European Talbot 2025 – Higher Algebra and Chromatic Homotopy Theory

Qi Zhu

30 June 2025 – 04 July 2025

Abstract

These are my (live) TeX'd notes of the European Talbot 2025 mentored by Gijs Heuts and Ishan Levy and *Higher Algebra and Chromatic Homotopy Theory*.

Here is the description of the workshop copied from the official website:

Chromatic homotopy theory originated from Quillen's pioneering work on the connection between complex-oriented cohomology theories and formal group laws. The development of higher algebra has since provided a powerful framework to deepen this interplay, leading to major breakthroughs in stable homotopy theory. Chromatic methods have proven essential in understanding the structure of stable homotopy groups of spheres. This workshop will first introduce the foundations of higher algebra before exploring the structural properties of key objects in chromatic homotopy theory, such as Morava E- and K-theories.

We plan to cover the following content:

- The fundamentals of higher algebra, including stable infinity-categories, (symmetric) monoidal structures, (commutative) algebras, operads, and Koszul duality.
- Construction of the basic algebras of interest in chromatic homotopy theory, such as MU, BP, and the Morava E-and K-theories.
- Power operations: the Dyer-Lashof algebra and power operations in the $K(n)$-local setting.
- The structure of $K(n)$-local $\mathbb{E}_\infty$-rings and their applications.

Please contact me at qzhu@mpim-bonn.mpg.de (or over social media) for comments or suggestions.

Contents

0 Intro

We (around 40 young topologists) met in Kolding in Denmark to learn about higher algebra and chromatic homotopy theory in this year's European talbot. Mentored by Gijs Heuts and Ishan Levy, we learned about the recent advances of this field.

Roughly, the following happened on the main mathematical sessions of each day:

- Day 1: Overview of higher categorical language (spectra, presentability, symmetric monoidality, operads).
- Day 2: Most important players in chromatic homotopy theory ($K(n), E_n$, BP and their multiplicative structures, monochromatic layers, descendability, ambidexterity).
- Day 3: Power operations (overview, Dyer-Lashof operations, $\mathbb{E}_3$-MU-structure on BP).
- Day 4: Synthetic methods (filtered spectra, spectral sequences, synthetic spectra and their applications to algebraicity and obstruction theory to multiplicative structures).
- Day 5: Multiplicative structures on monochromatic layers ($K(1)$-local power operations, chromatic Nullstellensatz and its applications).

Besides that, there were also various social events (canoe/kayak, independence day American bbq, karaoke, ...) as well as mathematical evening sessions. This includes an improvised crash course on complex-oriented cohomology theories, two question sessions and a chili session!

Remark 0.1. My notation and language is not always consistent with the speakers' choices. I also occassionally added some parts which were not included in the actual talks; such parts will always be indicated by a star like Lemma*.

Acknowledgements. We thank Gijs Heuts and Ishan Levy for being wonderful mentors leading us through the entire week.

Figure 1: Ishan answering questions at midnight.

We also thank the organizers Daniel Bermudez, Marie-Camille Delarue, Joao Fernandes, Hyeonhee Jin, Filippos Sytilidis and Maxime Wybouw for all their effort making this event possible! Of course, we also thank all the speakers for preparing and giving nice talks throughout the week. I'd also like to simply thank all participants all of whom I enjoyed interacting with.

1 Overview (Ishan Levy)

This workshop is about higher algebra and chromatic homotopy theory. What is it even about? TALK 1 30.06.2025

1.1 Higher Algebra

In usual algebra you study algebraic structures in **Set** like **Ring** or **Ab**. In higher algebra **Set** is replaced by the (∞-)category of homotopy types or also called anima these days. But why?

Homotopy types show up quite a lot:

- geometric topology (e.g. surgery theory),
- algebraic invariants of varieties or arithmetic objects,
- natural from the viewpoint of category theory.

What is higher algebra? In usual linear algebra you work in abelian categories while in higher algebra we work in stable ∞-categories. In linear algebra one cares about short exact sequences while in higher algebra the analog is fiber sequences. There is an universal abelian category which acts on others (admitting colimits), namely **Ab** and similarly in the higher algebra story it is **Sp**. There is an embeddding $\mathbf{Ab} \hookrightarrow \mathbf{Sp}$ and both are symmetric monoidal with unit $\mathbb{Z}$ resp. $\mathbb{S}$.

Taking maps we get $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$ and $\mathrm{Map}_{\mathbf{Sp}}(\mathbb{S}, \mathbb{S}) \simeq \mathrm{colim}_n\, \Omega^n S^n$. Note that $\mathbb{Z}$ is the initial associative/commutative ring while commutativity is more subtle in higher algebra. There are operads parametrizing multiplications, particularly there is a family of operads $\mathbb{E}_1, \mathbb{E}_2, \cdots, \mathbb{E}_\infty$ starting from the associative operad to the (most) commutative operad. There are also some others like the Lie operad **Lie**. But you really need a lot of data to realize these things. Working with operads is where a lot of the technicalities in higher algebra comes in.

