verdier duality is natural with respect to pushforward of sheaves along proper maps $f : X \to Y$.

Proof. There exists a natural transformation $f_*\mathbb{D} \to \mathbb{D}f_*$ (even when f is not proper). Let $V \in \mathbf{Open}(Y)$ and $K \in \mathcal{K}(X)$ such that $K \subseteq f^{-1}(V)$. We have a diagram

$$\begin{tikzcd}
\Gamma(X \setminus K, F) \arrow[r] & \Gamma(X \setminus f^{-1}(f(X)), F) \arrow[r, "\simeq"] & \Gamma(Y \setminus f(K), f_*F) \\
\Gamma(X, F) \arrow[u] \arrow[r, equal] & \Gamma(X, F) \arrow[u] \arrow[r, "\simeq"] & \Gamma(Y, f_*F) \arrow[u] \\
\Gamma_K(X, F) \arrow[u] \arrow[r] & \Gamma_{f^{-1}(f(K))}(X, F) \arrow[u] \arrow[r, "\simeq"] & \Gamma_{f(K)}(Y, f_*F) \arrow[u]
\end{tikzcd}$$

So we get $\Gamma_K(X, F) \to \Gamma_c(V, f_*F)$. Taking a colimit over compact $K \subseteq f^{-1}(V)$ yields

$$\Gamma_c(f^{-1}(V), F) \to \Gamma_c(V, f_*F).$$

Now, when f is proper, each compact $K \subseteq X$ is contained in the compact $f^{-1}(f(K))$. So by cofinality

$$\operatorname*{colim}_{K \subseteq f^{-1}(V)} \Gamma_K(X, F) \simeq \operatorname*{colim}_{K' \subseteq V} \Gamma_{K'}(V, f_*F).$$

□

7.3 Dualizability of Spectral Sheaves

Lemma 7.16. There exists a functor $\varphi : \mathbf{Sh}(\mathcal{K}(X), \mathscr{C}) \to \mathbf{Sh}_{\mathcal{K}}(X, \mathscr{C})$ satisfying:

(i) Let $F \in \mathbf{Sh}(\mathcal{K}(X), \mathscr{C})$ and $K \in \mathcal{K}(X)$. We have $\Gamma(K, \varphi F) \simeq \operatorname{colim}_{K \Subset K'} \Gamma(K', F)$.

(ii) The functor φ preserves filtered colimits.

(iii) It is a retract of the inclusion $\mathbf{Sh}_{\mathcal{K}}(X, \mathscr{C}) \to \mathbf{Sh}(\mathcal{K}(X), \mathscr{C})$.

Definition 7.17. Let $\mathscr{C}$ be a closed symmetric monoidal ∞-category. For $y \in \mathscr{C}$ consider the counit $\mathrm{ev} : \underline{\mathrm{Map}}(y, \mathbb{1}) \otimes y \to \mathbb{1}$. Then, $x \in \mathscr{C}$ is called **strongly dualizable** if for all $y \in \mathscr{C}$ the map

$$y \otimes \underline{\mathrm{Map}}(x, \mathbb{1}) \to \underline{\mathrm{Map}}(x, y)$$

is an equivalence. The map is adjoint to

$$1 \otimes \mathrm{ev}_x : y \otimes \underline{\mathrm{Map}}(x, \mathbb{1}) \otimes x \to y \otimes \mathbb{1} \simeq y.$$

Remark 7.18. Dualizable and strongly dualizable coincide in closed categories.

Definition 7.19. We call an ∞-category **dualizable** if it's a dual object in $\mathbf{Cocont}_{\infty}^{\mathrm{st}}$.

Example 7.20.

(i) Compactly generated ∞-categories are dualizable. If $\mathscr{C} = \mathrm{Ind}(\mathscr{C}_0)$, then $\mathscr{C}^{\vee} \simeq \mathrm{Ind}(\mathscr{C}_0^{\mathrm{op}})$.

(ii) Retracts of dualizable objects are dualizable.

Theorem 7.21. Let X be a topological space. Then, $\mathbf{Sh}(X, \mathbf{Sp})$ is dualizable.

Proof. We do this for $\mathbf{Sh}_{\mathcal{K}}(X, \mathbf{Sp})$ which is equivalent. It is a retract of $\mathbf{Sh}(\mathcal{K}(X), \mathbf{Sp}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{K}(X)^{\mathrm{op}}, \mathbf{Sp})$ which is compactly generated. □

Corollary 7.22. There is an equivalence $\mathbf{CoSh}(X, \mathbf{Sp}) \otimes \mathscr{C} \xrightarrow{\simeq} \mathbf{CoSh}(X, \mathscr{C})$.

You can also write this as the composition of functors

$$\mathrm{Fun}_!(\mathbf{Sh}(X, \mathbf{Sp}), \mathbf{Sp}) \times \mathrm{Fun}_!(\mathbf{Sp}, \mathscr{C}) \to \mathrm{Fun}_!(\mathbf{Sh}(X, \mathbf{Sp}), \mathscr{C}).$$

7.4 Functorialities

Proposition 7.23. Let $f : X \to Y$ be proper. The diagram

$$\begin{tikzcd} \mathbf{Sh}(X, \mathbf{Sp}) \otimes \mathscr{C} \arrow[r, "\eta"] \arrow[d, "f_*^{\mathbf{Sp}} \otimes \mathscr{C}"'] & \mathbf{Sh}(X, \mathscr{C}) \arrow[d, "f_*^{\mathscr{C}}"] \\ \mathbf{Sh}(Y, \mathbf{Sp}) \otimes \mathscr{C} \arrow[r, "\eta"'] & \mathbf{Sh}(Y, \mathscr{C}) \end{tikzcd}$$

commutes. So $f_*^{\mathscr{C}}$ has a left adjoint.

Proof. Use analogous diagrams for cosheaves and use functoriality of Verdier duality for proper maps. □

Proposition 7.24. Let $j : U \hookrightarrow X$ be an open immersion. Then, the diagram

$$\begin{tikzcd} \mathbf{Sh}(X, \mathbf{Sp}) \otimes \mathscr{C} \arrow[r] \arrow[d, "j^*_{\mathbf{Sp}} \otimes \mathscr{C}"'] & \mathbf{Sh}(X, \mathscr{C}) \arrow[d, "j^*_{\mathscr{C}}"] \\ \mathbf{Sh}(U, \mathbf{Sp}) \otimes \mathscr{C} \arrow[r] & \mathbf{Sh}(U, \mathscr{C}) \end{tikzcd}$$

commutes.

Theorem 7.25. Let $f : X \to Y$ be a map of locally compact T_2-spaces. Then, the diagram

$$\begin{tikzcd} \mathbf{Sh}(Y, \mathbf{Sp}) \otimes \mathscr{C} \arrow[r] \arrow[d, "f^*_{\mathbf{Sp}} \otimes \mathscr{C}"'] & \mathbf{Sh}(Y, \mathscr{C}) \arrow[d, "f^*_{\mathscr{C}}"] \\ \mathbf{Sh}(X, \mathbf{Sp}) \otimes \mathscr{C} \arrow[r] & \mathbf{Sh}(X, \mathscr{C}) \end{tikzcd}$$

commutes.

Proof. Consider

$$\begin{tikzcd} X \times Y \arrow[r, hook] & \overline{X} \times Y \arrow[d] \\ X \arrow[u] \arrow[r] & Y \end{tikzcd}$$

where $\overline{X}$ is the one-point compactification. The left map is the graph, the right map is proper. □

8 Six Functors on LCH (Emma Brink)

8.1 Proper Geometric Morphisms

We write **RTop** the ∞-category of topoi and the morphisms as the right adjoints of geoemtric morphisms. TALK 8 11.12.2025

Lemma 8.1 ([Lur09, 6.3.4.6]). The ∞-category **RTop** has pullbacks. It is computed via the functor $(\mathbf{Cat}^{\mathrm{lex}})^{\mathrm{op}} \to \mathbf{RTop}$, $(\mathscr{C} \to \mathscr{D}) \mapsto (\mathbf{PSh}(\mathscr{C}) \leftarrow \mathbf{PSh}(\mathscr{D}))$ by computing the pullback on presheaves and taking the suited local subcategory.

Definition 8.2. A functor $p_* : \mathscr{X} \to \mathscr{Y}$ is **proper** if

$$\begin{tikzcd} \mathscr{A} \arrow[r] \arrow[d] & \mathscr{S} \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \\ \mathscr{B} \arrow[r] & \mathscr{T} \arrow[r] & \mathscr{Y} \end{tikzcd}$$

consists of two pullbacks in **RTop**, then the left square is left adjointable.

Lemma 8.3 (Lurie, Martini–Wolf)**.** Let p_* be proper. Then, p_* preserves filtered colimits.

Proof Sketch. Let I be filtered. This consists of two pullbacks.

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "\mathrm{const}"] \arrow[d] & \mathrm{Fun}(I, \mathscr{X}) \arrow[r, "\lim"] \arrow[d, dashed, "p_*"] & \mathscr{X} \arrow[d, "p_*"] \\
\mathscr{Y} \arrow[r, "\mathrm{const}"'] & \mathrm{Fun}(I, \mathscr{Y}) \arrow[r, "\lim"'] & \mathscr{Y}
\end{tikzcd}$$

The right pullback square is in the parametrized presentability paper, Ex. 3.2.7.5 from Martini–Wolf. We use filteredness to show that const is a map in **RTop** and that $\lim \circ \mathrm{const} \simeq \mathrm{id}$. Use properness now. ☐

Proposition 8.4. Let $p : X \to Y$ in **LCH**. Consider

$$p_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y),\ F \mapsto (U \mapsto F(p^{-1}(U))).$$

Then, p_* is proper if and only if p is proper.