Encoding algebras and algebra maps requires a lot of data. Here are some methods:

- Constructions: Thom spectra, K-theory, ...
- Deformation theory: Trying to build nonlinear objects like $\mathbb{E}_1$-algebras using linear data.
- Obstruction theory: Say you want to understand $\mathbb{E}_n$-algebras. The key thing to understand is free algebras, e.g. by computing homotopy groups of these. By the Yoneda lemma this is essentially about understanding (power) operations. Algebras are built under colimits from free ones. One example is to try taking pushouts

$$\begin{tikzcd}
\mathrm{Free}\{x\} \arrow[r, "r"] \arrow[d, "0"'] & R \arrow[d] \arrow[ddr, bend left] & \\
\mathbb{1} \arrow[r] \arrow[drr, bend right] & R' \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed] & \\
& & S
\end{tikzcd}$$

  and maps out of them. So the top arrow picks out an element $r \in R$ and the existence of a map $R' \to S$ is obstructed by x becomes nullhomotopic in S.
- Descent: If $f : R \to S$ is faithfully flat, then $\mathbf{Mod}_R \simeq \lim_{[n]\in\Delta} \mathbf{Mod}(S^{\otimes_R n})$. In higher algebra descent is effective way beyond faithfully flatness! For example one can try to understand descent along $\mathbb{S} \to \mathrm{MU}$ resp. $\mathbb{S} \to \mathbb{F}_p$ which leads to the ANSS resp. ASS.

In higher algebra one often tries to study some object $X \in \mathscr{C}$ by finding a filtration on it. A *filtration* on X is a functor $\mathbb{Z} \to \mathscr{C}$, perhaps written as

$$\cdots \longrightarrow X_i \longrightarrow X_{i-1} \longrightarrow \cdots$$

with $\operatorname{colim}_i X_i \simeq X$. Here, one should particularly study the associated graded gr X_i/X_{i+1}.

Example 1.1. Take the filtration $(\tau_{\geq n} X)_n$, then its associated graded is $\pi_\bullet X$.

A principle is that all spectral sequences come from filtered objects. In particular, one can apply techniques from higher algebra to filtered objects!

1.2 Chromatic Homotopy Theory

What does an abelian group look like? One of the first things you learn is the structure theorem of finitely generated abelian groups which leads to a primary decomposition of abelian groups. So what does a spectrum look like?

Classically, we could consider $\mathbb{Z} \to \mathbb{Q}$ but this loses information which is captured in the cokernel

$$\mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \cong \textstyle\bigoplus_p \operatorname{colim}_k \mathbb{Z}/p^k.$$

In higher algebra

$$\begin{tikzcd} \mathbb{S} \arrow[r] & \mathbb{S}[p^{-1} : p \text{ prime}] \arrow[r] \arrow[d, dash, "\simeq"'] & \mathbb{Q}/\mathbb{S} \simeq \bigoplus_p \operatorname{colim}_k \mathbb{S}/p^k \\ & \mathbb{Q} & \end{tikzcd}$$

where we essentially use Serre's finiteness theorem for the middle equivalence. In classical algebra we would already be done but in higher algebra $\pi_\bullet \mathbb{S}/p^k$ has an element $v_1^{k_1}$ which generates $\pi_\bullet$/nilpotent. This leads to

$$\Sigma^? \mathbb{S}/p^{k_0} \xrightarrow{v_1^{k_1}} \mathbb{S}/p^{k_0} \longrightarrow \mathbb{S}/(p_0^{k_0}, v_1^{k_1})$$

and $(\pi_\bullet \mathbb{S}/p^{k_0})[v_1^{-1}]$ which is completely understood: It is a local ring of Krull dimension 0, i.e. every element is a unit or nilpotent. We are not done: There is also $v_2^{k_2} \in \pi_\bullet \mathbb{S}/(p^{k_0}, v_1^{k_1})$. If we keep going, we obtain a type n complex

$$X_n = \mathbb{S}/(p^{k_0}, v_1^{k_1}, \cdots, v_{n-1}^{k_{n-1}})$$

and $X_n[v_n^{k_n - 1}]$ is a so-called *telescope of height n*.

This was not very canonical but here is a functor one could write done:

$$\mathbf{Sp} \to \mathbf{Sp}_{T(n)},\ X \mapsto \lim_{k_0, \cdots, k_{n-1}} \left(X \otimes \mathbb{S}/(p^{k_0}, \cdots, v_{n-1}^{k_{n-1}}) \right) \left[(v_n^{k_n})^{-1} \right].$$

This is now 'more canonical', it is for example symmetric monoidal. These gadgets essentially capture the v_n-periodic information.