Proof. Consider a pullback

$$\begin{tikzcd}
X_Z \arrow[r, "i_X"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Z \arrow[r, "i"'] & Y
\end{tikzcd}$$

in **LCH** where i is a closed immersion. Then,

$$\begin{tikzcd}
\mathbf{Sh}(X_Z) \arrow[r] \arrow[d] & \mathbf{Sh}(X) \arrow[d, "f_*"] \\
\mathbf{Sh}(Z) \arrow[r] & \mathbf{Sh}(Y)
\end{tikzcd}$$

is a pullback in **RTop**. Then,

$$\mathbf{Sh}(Z) \simeq \mathbf{Sh}(Y)_{/(Y \setminus Z)} \underset{(Y\setminus Z)\times -}{\overset{i_*}{\rightleftarrows}} \mathbf{Sh}(Y)$$

by [Lur09, Proposition 7.3.2.10].

Now we can do the only if part. Consider a pullback

$$\begin{tikzcd}
p^{-1}(K) \arrow[r] \arrow[d] & X \arrow[d] \\
K \arrow[r] & Y
\end{tikzcd}$$

giving

$$\begin{tikzcd}
\mathbf{Sh}(p^{-1}(K)) \arrow[r] \arrow[d] \arrow[dd, bend right=60, "\Gamma"'] & \mathbf{Sh}(X) \arrow[d, "p_*"] \\
\mathbf{Sh}(K) \arrow[r] \arrow[d, "\Gamma"'] & \mathbf{Sh}(Y) \\
\mathbf{An} &
\end{tikzcd}$$

So the composite Γ preserves filtered colimits with which p is proper.

For the if direction, factor p as

$$X \hookrightarrow X^+ \times Y \longrightarrow Y$$

Closed immersions give proper morphisms by [Lur09, 7.3.2.12]. Consider

$$\begin{tikzcd} \mathbf{Sh}(X^+ \times Y) \arrow[r] \arrow[d] & \mathbf{Sh}(X^+) \arrow[d] \\ \mathbf{Sh}(Y) \arrow[r] & \mathbf{Sh}(\infty) \simeq \mathbf{An} \end{tikzcd}$$

the pullback via certain locale considerations. The right map is proper by [Lur09, 7.3.5.3]. □

Corollary 8.5 (Proper base change)**.** Let $p : X \to Y$ in **LCH** be proper. Then, the square

$$\begin{tikzcd} \mathbf{Sh}(X_Z) \arrow[r] \arrow[d] & \mathbf{Sh}(X) \arrow[d] \\ \mathbf{Sh}(Z) \arrow[r] & \mathbf{Sh}(Y) \end{tikzcd}$$

is left adjointable.

Proof. Consider

$$\begin{tikzcd} \mathbf{Sh}(Z_X) \arrow[r] \arrow[d] & \mathbf{Sh}(X) \arrow[d] & \\ \mathbf{Sh}(Z \times X^+) \arrow[r] \arrow[d] & \mathbf{Sh}(Y \times X^+) \arrow[r] \arrow[d] & \mathbf{Sh}(X^+) \arrow[d] \\ \mathbf{Sh}(Z) \arrow[r] & \mathbf{Sh}(Y) \arrow[r] & \mathbf{Sh}(\infty) \end{tikzcd}$$

The right square is a pullback by the previous observation. So also the bottom left square. The upper left square is a pullback by the previous observation. Then, we are done by using properness. □

8.2 6FF on LCH

Lemma 8.6. The pair ($I =$ open immersions, $P =$ proper maps) forms a suitable decomposition of **LCH**.

Proof. Let $i : C \to \prod_{\mathrm{Hom}_{\mathbf{Top}}(C,[0,1])}[0,1]$, then $\beta C = \overline{i(C)}$. If C is regular, the $C \to \beta C$ is an open immersion. Check all the suitable decomposition conditions. □

Let $f : X \to Y$ in **LCH** with associated $f^* : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$. Since f^* is left-exact between ∞-categories with products, this gives a functor $\mathbf{LCH}^{\mathrm{op}} \to \mathbf{CAlg}(\mathbf{Cat}_\infty)$. The functors are also cocontinuous, so we get a factorization

$$\begin{tikzcd} & \mathbf{CAlg}(\mathbf{Pr}^L) \arrow[d] \\ \mathbf{LCH}^{\mathrm{op}} \arrow[r] \arrow[ur, dashed] & \mathbf{CAlg}(\mathbf{Cat}) \end{tikzcd}$$

Let $\mathscr{C} \in \mathbf{CAlg}(\mathbf{Cat}^{\mathrm{st}}_{\mathrm{colim}})$. We get $\mathbf{Sh}^*(-) \otimes_{\mathbf{CAlg}(\mathbf{Cat}_{\mathrm{colim}})} \mathscr{C} : \mathbf{LCH}^{\mathrm{op}} \to \mathbf{CAlg}(\mathbf{Cat})$. This extends to a 3-functor formalism $\mathbf{Span}(\mathbf{LCH}, \mathrm{all}, \mathrm{all}) \to \mathbf{Cat}_\infty$ if the following hold:

(i) Let $i : U \hookrightarrow X$ be in I. Then, i^* has a left adjoint $i_!$ and for pullbacks

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d, hook, "i"] \\ Z \arrow[r, "f"'] & X \end{tikzcd}$$

we have the BC condition

$$\begin{tikzcd} \mathbf{Sh}(U',\mathscr{C}) \arrow[d, "i'_!"'] & \mathbf{Sh}(U,\mathscr{C}) \arrow[l, "(f')^*"'] \arrow[d, "i_!"] \\ \mathbf{Sh}(Z,\mathscr{C}) & \mathbf{Sh}(X,\mathscr{C}) \arrow[l, "f^*"] \end{tikzcd}$$

and we need the projection formula.

(ii) If $p : X \to Y$ is proper, then $p^*_{\mathscr{C}}$ has a right adjoint $p^{\mathscr{C}}_*$ satisfying BC and PF.

(iii) Consider a square

$$\begin{tikzcd} X' \arrow[r, "i'"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ U \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

of open immersions and proper maps. Then, $i_! p'_* \xrightarrow{\simeq} p_* i'_!$.

Let's show these.

Proof. Last time, we wrote down an equivalence

$$\begin{tikzcd} \mathbf{Sh}(Y,\mathbf{Sp}) \otimes_{\mathbf{Cat}^{\mathrm{cocont}}_{\mathrm{st}}} \mathscr{C} \arrow[r, "\simeq"] \arrow[d, "f^*_{\mathbf{Sp}} \otimes \mathscr{C}"'] & \mathbf{Sh}(Y,\mathscr{C}) \arrow[d, "f^*_{\mathscr{C}}"] \\ \mathbf{Sh}(X,\mathbf{Sp}) \otimes \mathscr{C} \arrow[r, "\simeq"'] & \mathbf{Sh}(X,\mathscr{C}) \end{tikzcd}$$

We have to show that $p_{*,\mathbf{Sp}}$ is cocontinuous and the statement for $\mathscr{C} = \mathbf{Sp}$.

(i) Let $i : U \hookrightarrow X$. Consider

$$\begin{tikzcd} \mathbf{Sh}(X) \arrow[r, shift left=2, "i^*" description] \arrow[r, shift right=2, "-\times U"'] & \mathbf{Sh}(U) \simeq \mathbf{Sh}(X)_{/U} \arrow[l, shift right=5, "i_!"'] \end{tikzcd}$$

These satisfy BC and PF. We get BC and PF for $i^* \otimes \mathbf{Sp} \vdash i_! \otimes \mathbf{Sp}$.

(ii) Let $p : X \to Y$ be proper. We have $p^*_{\mathbf{An}} \dashv p_{*,\mathbf{An}}$. Apply $\mathrm{Fun}^{\mathrm{red,ex}}(\mathbf{An}^{\mathrm{fin}}_*, -)$ to get another adjunction. Plus this is the stabilization, so we get

$$\begin{tikzcd} \mathbf{Sh}(Y,\mathbf{Sp}) \arrow[r, shift right, "p_*"'] & \mathbf{Sh}(X,\mathbf{Sp}) \arrow[l, shift right, "p^*"'] \end{tikzcd}$$

By Yoneda extension we define

$$\begin{tikzcd} \mathbf{Sh}(X, \mathbf{Sp}) \times \mathbf{Sh}(Y, \mathbf{Sp}) \arrow[r, dashed, "- \boxtimes -"] & \mathbf{Sh}(X \times Y, \mathbf{Sp}) \\ \mathbf{Open}(X) \times \mathbf{Open}(Y) \arrow[u] \arrow[r] & \mathbf{Open}(X \times Y) \arrow[u] \end{tikzcd}$$

We have $\Delta^*(- \boxtimes -) \simeq - \otimes -$ and

$$p_*(-) \otimes (-) \simeq \Delta^*(p_*(-) \boxtimes (-)) \simeq \Delta^*(p \times \mathrm{id})_*(- \boxtimes -).$$

In the second equivalence we need that p_* preserves colimits which comes from stability. Consider

$$\begin{tikzcd} X \arrow[r, "\Gamma_p"] \arrow[d, "p"'] & X \times Y \arrow[d, "p \times \mathrm{id}_Y"] \\ Y \arrow[r] & Y \times Y \end{tikzcd}$$

So proper base change we get

$$\Delta^*(p \times \mathrm{id})_*(- \boxtimes -) \simeq p_* \Gamma^*(- \boxtimes -) \simeq p_*(- \otimes p^* -).$$

$\square$

Get six functors: $f^* \dashv f_*$ and $f_!$ characterized by $p_! = p_*$ and $i_! \dashv i^*$.

Example 8.7. Let $f : X \to *$ and $A \in \mathbf{Sh}(X, \mathbf{Sp})$. Then, $f_* A$ is cohomology of X with coefficients in A. Moreover, $f_! A$ is compactly supported cohomology. If $B \in \mathbf{Sp}$, then $f_! f^! B$ is the B-homology of X and $f_* f^! B$ is Borel–Moore homology. We have $f_* \underline{\mathrm{Map}}(F, f^! G) \simeq \underline{\mathrm{Map}}(f_! F, G)$. Plugging in $F = f^* \mathbb{1}$ and $G = \mathbb{1}$ we get

$$f_* \underline{\mathrm{Map}}(f^* \mathbb{1}, f^! \mathbb{1}) \simeq f_* f^! \mathbb{1}$$

which is Borel–Moore homology. On the other hand,

$$f_* \underline{\mathrm{Map}}(f^* \mathbb{1}, f^! \mathbb{1}) \simeq \underline{\mathrm{Map}}(f_! f^* \mathbb{1}, \mathbb{1})$$

which is the dual of compactly supported cohomology.

8.3 Relative Atiyah Duality

Let **PH** be the category of paracompact Hausdorff spaces. There is a Grothendieck topology given by opens. Put

$$\mathbf{Vect}(-) = \mathrm{const} \coprod_n \mathrm{BO}(n) : \mathbf{PH}^{\mathrm{op}} \to \mathbf{CMon}(\mathcal{S})$$

as the constant sheaf. So $\mathbf{Vect}(X) \simeq \mathrm{Sing}\left(\mathrm{Hom}_{\mathbf{Top}}(X, \coprod_n \mathrm{Gr}_n)\right)$.

Definition 8.8. The functor Th : $\mathbf{Vect}(-) \to \mathrm{Pic}(\mathbf{Sh}(-, \mathbf{Sp}))$ is the functor which by adjunction corresponds to $J : \mathbf{Vect}(*) \to \mathrm{Pic}(\mathbf{Sp})$.

Claim 8.9. Let $p : E \to X$ be in $\mathbf{Vect}(X)$. Then,

$$\mathrm{Th}(E) = p_\sharp \, \mathrm{cofib}(j_\sharp j^* \mathbb{S}_E \to \mathbb{S}_E).$$

Proof. Consider $\mathbf{Vect}(-) \to \mathbf{RTop}_{/\mathbf{Sh}(-)}$, $(E \to X) \mapsto (p_* : \mathbf{Sh}(E) \to \mathbf{Sh}(X))$. Wo postcompose by the shape functor $\mathbf{RTop}_{\mathbf{Sh}(-)} \to \mathbf{Sh}(-)$. It sends $p_* : \mathbf{Sh}(E) \to \mathbf{Sh}(X)$ to $p_\sharp \mathbb{S}_E$ and $\mathbf{Sh}(E \setminus 0) \to \mathbf{Sh}(X)$ to $\widetilde{p}_\sharp \mathbb{S}_{E\setminus 0} = p_\sharp j_\sharp h^* \mathbb{S}_E$ where $\widetilde{p} : E \setminus 0 \to E \to X$. Consider

$$\mathbf{Vect}(-) \longrightarrow \mathrm{Fun}([1], \mathbf{Sh}(-)) \xrightarrow{\Sigma^\infty \mathrm{cofib}} \mathbf{Sh}(-, \mathbf{Sp})$$

□

Exercise 8.10. Let $p : E \to X$ be a vector bundle and s be the zero section. Then, $\mathrm{Th}(E) \simeq p_\sharp s_! \mathbb{S}_X$ by recollement. Show $\mathrm{Th}(E)^{-1} \simeq s^! \mathbb{S}_E$.

Theorem 8.11 (Atiyah Duality)**.** Let $f : X \to Y$ be a smooth submersion. Consider the SES

$$0 \longrightarrow T_f \longrightarrow TX \longrightarrow f^*TY \longrightarrow 0.$$

Then, $f^! \mathbb{S}_Y \simeq \mathrm{Th}(T_f)$.

Proof. The SES splits, so by symmetric monoidality we get $\mathrm{Th}(T_f) \simeq \mathrm{Th}(TX) \otimes \mathrm{Th}(f^*TY)^{-1}$.

Claim 8.12. Let $m : M \to *$. Then, $\mathrm{Th}(TM) \simeq m^! \mathbb{S}$.

Proof. Consider an embedding

$$\begin{tikzcd} M \arrow[r, hook, "i"] \arrow[dr, "m"'] & \mathbb{R}^n \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

We obtain

$$m^! \mathbb{S} \simeq i^! \pi^! \mathbb{S} \simeq i^! \mathbb{S} \otimes i^* \pi^! \mathbb{S} \simeq i^! \mathbb{S} \otimes i^* \Sigma^n \pi^* \mathbb{S} \simeq i^! \mathbb{S} \otimes i^* \mathrm{Th}(\mathbb{R}^n)$$

using $i^! \simeq i^! \mathbb{S} \otimes i^*(-)$ and $\pi^! \simeq \Sigma^n \pi^*$. Choose a tubular neighbourhood

$$\begin{tikzcd} M \arrow[rr] \arrow[dr, "s"'] & & \mathbb{R}^n \\ & \mu_{\mathbb{R}^n}(M) \arrow[ur] & \end{tikzcd}$$

With the previous exercise we get $i^! \mathbb{S} \simeq s^! \mathbb{S}_{\mu_{\mathbb{R}^n}(M)} \simeq \mathrm{Th}(\mu_{\mathbb{R}^n}(M))^{-1}$. □

With this claim, we obtain $\mathrm{Th}(T_f) \simeq x^! \mathbb{S} \otimes (f^* y^! \mathbb{S})^{-1}$. For essential geometric morphisms one has $f^* \simeq f^!(-) \otimes (f^! \mathbb{S}_Y)^{-1}$. Plugging this in gives

$$\mathrm{Th}(Tf) \simeq x^! \mathbb{S} \otimes (f^! y^! \mathbb{S})^{-1} \otimes f^! \mathbb{S}_Y \simeq f^! \mathbb{S}_Y.$$

□

9 Analytic Manifolds (Daniel Arone)

No notes available unfortunately. The talk was on the technical foundations of analytic manifolds [Cla25, Section 2, 3]. TALK 9 18.12.2025

10 Étale Sheaves on Light Condensed Anima (Kaif Hilman)

No notes available unfortunately. The talk was about étale sheaves on light condensed anima. Kaif has written amazing notes which he has linked on his website https://sites.google.com/view/kaif-hilman/talknotes! TALK 10 08.01.2026

11 Six Functors for Étale Sheaves (Fabio Neugebauer)

See https://fneugebauer.github.io/files/six_ff_etale_sheaves.pdf for nice notes that Fabio wrote! TALK 11 15.01.2026

Example 11.1. Let M be a smooth $\mathbb{R}$-manifold. Consider the map $f : M \to *$. It participates in the 6FF for sheaves on spaces.

(i) f is smooth.

(ii) The sheaf $f^! \mathbb{1}$ lies in the image of

$$\mathrm{Fun}(M, \mathbf{Sp}) \hookrightarrow \mathbf{Sh}(M)$$

and identified with $\mathbb{S}^{TM} : M \to \mathbf{Sp}$, $m \mapsto \Sigma^\infty T_m M^+$.

Part (i) implies that f^* admits a left adjoint $f_\sharp$ given by the formula $f_\sharp \simeq f_!(f^!(\mathbb{S}) \otimes (-))$. Plugging in $\mathbb{S}$ gives

$$\Gamma_c(M; \mathbb{S}^{TM}) = f_\sharp \mathbb{S} \simeq \Sigma^\infty_+ M.$$

If $\mathbb{S}^{TM} \otimes \mathbb{Z} \simeq \mathbb{Z}[d]$, then $H_{d-*}(M; \mathbb{Z}) \cong H^*_c(M; \mathbb{Z})$.

Goal: Generalize this to smooth F-analytic Artin stacks for a local Field F.

This will allow us to apply the result to BG for a p-adic Lie group G. So one gets Atiyah duality for continuous group cohomology in this setting.

11.1 The 6FF

Construction 11.2. For $X \in \mathbf{Cond}(\mathbf{An})$ set

$$\mathbf{Sh}_v(X) = \mathbf{Cond}(\mathbf{An})_{/X} \quad \text{and} \quad \mathbf{Sh}_{\text{ét}}(X) \subseteq \mathbf{Cond}(\mathbf{An})_{/X}$$

where the latter is spanned by étale maps $Y \to X$.

Recall:

Definition 11.3. A map $f : Y \to X$ is **étale** if for every $S \in \mathrm{Pro}^\omega(\mathbf{Fin})$ and every map $S \to X$ the pullback $Y \times_X S \to S$ is in the image of

$$\mathbf{Sh}(X) \hookrightarrow \mathbf{Cond}(\mathbf{An})_{/S}, \; よ_U \mapsto (U \to S).$$

Definition 11.4. Let $\mathscr{C} \in \mathbf{CAlg}(\mathbf{Pr}^L_{\mathrm{st}})$ be compactly generated.[23] We define

$$\mathbf{Sh}_{\text{ét}}(X; \mathscr{C}) = \mathbf{Sh}_{\text{ét}}(X) \otimes \mathscr{C} \quad \text{and} \quad \mathbf{Sh}_v(X; \mathscr{C}) = \mathbf{Sh}_v(X) \otimes \mathscr{C}.$$

We imposed these conditions for the results that will show up. Recall:

Fact 11.5. Let $X \in \mathbf{Pr}^L$. There is an equivalence

$$X \otimes \mathscr{C} \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{lex}}(\mathscr{C}^{\omega,\mathrm{op}}, X)$$

naturally in colimit-preserving and finite-limit-preserving maps $X \to Y$.

Lemma 11.6.

(i) Let $X \in \mathbf{Cond}(\mathbf{An})$. The functor $\mathbf{Sh}_{\text{ét}}(X; \mathscr{C}) \to \mathbf{Sh}_v(X; \mathscr{C})$ is fully faithful.

[23] More generally, compactly assembled.

(ii) The functor $\mathbf{Cond}(\mathbf{An})^{\mathrm{op}} \to \mathbf{Pr}^L$, $X \mapsto \mathbf{Sh}_{\text{ét}}(X;\mathscr{C})$ is limit-preserving.

(iii) Let $X \in \mathbf{Cond}(\mathbf{An})$ and $f : \mathcal{F} \to \mathcal{G}$ be a map in $\mathbf{Sh}_{\text{ét}}(X;\mathscr{C})$. Then, f is an equivalence if and only if $x^*\mathcal{F} \to x^*\mathcal{G}$ is an equivalence for all $x : * \to X$.

Proof Sketch. Check on profinite sets and apply 11.5. □

By (ii) we see in particular that this specializes to the 6FF from Heyer–Mann.

Lemma 11.7. Let $f : T \to S$ be in **ProFin**. Then:

(i) the right adjoint f_* of $f^* : \mathbf{Sh}_{\text{ét}}(S;\mathscr{C}) \to \mathbf{Sh}_{\text{ét}}(T;\mathscr{C})$ exists,

(ii) the adjunction $f^* \dashv f_*$ satisfies the projection formula,

(iii) f_* satisfies base change along any map of profinite sets.

Construction 11.8. We obtain a unique 6FF on **ProFin** such that:

(i) the underlying functor is $\mathbf{Sh}_{\text{ét}}(-;\mathscr{C})$,

(ii) all maps are proper, i.e. $f_! \simeq f_*$.

Now Heyer–Mann provides a class of $\mathscr{C}$-!-able maps E in $\mathbf{Cond}(\mathbf{An})$ together with a unique sheafy extension $\mathbf{Span}_E(\mathbf{Cond}(\mathbf{An})) \to \mathbf{Pr}^L$ of the previous 6FF. Sheafy implies that the underlying functor is $\mathbf{Sh}_{\text{ét}}(-;\mathscr{C})$.

Definition 11.9. A map $f : X \to Y$ is called **$\mathscr{C}$-smooth ($\mathscr{C}$-étale, ...)** if it is smooth (étale, ...) for this 6FF.

Remark 11.10. Consider a map of coefficient categories $\mathscr{C} \to \mathscr{C}'$. If f is $\mathscr{C}$-!-able, then it is $\mathscr{C}'$-!-able.

So shriekability of **Sp** is the strongest shriekability.

Corollary 11.11.

(i) If we restrict the $\mathscr{C}'$-6FF to $\mathscr{C}$-!-able maps, then it is $\mathbf{Sh}_{\text{ét}}(-,\mathscr{C}) \otimes_{\mathscr{C}} \mathscr{C}'$.

(ii) If f is $\mathscr{C}$-smooth (étale, proper, ...), then it is $\mathscr{C}'$-smooth (étale, proper, ...).

Remark 11.12. Let X admit a basis by second-countable $Y_i \in \mathbf{CHaus}$. Then, $\mathbf{Sh}(X) \to \mathbf{Sh}_{\text{ét}}(X)$ is Postnikov-completion. If $\dim(X) < \infty$, then $\mathbf{Sh}(X)$ is Postnikov-complete, e.g. when $X \in \mathbf{Man}_F$. So the 6FFs agree here.

11.2 Examples of Sp-Étale Maps

Proposition 11.13. Any truncated étale map $f : X \to Y$ is **Sp**-étale.

Lemma 11.14. Let $S \in$ **ProFin** and $f : U \hookrightarrow S$ be clopen. Then, f is étale.

Proof Sketch. The map f is proper. Then, $\mathbf{Sh}(S;\mathbf{Sp}) \simeq \mathbf{Sh}(U;\mathbf{Sp}) \times \mathbf{Sh}(S - U;\mathbf{Sp})$. You can check that $(\mathrm{id}, 0)$ is both a left and right adjoint of f^*. Play around more. □

Proof of 11.13. Assume $Y \in$ **ProFin**. Suppose that 11.13 holds for n-truncated maps. Let f be $(n+1)$-truncated. Because f is étale, we can find a collection of clopens $U_i \subseteq Y$ and local sections

$$\begin{tikzcd} & X \arrow[d] \\ U_i \arrow[ur, dashed, "s_i"] \arrow[r, hook] & Y \end{tikzcd}$$

such that $(s_i)_i : \coprod_i U_i \to X$ is surjective. Consider

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, dashed, "\Delta_f"] & \\ ? \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\ U_i \arrow[r, "s_i"'] & X \arrow[r] & Y \end{tikzcd}$$

Induction hypothesis gives that Δ_f is **Sp**-étale. The composite $? \to X \times_Y X \to X$ is m_i and we get that $s_i = m_i \circ d_i$ which is **Sp**-étale. □

11.3 Examples of Proper Maps

Lemma 11.15. Suppose that $g\colon X' \to X$ is $\mathscr{C}$-proper and that $g_*\mathbb{1} \in \mathbf{Sh}_{\text{ét}}(X; \mathscr{C})$ is descendable. If $f\colon X \to Y$ is truncated and $f \circ g$ is proper, then f is proper.

Theorem 11.16. Let $X \in \mathbf{CHaus}$ be second countable and of finite cohomological dimension. Then, $f\colon X \to *$ is **Sp**-proper.

Proof. Pick $S \in \mathbf{ProFin}$ with $g\colon S \twoheadrightarrow X$ such that for all $T \in \mathbf{ProFin}$ we have $S \times_X T \in \mathbf{ProFin}$. Let's blackbox the following:

> **Stalkwise Criterion for Descendability.** If $\mathbf{Sh}_{\text{ét}}(X; \mathbf{Sp})$ is countably-assembled, rigid, and of finite cohomological dimension, then $A \in \mathbf{Sh}_{\text{ét}}(A; \mathbf{Sp})$ is descendable if and only if there exists $N \in \mathbb{N}$ such that x^*A is descendable of index $\leq n$ for all $x\colon * \to X$.

Now g is proper and $f \circ g$ is proper. It remains to show that $g_*\mathbb{1}$ is descendable, and then we are done by 11.15. We do this stalkwise via the criterion mentioned about. See the following Lemma. □

Lemma 11.17. Let $g\colon Y \to X$ in $\mathbf{Cond}(\mathbf{An})$ be surjective and $\mathscr{C}$-proper. Then, for every $x\colon * \to X$ the algebra $x^*g_*\mathbb{1}$ is descendable of index ≤ 1.

Proof. Proper base change gives $x^*g_*(\mathbb{1}) = (g_x)_*(\mathbb{1})$ for $g_x : g^{-1}(x) \to *$. But g_x admits a section. So $\mathbb{1} \to (g_x)_*\mathbb{1}$ admits a section. □

Corollary 11.18 (Locality)**.** Let $f\colon X \to Y$ be a map in $\mathbf{Cond}(\mathbf{An})$. If $\{Y_i \to Y\}$ is jointly surjective and $X \times_Y Y_i \to Y_i$ are $\mathscr{C}$-!-able ($\mathscr{C}$-proper, ...), then so is f.

Proof. We have seen that for any $T \twoheadrightarrow S$ in **ProFin** the object $g_*(\mathbb{1})$ is descendable after some reductions in profinite sets. So g is a !-cover. Thus, any surjective cover in **ProFin** is a !-cover. □

Corollary 11.19. Let $f\colon X \to Y$ be

(i) in **Top** with each X, Y admitting a basis by **CHaus** of second countable and finite cohomological dimension spaces

(ii) or a representable map in $\mathbf{Sh}(\mathbf{Man}_F)$.

Then, f is **Sp**-!-able.

Proof.

(i) It suffices to check this for $X \to *$. It also suffices to check this for $X \in \mathbf{CHaus}$ which is second-countable and of finite cohomology dimension. We have checked this.

(ii) There exist manifolds M_i such that $\operatorname{colim}_i M_i \simeq Y$. So it suffices to check map of manifolds. Use (i).

□

11.4 Smooth Morphisms

Lemma 11.20. Let $\mathscr{C}$ be a coefficient category which is semi-rigid, e.g. rigid like $\mathbf{Sp}_p^\wedge$. Let $f\colon X \to S$ with $S \in \mathbf{ProFin}$. If f is !-able, then TFAE:

(i) f is smooth,

(ii) $f^!$ preserves colimits and $f^!\mathbb{1}$ is invertible.

Proof Sketch. The semi-rigidity basically allows you to reduce to dualizable objects. □

Theorem 11.21 ($\mathbb{R}$-case)**.**

(i) Let $f : X \to Y$ be a representable submersion in $\mathbf{Sh}(\mathbf{Man}_{\mathbb{R}})$. Then, f is smooth.

(ii) For any $\mathbb{R}$-analytic smooth Artin stack $M \twoheadrightarrow X$ the map $X \to *$ is smooth.

Proof. By locality (11.18) we can reduce to X, Y, M being manifolds. But we have an equivalence $\mathbf{Sh}_{\text{ét}}(M; \mathbf{Sp}) \simeq \mathbf{Sh}(M; \mathbf{Sp})$. □

Theorem 11.22 ($\mathbb{Q}_p$-case)**.** Let X be a $\mathbb{Q}_p$-manifold and $G \to X$ be a group object in representable submersions to X. Let $BG \to X$ be a relative classifying stack.

(i) f is $\mathbf{Sp}_p^\wedge$-smooth.

(ii) If $G \to X$ is **Top**-proper and each fiber G_x is p-torsionfree, then f is $\mathbf{Sp}_p^\wedge$-proper.

Proof.

(ii) Reduce to $X = S \in \mathbf{ProFin}$. Consider the retraction

$$S \xrightarrow{e} BG \longrightarrow S$$

It suffices to check:

- e is proper: We know that $G = S \times_{BG} S \to S$ is **Sp**-proper.
- $e_*\mathbb{1}$ is descendable: Use a p-adic version of the stalkwise criterion. Need:
  - $e_*\mathbb{1}$ is uniformly stalkwise descendable.
  - $\mathbf{Sh}_{\text{ét}}(BG, \mathbf{Sp}_p^\wedge)$ is rigid and countably assembled.
  - finite cohomological dimension.

(i) Replace X be a profinite set S and G by a uniform pro-p-group. Check:

- f is !-able.
- The functor $f^!$ is cocontinuous which is if and only if $f_!$ preserves compacts.
- The object $f^!\mathbb{1}$ is invertible. Reduce to $f^!\mathbb{F}_p$. By Schwede–Shipley we get
$$\mathbf{Sh}_{\text{ét}}(BG, \mathcal{D}(\mathbb{F}_p)) \simeq \mathbf{Mod}_{f_*\mathbb{F}_p}\left(\mathbf{Sh}(S, \mathcal{D}(\mathbb{F}_p))\right).$$

We $\underline{\operatorname{Map}}_{\mathbb{F}_p}(f_*\mathbb{F}_p, \mathbb{F}_p) \simeq f^!\mathbb{F}_p$.

□

11.5 On Nonstandard Paths

Let $f\colon X \to Y$ be a smooth map of F-analytic Artin stack. Our goal is to identify $f^*\mathbb{1}$ in terms of tangent bundle.

Strategy: Deformation to the tangent bundle yields an $F^\times$-equivariant bundle on $F \times X$, hence a sheaf $\mathcal{F} \in \mathbf{Sh}_{\text{ét}}(X \times F/F^\times, \mathbf{Sp})$ such that:

- $\mathcal{F}$ at 0 yields tangent info.
- $\mathcal{F}$ at 1 yields $f^!\mathbb{1}$.

Theorem 11.23. Let F be a local field and p be a prime. Let $X \in \mathbf{Cond}(\mathbf{An})$ and moreover let $\mathcal{F} \in \mathbf{Sh}_{\text{ét}}(X \times F/F^\times, \mathbf{Sp}_p^\wedge)$ be invertible. Then, $0^*\mathcal{F} \simeq 1^*\mathcal{F}$ in $\mathbf{Sh}_{\text{ét}}(X, \mathbf{Sp}_p^\wedge)$.

Proof. Choose a map of condensed anima $\gamma\colon I \to F/F^\times$ and lifts of $0, 1 \in F$ along γ, such that $\mathrm{pr}^*\colon \mathbf{Sh}_{\text{ét}}(X, \mathbf{Sp}_p^\wedge) \to \mathbf{Sh}_{\text{ét}}(X \times I, \mathbf{Sp}_p^\wedge)$ is fully faithful and hits all invertible sheaves. Consider the diagram

$$X \xrightarrow{\quad 0 \quad} I \times X \xrightarrow{\quad \mathrm{pr} \quad} X.$$

Choose $\mathcal{F}'$ with $\mathrm{pr}^*\mathcal{F}' \simeq \mathcal{F}$. Then, $0^*\mathcal{F} \simeq 0^*\mathrm{pr}^*\mathcal{F}' \simeq \mathcal{F}' \simeq 1^*\mathrm{pr}^*\mathcal{F}' \simeq 1^*\mathcal{F}$. □

Let's discuss exotic intervals.

Definition 11.24. An object $I \in \mathbf{Cond}(\mathbf{An})$ is called **$\mathscr{C}$-interval** if $f\colon I \to *$ is proper and $f^*\colon \mathscr{C} \to \mathbf{Sh}_{\text{ét}}(I, \mathscr{C})$ is fully faithful, and every invertible sheaf lies in the image of f^*.

Example 11.25. The object $[0,1]$ is an $\mathbf{Sp}$-interval.

Lemma 11.26. Let I be a $\mathscr{C}$-interval and $X \in \mathbf{Cond}(\mathbf{An})$. Then,

$$\mathrm{pr}^*\colon \mathbf{Sh}_{\text{ét}}(X, \mathscr{C}) \to \mathbf{Sh}_{\text{ét}}(X \times I, \mathscr{C})$$

is fully faithful and hits every invertible object.

Lemma 11.27. Let $I \in \mathbf{CHaus}$ be second countable and of finite cohomological dimension. Suppose:

(i) $f\colon I \to *$ is an $\mathbb{F}_p$-cohomology isomorphism,

(ii) any continuous map $I \to BG$ for finite G is nullhomotopic.

Then, I is an $\mathbf{Sp}_p^\wedge$-interval.

Need $\gamma : I \to F/F^\times$. Put

$$K = \mathbb{Z} \cup \{\infty\} \to F, \ \frac{1}{n} \mapsto q^{-n}.$$

There is a $\mathbb{Z}$-action on K. Can then get $K/\mathbb{Z} \to F/\mathbb{Z} \to F/F^\times$. Have a map $(K \times \mathbb{R})/\mathbb{Z} \to K/\mathbb{Z}$. Cut off somewhere to make this compact. Consider the solenoid

$$S^\wedge = \lim\left(S^1 \xleftarrow{(-)^2} S^1 \xleftarrow{(-)^3} S^1 \xleftarrow{(-)^4} \cdots\right) \twoheadrightarrow S^1.$$

In the middle we change the object into a solenoid. Get

$$I \twoheadrightarrow (K \times \mathbb{R})/\mathbb{Z} \longrightarrow K/\mathbb{Z} \longrightarrow F/\mathbb{Z} \longrightarrow F/F^\times.$$

Let G be finite. Then we need to check that $S^\wedge \to BG$ is null.

12 Duality for p-adic Lie Groups (Shay Ben-Moshe)

See https://shaybm.com/pdfs/notes/2026-01-22-lie-group-duality.pdf for nice notes that Shay wrote! TALK 12 22.01.2026

12.1 Overview

Theorem 12.1. Let M be a smooth manifold. Write $f : M \to *$.

(i) Then, f is $\mathbf{Sp}$-smooth, that is, for all $E \in \mathbf{Sp}$ the map $f^*E \otimes \omega_f \to f^!E$ is an equivalence where $\omega_f = f^!\mathbb{S}$, and ω_f is invertible.

(ii) There is an equivalence $\omega_f \simeq \mathbb{S}^{TM}$.

Example 12.2. Suppose that M is compact and f is $\mathbf{Sp}$-proper. Take $E = \mathbb{Z}$. Then,

$$C^*(M;\mathbb{Z})^\vee \simeq C^*\left(M;\Lambda^d\mathbb{Z}^{TM}\right)$$

where $\Lambda^d\mathbb{Z}^{TM} = \mathrm{Or}_M$ is the *orientation bundle*.

We will prove a variant, namely for $f : BG \to *$ where G is a real or p-adic Lie group, i.e. $G \in \mathbf{Grp}(\mathbf{Man}_F)$. With integral coefficients this is due to Lazard. In the previous talk we saw the $\mathbf{Sp}_p$-smoothness of f. In this talk, we will focus on ω_f.

Theorem 12.3.

(i) Let G be a real Lie group and $f : \mathrm{BG} \to *$. Then,[24] $\omega_f = f^!\mathbb{S} \simeq \mathbb{S}^{-\mathrm{ad}_G} \in \mathbf{Sh}_{\text{ét}}(\underline{\mathrm{BG}};\mathbf{Sp})$.

(ii) If G is a p-adic Lie group, then $\omega_f = f^!\mathbb{S}_p \simeq \mathbb{S}_p^{\mathrm{ad}_G} \in \mathbf{Sh}_{\text{ét}}(\underline{\mathrm{BG}};\mathbf{Sp}_p)$.

Example 12.4. Let G be compact and p-torsionfree. Then, $f : \mathrm{BG} \to *$ is $\mathbf{Sp}_p$-proper. After base change to $\mathbb{Z}_p$ we recover Lazard's theorem

$$C^*(G;\mathbb{Z}_p)^\vee \simeq C^*\left(G;(\Lambda^d\,\mathrm{ad}_G)_{\mathbb{Z}_p}[d]\right).$$

We will discuss the construction $V \mapsto \mathbb{S}^V$ more in the second part, in particular a p-adic analogue of the J-homomorphism. Today's plan is the following:

1. A uniform proof of Atiyah duality and a duality for Lie groups.
2. J-homomorphism, reciprocity law.
3. Artin homomorphism.

12.2 Atiyah Duality & Duality for Lie Groups

Deformation to tangent bundle

Let $X \in \mathbf{Man}_F$.

- We have defined another manifold

$$TX \underset{\longleftarrow}{\xrightarrow{F}} X,$$

the tangent bundle.

[24]You compute BG is $\mathbf{Sh}(\mathbf{Man}_F)$.

- We have defined the deformation space

$$DX \underset{\Sigma}{\overset{\Pi}{\rightleftarrows}} F \times X$$

with $DX \times_F \{0\} \simeq TX \rightleftarrows X$ and $DX \times_{F^\times} F^\times \simeq F^\times \times X \times X \rightleftarrows F \times X$. The latter maps are projection and diagonal. This is all $F^\times$-equivariant.

Relativize, so put $f : X \to Y$ in $\mathbf{Man}_F$ which is a submersion. We extend to

$$T(X/Y) = \ker(TX \to f^*TY),$$
$$D(X/Y) = \cdots.$$

This construction is local on Y, so it extends to representable submersions $f : X \to Y$ in $\mathbf{Sh}(\mathbf{Man}_F)$.

Thom Objects

Let's start with the real case.

Definition 12.5. Let $X \in \mathbf{Sh}(\mathbf{Man}_\mathbb{R})$ and let $f : V \rightleftarrows X : e$ be a vector bundle. Then,

$$\mathbb{S}^V = e^* f^! \mathbb{S} \in \mathbf{Sh}_{\text{ét}}(\underline{X}; \mathbf{Sp}).$$

Since f is $\mathbf{Sp}$-smooth, $f^!\mathbb{S}$ is invertible and by symmetric monoidality of e^* we get that $\mathbb{S}^V$ is invertible. Thus, it is locally constant, i.e. it comes from

$$\mathbf{An}_{/|X|} \otimes \mathbf{Sp} \simeq \mathbf{Sh}_{\text{ét}}(|X|; \mathbf{Sp}) \hookrightarrow \mathbf{Sh}_{\text{ét}}(\underline{X}; \mathbf{Sp}).$$

Remark 12.6. Consider $V \setminus 0 \hookrightarrow V$. Then, $\mathbb{S}^V \simeq \Sigma^\infty(|V|/|V \setminus 0|)$ by a standard 6FF argument.

Atiyah Duality

Theorem 12.7. Let $f\colon X \to Y$ in $\mathbf{Sh}(\mathbf{Man}_\mathbb{R})$ be a representable submersion. Then,

$$\omega_f = f^! \mathbb{S} \simeq \mathbb{S}^{T(X/Y)}$$

in $\mathbf{Sh}_{\text{ét}}(\underline{X}; \mathbf{Sp})$.

Lemma 12.8. Let $f\colon X \to Y$ be smooth for some 6FF. Then,

$$\omega_f = f^! \mathbb{1} \simeq \delta^* \pi_1^! \mathbb{1} \in D(X)$$

with $\pi_1 : X \times_Y X \rightleftarrows X : \delta$.

Proof of 12.7. Consider

$$D(X/Y) \underset{\Sigma}{\overset{\Pi}{\rightleftarrows}} \mathbb{R} \times X$$

which is a representable submersion. Let's write

$$\mathbb{S}^{D(X/Y)} = \Sigma^* \Pi^! \mathbb{S} \in \mathbf{Sh}_{\text{ét}}(\underline{\mathbb{R} \times X}; \mathbf{Sp})$$

which is locally constant. Moreover, $|\mathbb{R}| \simeq *$. This implies

$$\mathbb{S}^{T(X/Y)} \simeq 0^* \mathbb{S}^{D(X/Y)} \simeq 1^* \mathbb{S}^{D(X/Y)} \simeq \delta^* \pi_1^! \mathbb{S} \simeq f^* \mathbb{S} = \omega_f.$$

via base change and the previous lemma. ∎

Linearization of real Lie groups

There are two steps in the proof.

1. Without G-action but in families.

2. Deduce main theorem.

Proposition 12.9. Let $X \in \mathbf{Sh}(\mathbf{Man}_{\mathbb{R}})$ and let $G \to X$ be a Lie group, i.e. a group object in representable submersions over X. Pass to relative the classifying object $f\colon \mathrm{B}G \to X$ which comes with a base point $e\colon X \to \mathrm{B}G$. Let $\mathfrak{g} = T(G/X) \to X$. Then,

$$e^*\omega_f = e^* f^! \mathbb{S} \simeq \mathbb{S}^{-\mathfrak{g}}$$

in $\mathbf{Sh}_{\text{ét}}(\underline{X}; \mathbf{Sp})$.

Proof. Same as before but instead of taking $D(\mathrm{B}G/X)$, take $D(G/X)$ which is still a group, take B. The fibers over 0 resp. 1 are $\mathrm{B}\mathfrak{g}$ and $\mathrm{B}G \times_X \mathrm{B}G$. Proceeding as before gets us to $e^*\omega_f \simeq \mathbb{S}^{\mathrm{B}\mathfrak{g}}$. □

Lemma 12.10. Let $X \xrightarrow{e} V \xrightarrow{f} X$ be a VB and write $X \xrightarrow{e_B} \mathrm{B}V \xrightarrow{f_B} X$. We write $\mathbb{S}^{\mathrm{B}V} = e_B^* f_B^! \mathbb{S}$ dual to $\mathbb{S}^V$.

Proof. We get

$$S \simeq e_B^! f_B^! \mathbb{S} \simeq e_B^* f_B^! \mathbb{S} \otimes e_B^! \mathbb{S} = \mathbb{S}^{\mathrm{B}V} \otimes e_B^! \mathbb{S}.$$

It remains to show $e_B^! \mathbb{S} \simeq \mathbb{S}^V$. Consider the diagram

$$\begin{tikzcd} X \arrow[dr, "e=\delta"] & & \\ & V \arrow[r, "f"] \arrow[d, "f"'] & X \arrow[d, "e_B"] \\ & X \arrow[r, "e_B"'] & \mathrm{B}V \end{tikzcd}$$

to apply 12.8. □

Proof of 12.3(i). Understand what ad_G is. Start with a discrete group G. Since G acts on itself by conjugation, it yields a functor $G^{\mathrm{ad}}\colon \mathrm{B}G \to \mathbf{Grp}$. So we obtain a functor

$$\mathrm{B}G \xrightarrow{G^{\mathrm{ad}}} \mathbf{Grp} \xrightarrow{\simeq} \mathbf{Grpd}_{*,\geq 1} \longrightarrow \mathbf{Grpd}$$

which is constant on BG. Unstraightening thus gives $\pi_1\colon \mathrm{B}G \times \mathrm{B}G \to \mathrm{B}G$. Functoriality in G extends this to any topos.

Consider $\mathrm{B}G \times G \to \mathrm{B}G$ and $T(\mathrm{B}G \times G/\mathrm{B}G) \simeq \mathrm{ad}_G \simeq \mathrm{B}G \times \mathrm{B}\mathfrak{g} \xrightarrow{\pi_1} \mathrm{B}G$. Apply the previous proposition to this family.

□

The p-adic case

The map $V \to X$ is not $\mathbf{Sp}_p$-smooth. Neight is $\mathbb{Q}_p \to *$. But $\mathrm{B}V \to X$ is $\mathbf{Sp}_p$-smooth.

Definition 12.11. Consider $f\colon V \to X$ and $X \xrightarrow{e_B} \mathrm{B}V \xrightarrow{f_B} X$. Then,

$$\mathbb{S}_p^V = e_B^* f_B^! \mathbb{S}_p \in \mathbf{Sh}_{\text{ét}}(X; \mathbf{Sp}_p).$$

Theorem 12.12. Same but $\omega_f = \mathbb{S}_p^{\mathrm{ad}_G}$.

Proof. Same. Except $0^* \simeq 1^*$ becomes $|\mathbb{R}| \simeq *$, use the non-standard path from last talk. □

12.3 J-Homomorphism and Reciprocity

Fix X. We want to look at $(V \to X) \mapsto \mathbb{S}^V$. We saw that $\mathbb{S}^{BV}$ is dual to $\mathbb{S}^V$. Then,

$$0 \longrightarrow V \longrightarrow W \longrightarrow W/V \longrightarrow 0$$

which implies $\mathbb{S}^W \simeq \mathbb{S}^V \otimes \mathbb{S}^{W/V}$. So this is K-theoretic of natural. One obtain $J : \mathrm{ko} \to \mathrm{Pic}(\mathbb{S})$.

Generalize.

1. p-adic,
2. bigger objects.

Vectorial Objects

Let $f\colon V \to X$ be a real VB. Then, f^* is FF. This is false p-adically, but true for B$V \to X$. Take B again, then f^* is an equivalence. Fix a six functor formalism $D : \mathbf{Span}_E(\mathscr{C}) \to \mathbf{Pr}^L$.

Definition 12.13.

(i) We say that $f\colon X \to Y$ in $\mathscr{C}$ is **vectorial** if f is in E and for every base change $\widetilde{f}$ of f we have that $\widetilde{f}^*$ and $\widetilde{f}_!$ are equivalences. This implies that f is smooth.

(ii) We say that $X \in \mathscr{C}$ is **vectorial** if $X \to *$ is.

Definition 12.14. Let X be vectorial with $f\colon X \to *$. Then, $J(X) = f_! f^* \mathbb{1} \in D(*)$.

Proposition 12.15. If f has a section $e\colon * \to X$, then $J(X) \simeq f_! f^* \mathbb{1} \simeq (e^* f^! \mathbb{1})^{-1}$.

Construction 12.16. We define $J\colon K(\mathbf{Vect}_D) \to \mathrm{Pic}(D(*))$. Put

$$Q(\mathbf{Vect}_D) = \mathbf{Span}_V(\mathbf{Vect}_D) \subseteq \mathbf{Span}_E(\mathscr{C}) \quad \text{and} \quad K(\mathbf{Vect}_D) = \Omega|Q(\mathbf{Vect}_D)|.$$

We obtain a map

$$\Omega|Q(\mathbf{Vect}_D)| \to \mathrm{Pic}\left(\mathbf{Mod}_{D(*)}(\mathbf{Pr}^L)\right).$$

Definition 12.17. An object $X \in \mathscr{C} \otimes \mathbf{Sp}$ is **vectorial** if it is bounded below, and $\Omega^\infty \Sigma^n X$ is vectorial.

Assume that D satisfies good descent and there is a stable version $\mathbf{vect}_D \subseteq \mathscr{C} \otimes \mathbf{Sp}$. There is a composite

$$\mathbf{vect}_D \longleftarrow \mathbf{Vect}_{D,\geq 1} \xrightarrow{\Omega^\infty} \mathbf{Vect}_D$$

and the left map induces an equivalence on $\Omega|Q(-)|$. So we get a map $J\colon K(\mathbf{vect}_D) \to \mathrm{Pic}(D(*))$.

Proper & Étale Triviality

Let f be vectorial and proper, so $f_! \simeq f_*$ is an inverse of f^*. Thus, $J(X) \simeq f_! f^* \mathbb{1} \simeq \mathbb{1}$.

Theorem 12.18. The composite

$$K(\mathbf{Vect}^\Pi) \longrightarrow K(\mathbf{Vect}_D) \xrightarrow{J} \mathrm{Pic}(D(*))$$

is nullhomotopic.

Proof. WLOG assume that all objects of $\mathscr{C}$ are proper. Let $\widetilde{\mathscr{C}} = \mathrm{Pro}(\mathscr{C})$. Want a functor $\widetilde{D}\colon \mathbf{Span}(\widetilde{\mathscr{C}}) \to \mathbf{Pr}^L$. Do some stuff to get countable products. Now do an Eilenberg-swindle argument. □

13 Chromatic Applications (Sven van Nigtevecht)

Sven: *As with all talks about chromatic stuff, this is about something that Hopkins already knew.* TALK 13 29.01.2026

Hopkins observed: $\mathrm{map}(E_n, \mathbb{S}_{K(n)}) \simeq \Sigma^{-n^2} E_n$ but this is not true $\mathbb{G}_n$-equivariantly. There homotopy groups are actually $\mathbb{G}_n$-equivariantly isomorphic, and you see this by computing some homotopy fixed points.

Beaudry–Goerss–Hopkins–Stojanoska filter $\mathbb{G}_n$ by normal subgroups

$$\cdots \trianglelefteq \Gamma_2 \trianglelefteq \Gamma_1 \trianglelefteq \Gamma_0 = \mathbb{G}_n.$$

Definition 13.1. Let $B\Gamma_i = \lim_j B(\Gamma_i/\Gamma_j)$ and $I_{\mathbb{G}} = \mathrm{colim}_i \Sigma^\infty_+ B\Gamma_i$.

Then, $I_{\mathbb{G}}$ is a p-complete sphere and $\mathrm{map}(E, \mathbb{S}_{K(n)}) \simeq \mathrm{map}(I_{\mathbb{G}}, E_n)$.

Remark 13.2. The *linearization hypothesis* described $I_{\mathbb{G}}$ in terms of $\mathbf{Lie}(\mathbb{G}_n)$.

Theorem 13.3 (Clausen)**.** There is an equivalence $E_n \simeq \mathrm{map}(E_n, L_{K(n)}\mathbb{S}^{\mathrm{ad}\,\mathbb{G}})$.

More precisely, $\mathbb{E}_n \simeq \underline{\mathrm{map}}_v(\mathbb{E}_n, L_{K(n)}\mathbb{S}^{\mathrm{ad}\,\mathbb{G}})$. Here is an outline of the proof.

1. Construct $\mathbb{E}_n$.
2. Prove it first for $\underline{\mathrm{map}}_{\text{ét}}$, then go to $\underline{\mathrm{map}}_v$.
3. Show $L_{K(n)}q^!\mathbb{S} \simeq L_{K(n)}\mathbb{S}^{\mathrm{ad}\,\mathbb{G}_n}$ where $q\colon B\mathbb{G}_n \to *$.

There is another main result: Clausen gives an explicit description of $q^!$ on solid sheaves.

Recollection 13.4.

(i) Let $X \in \mathbf{Cond}(\mathbf{An})$, then $\mathbf{Sh}_v(X) = \mathbf{Cond}(\mathbf{An})_{/X}$.

(ii) Let $\mathscr{C}$ be compactly generated, then $\mathbf{Sh}_v(X;\mathscr{C}) = \mathbf{Sh}_v(X) \otimes \mathscr{C}$ in $\mathbf{Pr}^L$. Today, $\mathscr{C}$ will always be stable, so this will also stabilize.

We have $\mathbf{Sh}_{\text{ét}}(X) \subseteq \mathbf{Sh}_v(X)$, and put $\mathbf{Sh}_{\text{ét}}(X;\mathscr{C}) = \mathbf{Sh}_{\text{ét}}(X) \otimes \mathscr{C}$. Then, $\mathbf{Sh}_{\text{ét}}(X;\mathscr{C}) \hookrightarrow \mathbf{Sh}_v(X;\mathscr{C})$ is fully faithful with image those F such that for all $A \in \mathscr{C}^\omega$ we have $\mathrm{map}(A, F) \in \mathbf{Sh}_{\text{ét}}(X)$.

Example 13.5. Let $\mathscr{C} = \mathbf{Sp}_p^\wedge$. Then, $F \in \mathbf{Sh}_{\text{ét}}(X; \mathbf{Sp}_p^\wedge) \iff F/p \in \mathbf{Sh}_{\text{ét}}(X; \mathbf{Sp})$.

13.1 Condensed Morava E-Theory

Theorem 13.6 (Lurie)**.** Write

$$\mathbf{Perf}_{/\mathcal{M}^n_{\mathrm{fg}}} = \{\mathrm{Spec}\, R, f\colon \mathrm{Spec}\, R \to \mathcal{M}^\wedge_{\mathrm{fg}} : R \text{ perfect } \mathbb{F}_p\text{-algebra}\}.$$

Then, there is a fully faithful functor

$$\mathbf{Perf}_{/\mathcal{M}^n_{\mathrm{fg}}} \to \mathbf{CAlg}(\mathbf{Sp}),\ (R, \widehat{G}) \mapsto E(R, \widehat{G}).$$

This is a pro-étale hypersheaf.

Construction 13.7. Define $\mathbf{ProFin} \to \mathbf{Perf}$, $S = \lim_i S_i \mapsto \mathrm{Spec}(C(S, \mathbb{F}_p)) = \lim_i \coprod_{S_i} \mathrm{Spec}\, F_p$. This induces a functor $\mathbf{Cond}(\mathbf{An}) \to \widehat{\mathbf{Sh}}(\mathbf{Perf})$. This yields a map

$$\mathbf{Cond}(\mathbf{An})_{/B\mathbb{G}_n} \to \widehat{\mathbf{Sh}}(\mathbf{Perf})_{/B\mathbb{G}_n}.$$

Pull back along $\mathcal{M}^n_{\mathrm{fg}} = \mathrm{Spec}(\mathbb{F}_{p^n})/\mathbb{G}_n \to B\mathbb{G}_n$ and stabilize to get

$$\alpha^*\colon \mathbf{Sh}_v(B\mathbb{G}_n; \mathbf{Sp}) \hookrightarrow \widehat{\mathbf{Sh}}(\mathbf{Perf})_{/\mathcal{M}^n_{\mathrm{fg}}}.$$

Definition 13.8. Let $\mathbb{E}_n = \alpha_*(E(-,-))$.

Example 13.9. We get

$$E_n = \mathbb{E}_n(* \to \mathrm{B}\mathbb{G}_n) \simeq E(\operatorname{Spec}\mathbb{F}_{p^n} \to \mathcal{M}^n_{\mathrm{fg}}) \simeq E(\mathbb{F}_{p^n}, \text{Honda formal group}).$$

You can view this as a condensed ring $\{\mathbb{E}_n(S \to \mathrm{B}\mathbb{G}_n)\}_{S\in\mathbf{ProFin}}$.

Proposition 13.10. The map $\delta^* E_n \to \mathbb{E}_n$ is $(p, v_1, \cdots, v_{n-1})$-completion in $\mathbf{Cond}(\mathbf{Sp})$.

Corollary 13.11. The object $\mathbb{E}_n$ is étale in $\mathbf{Sh}_v(\mathrm{B}\mathbb{G}_n; \mathbf{Sp}^\wedge_{p,v_1,\cdots,v_{n-1}})$.

Let $S \to \mathrm{B}\underline{\mathbb{G}}_n$. For $T \in \mathbf{ProFin}_{\mathbb{G}_n}$ we get a map $T/\mathbb{G}_n \to \mathrm{B}\mathbb{G}_n$. Recall that

$$f^* \colon \mathbf{Sh}(\mathbf{Fin}_{\mathbb{G}_n}; \mathbf{Sp}) \to \mathbf{Sh}_v(\mathrm{B}\mathbb{G}_n; \mathbf{Sp})$$

exhibits $\mathbf{Sh}_{\text{ét}}$ as the Postnikov completion of $\mathbf{Sh}(\mathbf{Fin}_{\mathbb{G}}, \mathbf{Sp})$.

Theorem 13.12. There are equivalences $f_*\mathbb{E}_n \simeq \mathcal{E}^{\mathrm{DH}}$ and $f^*\mathcal{E}^{\mathrm{DH}} \simeq \mathbb{E}_n$.

Itamar's definition is $f^*\mathcal{E}^{\mathrm{DH}}$.

Lemma 13.13. The map

$$\mathbb{E}_n(T/\mathbb{G}_n) \otimes \mathbb{E}_n(T'/\mathbb{G}_n) \to \mathbb{E}_n(T \times T'/\mathbb{G}_n)$$

is a $(p, v_1, \cdots, v_{n-1})$-completion.

Corollary 13.14. There is an equivalence $\mathbb{E}_n(\mathrm{B}\mathbb{G}_n) \simeq \mathbb{E}_n(*/\mathbb{G}_n) \simeq \mathbb{S}_{K(n)}$.

Corollary 13.15. There is an equivalence

$$L_{K(n)}(\mathbb{E}_n \otimes q^* E_n) \simeq L_{K(n)}(\mathbb{E}_n \otimes e_*\mathbb{S})$$

where $* \xrightarrow{e} \mathrm{B}\underline{\mathbb{G}}_n \xrightarrow{q} *$.

13.2 Solidity

Definition 13.16. Write $P = \mathbb{Z}[\mathbb{N}_\infty]/\mathbb{Z}[\infty]$ and $f = \mathrm{id} - \text{shift}$. Then, $A \in \mathbf{Cond}(\mathbf{Ab})$ is **solid** if

$$f^* \colon \underline{\mathrm{Hom}}(P, A) \to \underline{\mathrm{Hom}}(P, A),\ (x_1, x_2, \cdots) \mapsto (x_1 - x_2, x_2 - x_3, \cdots)$$

is an isomorphism.

Theorem 13.17.

(i) The inclusion $\mathbf{Solid}(\mathbf{Ab}) \subseteq \mathbf{Cond}(\mathbf{Ab})$ is closed under limits, colimits, extensions, $\underline{\mathrm{Ext}}^i$, and contains $\mathbb{Z}$.

(ii) The inclusion has a left adjoint $(-)^\square$, the solidification functor.

(iii) The localization is symmetric monoidal, so it gives a symmetric monoidal structure on $\mathbf{Solid}(\mathbf{Ab})$.

Example 13.18. Let $\mathbb{Z}_p = \lim_n \mathbb{Z}/p^n \in \mathbf{Solid}(\mathbf{Ab})$.

Definition 13.19. An object $X \in \mathbf{Cond}(\mathbf{Sp})$ is **solid** if $\pi_n X \in \mathbf{Cond}(\mathbf{Ab})$ is solid for all n.

Theorem 13.20. The inclusion $\mathbf{Solid}(\mathbf{Sp}) \subseteq \mathbf{Cond}(\mathbf{Sp})$ is closed under limits, colimits, retracts, and contains $\mathbb{S}$.

Thus, the adjoint functor theorem gives a solidification functor $\mathbf{Cond}(\mathbf{Sp}) \to \mathbf{Solid}(\mathbf{Sp})$ and it yields $\otimes^{\square}$ on $\mathbf{Solid}(\mathbf{Sp})$.

Construction 13.21. The functor $\delta^*\colon \mathbf{Sp} \to \mathbf{Cond}(\mathbf{Sp})$ lands in $\mathbf{Solid}(\mathbf{Sp})$. So if $X\colon I \to \mathbf{Sp}$, then $\lim_I \delta^* X \in \mathbf{Solid}(\mathbf{Sp})$.

Example 13.22. There is an equivalence $\mathbb{S}_p \simeq (\delta^*\mathbb{S}_p)^{\wedge}_p \in \mathbf{Solid}(\mathbf{Sp})$. The interesting part is that $\mathbb{S}_p \in \mathbf{Solid}(\mathbf{Sp})$ is idempotent as opposed to the classical setting.

This follows from:

Lemma 13.23. Consider towers of connective spectra $\{N_i\}_i$ and $\{M_j\}_j$. Write $N = \lim_i \delta^* N_i$ and $M = \lim_j \delta^* M_j$. Then,

$$M \otimes^{\square} N \simeq \lim_{i,j} (\delta^* M_j \otimes \delta^* N_i)$$

Upshot: $\mathbb{S}_p \otimes^{\square} \mathbb{S}_p$ is p-complete. In particular, there is a fully faithful functor

$$\mathbf{Mod}_{\mathbb{S}_p}(\mathbf{Solid}(\mathbf{Sp})) \hookrightarrow \mathbf{Solid}(\mathbf{Sp}),$$

so it is a condition to be a module.

Definition 13.24. Let $\mathbf{Sh}_{\square}(B\mathbb{G}; \mathbf{Sp}) \subseteq \mathbf{Sh}_v(B\mathbb{G}; \mathbf{Sp})$ be the full subcategory on those F such that $e^*F \in \mathbf{Sh}_v(*; \mathbf{Sp}) = \mathbf{Cond}(\mathbf{Sp})$ is solid.

Proposition 13.25. There is an adjunction

$$\mathbf{Sh}_v(B\mathbb{G}_n, \mathbf{Sp}) \rightleftarrows \mathbf{Sh}_{\square}(B\mathbb{G}_n, \mathbf{Sp}).$$

Proof. A Barr–Beck argument yields $\mathbf{Sh}_v(BG) \simeq \mathbf{Mod}_G(\mathbf{Cond}(\mathbf{An}))$. Then,

$$\mathbf{Sh}_v(BG, \mathbf{Sp}) \simeq \mathbf{Mod}_{\mathbb{S}[G]}(\mathbf{Cond}(\mathbf{Sp})).$$

Then, $\mathbf{Sh}_{\square}(BG; \mathbf{Sp}) \simeq \mathbf{Mod}_{\mathbb{S}[G]}(\mathbf{Solid}(\mathbf{Sp}))$. Also see that e^* commutes with solidification. □

Next goal: Construct $q^!$ for $\mathbf{Sh}_{\square}(B\mathbb{G}, \mathbb{S}_p) \simeq \mathbf{Mod}_{\mathbb{S}_p[\mathbb{G}]^{\text{solid}}}(\mathbf{Solid}(\mathbf{Sp}))$.

Notation 13.26. Write $\mathbb{F}_p[\![G]\!] = \lim_i \mathbb{F}_p[G/N_i] \in \mathbf{Sp}$.

Lemma 13.27. If G is a profinite group such that $\mathbb{F}_p \in \mathbf{Perf}(\mathbb{F}_p[\![G]\!])$, then

$$q_*\colon \mathbf{Sh}_{\square}(BG; \mathbb{S}_p) \to \mathbf{Sh}_{\square}(*; \mathbb{S}_p)$$

preserves étale sheaves.

Proof Sketch. Let q^* be the restriction of scalars along $\mathbb{S}_p[G]^{\square} \to \mathbb{S}_p$. Then, $q_* = \underline{\mathrm{map}}_{\mathbb{S}_p[G]^{\square}}(\mathbb{S}_p, -)$. Need

$$\mathbb{S}_p \in \mathbf{Perf}(\mathbb{S}_p[G]^{\square}) \iff \mathbb{F}_p \in \mathbf{Perf}(\mathbb{F}_p[G]^{\square}).$$

Now reduce to the usual spectral case by some solid blabla. □

Definition 13.28. Define $q_!$ as the left Kan extension

$$\begin{tikzcd}
\mathbf{Sh}_{\square}(BG, \mathbb{S}_p)^{\omega} \arrow[r, "q_*"] \arrow[d] & \mathbf{Sh}_{\square}(*; \mathbb{S}_p) \\
\mathbf{Sh}_{\square}(BG; \mathbb{S}_p) \arrow[ur, dashed, bend right, "q_!"'] &
\end{tikzcd}$$

Also get $q_! \to q_*$.

Proposition 13.29. Suppose that G has an open normal subgroup $N \trianglelefteq G$ such that we have $\mathbb{F}_p \in \mathbf{Perf}(\mathbb{F}_p[\![N]\!])$. Then,

$$\begin{tikzcd} \mathbf{Sh}_{\text{ét}}(\mathrm{B}G; \mathbb{S}_p) \arrow[r, "q_!^{\text{ét}}"] \arrow[d, hook] & \mathbf{Sh}_{\text{ét}}(*; \mathbb{S}_p) \arrow[d, hook] \\ \mathbf{Sh}_{\square}(\mathrm{B}G; \mathbb{S}_p) \arrow[r, "q_!"'] & \mathbf{Sh}_{\square}(*; \mathbb{S}_p). \end{tikzcd}$$

commutes.

Proof Sketch. We start with:

1. Consider $\mathbf{Sh}_{\text{ét}}(\mathrm{B}G; \mathbb{S}_p) \subseteq \mathbf{Sh}_{\square}(\mathrm{B}G; \mathbb{S}_p)$.
2. The functor $q_!$ is a $\mathbf{Sh}_{\square}(*; \mathbb{S}_p)$-linear functor.

By Morita theory one gets an equivalence $q_! \simeq \mathbb{D}_G \otimes^{\square}_{\mathbb{S}_p[G]^{\square}} (-)$ as a functor

$$\mathbf{Mod}_{\mathbb{S}_p[G]^{\square}}(\mathbf{Solid}(\mathbf{Sp})) \to \mathbf{Mod}_{\mathbb{S}_p}(\mathbf{Solid}(\mathbf{Sp}))$$

where $\mathbb{D}_G = q_*(\mathbb{S}_p[G]^{\square}) = \mathrm{map}_{\mathbb{S}_p[G]^{\square}}(\mathbb{S}_p, \mathbb{S}_p[G]^{\square})$. Now look at

$$q_!^{\text{ét}} \longrightarrow q_*^{\text{ét}} \longrightarrow q_* \longleftarrow q_!$$

□

Now we have $q^! = \mathrm{map}(\mathbb{D}_G, -) \colon \mathbf{Mod}_{\mathbb{S}_p} \to \mathbf{Mod}_{\mathbb{S}_p[G]^{\square}}$.

Proposition 13.30. Let G be as before. Then, $q^!$ also preserves étale sheaves and $(q^!)^{\text{ét}} \simeq q^!$.

Corollary 13.31. Let G be as before.

(i) The étale dualizing sheaf $q^!\mathbb{S}_p$ identifies with $\mathrm{map}(\mathbb{D}_G, \mathbb{S}_p)$.

(ii) If G is a virtual Poincaré duality group, then $\mathbb{D}_G$ is étale and is the inverse of $q^!\mathbb{S}_p$.

Theorem 13.32. There is an equivalence $\mathbb{E}_n \simeq \underline{\mathrm{map}}_v(\mathbb{E}_n, L_{K(n)} q^! \mathbb{S}_p)$.

Proof Sketch.

1. Show $\mathbb{E}_n \simeq \mathrm{map}_{\text{ét}}(\mathbb{E}_n, L_{K(n)} q^! \mathbb{S}_p)$. The input is a descent result, i.e. that

$$L_{K(n)}(\mathbb{E}_n \otimes -) \colon \mathbf{Sp}_{K(n)} \to \mathbf{Mod}_{\mathbb{E}_n}\left(\mathbf{Sh}_{\text{ét}}(\mathrm{B}\mathbb{G}_n; \mathbf{Sp}_{K(n)})\right)$$

   is an equivalence. This is the usual descent result phrased with condensed things. The inverse is q_*.

2. The object $\underline{\mathrm{map}}_v(\mathbb{E}_n, L_{K(n)} q^! \mathbb{S}_p)$ is $K(n)$-locally étale, i.e. is discrete mod $(p, v_1, \cdots, v_{n-1})$. Now pass to the solid setting. The input there is that

$$\mathrm{map}\left(\prod_I \mathbb{S}_p, \mathbb{S}_p\right) \simeq \bigoplus_I \mathbb{S}_p$$

   in $\mathbf{Solid}(\mathbf{Sp})$. But Clausen writes something about this part being special to Morava E-theory.

□

References

[Cla25] Dustin Clausen. Duality and linearization for p-adic lie groups, 2025. (Cited on pages 0 and 43.)

[CLL25] Bastiaan Cnossen, Tobias Lenz, and Sil Linskens. Universality of span 2-categories and the construction of 6-functor formalisms, 2025. (Cited on page 6.)

[DK25] Adam Dauser and Josefien Kuijper. Uniqueness of six-functor formalisms, 2025. (Cited on page 6.)

[GH15] David Gepner and Rune Haugseng. Enriched ∞-categories via non-symmetric ∞-operads. *Adv. Math.*, 279:575–716, 2015. (Cited on pages 7 and 8.)

[HM24] Claudius Heyer and Lucas Mann. 6-functor formalisms and smooth representations, 2024. (Cited on pages 0, 4, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, and 28.)

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009. (Cited on pages 23, 26, 35, 38, 39, and 40.)

[Lur17] Jacob Lurie. Higher algebra, September 2017. Unpublished notes. (Cited on pages 5 and 15.)

[Lur18] Jacob Lurie. Spectral algebraic geometry, February 2018. Unpublished notes. (Cited on page 17.)

[Mat16] Akhil Mathew. The Galois group of a stable homotopy theory. *Adv. Math.*, 291:403–541, 2016. (Cited on pages 15 and 16.)

[NSS15] Thomas Nikolaus, Urs Schreiber, and Danny Stevenson. Principal ∞-bundles: general theory. *J. Homotopy Relat. Struct.*, 10(4):749–801, 2015. (Cited on page 22.)

[Vol21] Marco Volpe. The six operations in topology. *arXiv preprint*, 2021. arXiv:2110.10212. (Cited on pages 0 and 33.)