Why does this happen? Consider

$$\mathbb{S} \longrightarrow \mathrm{MU} \rightrightarrows \mathrm{MU} \otimes \mathrm{MU} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots$$

which induces a spectral sequence $H^s(\mathcal{M}_{\mathrm{fg}};\omega^{\otimes t}) \Rightarrow \pi_{2t-s}\mathbb{S}$, the Adams-Novikov spectral sequence (ANSS). For example, one now gets

$$v_n \in H^0\left(\mathcal{M}_{\mathrm{fg}}, \omega^{\otimes(p^n-1)}/(v_0,\cdots,v_{n-1})\right)$$

for which one needs to obtain some understanding of the classification of formal groups.

One obtains[1]

$$L_{K(n)}X \simeq \lim_{\Delta} L_{T(n)}(X \otimes \mathrm{MU}^{\otimes \bullet+1}).$$

The point is that the left side is a more computable version of $T(n)$-localization. One reason for this is that it can be understood in terms of group cohomology: namely we have $L_{K(n)}X \simeq (X\widehat{\otimes}E_n)^{h\mathbb{G}_n}$ where one completely understands the homotopy groups

$$\pi_\bullet E_n \cong \mathbb{W}(F_{p^n})[\![v_1,\cdots,v_{n-1}]\!][\beta^{\pm 1}]$$

and in practice $(-)^{h\mathbb{G}_n}$ means that there is some spectral sequence starting in group cohomology which one can then try to run.

The localizations L_n, L_n^f glue together info of $L_{K(i)}$ resp. $L_{T(i)}$ for $i \leq n$. You can recover the sphere spectrum from such local information!

Theorem 1.2 (Chromatic Convergence)**.** There is an equivalence $\mathbb{S}_{(p)} \simeq \lim_n L_n\mathbb{S}$.

So L_n and L_n^f in principle see everything.[2] The finiteness in L_n^f is that it is the universal functor yielding $\mathbb{S} \mathbin{/\mkern-6mu/} X$ for some type $n+1$ (finite!) complex X.

There are many cool properties of the *monochromatic categories* $L_{T(n)}\mathbf{Sp}$ resp. $L_{K(n)}\mathbf{Sp}$.

- Ambidexterity: It feels a lot like working in characteristic 0. Let V be a rational vector space with a G action, then the norm map $V_G \to V^G$, $x \mapsto \sum_g gx$ is an isomorphism.

  The same thing happens in $L_{T(n)}\mathbf{Sp}$ or $L_{K(n)}\mathbf{Sp}$. For a π-finite group G acting on X there is a norm map $X_{hG} \to X^{hG}$ which is an equivalence.

- The $K(n)$-local $\mathbb{E}_\infty$-rings are really nice. For example, a free $K(n)$-local $\mathbb{E}_\infty$-E_n-algebra is at the level of $\pi_\bullet$ a completed polynomial algebra over $\pi_\bullet E_n$.

- Bousfield-Kuhn functors: Usually, you cannot recover a spectrum from its underlying space. By you can recover its $T(n)$-localization!

$$\begin{tikzcd} \mathbf{Sp} \arrow[rr, "L_{T(n)}"] \arrow[dr, "\Omega^\infty"'] & & \mathbf{Sp}_{T(n)} \\ & \mathcal{S} \arrow[ur, "\Phi_n"'] & \end{tikzcd}$$

  where Φ_n is the *Bousfield-Kuhn functor*.

2 Spectra and Stabilization (Yuqin Kewang)

2.1 Stable ∞-Categories

TALK 2
30.06.2025

This is a better notion than that of triangulated categories.

[1] Essentially, the reason is that $L_{K(n)}$ and $L_{T(n)}$ agree in $\mathbf{Mod}_{\mathrm{MU}}$ and so it's a question about showing that this descent thing of $\mathbb{S} \to \mathrm{MU}$ converges in the $K(n)$-local category.

[2] In fact, L_n^f sees strictly more than L_n.

Definition 2.1. Let $\mathscr{C}$ be an ∞-category.

(i) The ∞-category $\mathscr{C}$ is **pointed** if $\mathscr{C}$ has a zero object $0 \in \mathscr{C}$,

(ii) Let $\mathscr{C}$ be a pointed ∞-category. A **triangle** in $\mathscr{C}$ is a diagram $[1] \times [1] \to \mathscr{C}$ of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

(iii) A triangle is a **fiber sequence** resp. **cofiber sequence** if it is a pullback resp. pushout square of the above form.

Example 2.2. Let $\mathscr{C} = \mathcal{S}_*$, so $0 = *$. For $X \in \mathcal{S}_*$ we get **loop space** and **suspension**

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

so we have a fiber sequence $\Omega X \to 0 \to X$ and a cofiber sequence $X \to 0 \to \Sigma X$.

More generally, this can be performed in any pointed ∞-category.

Fact 2.3. There is an adjunction $\Sigma \dashv \Omega$.

Definition 2.4. An ∞-category $\mathscr{C}$ is **stable** if

(i) the category $\mathscr{C}$ is pointed,

(ii) every morphism in $\mathscr{C}$ has a fiber and a cofiber,

(iii) every triangle in $\mathscr{C}$ is a pullback if and only if it is a pushout.

Lemma 2.5. If $\mathscr{C}$ is stable, then $\Sigma \dashv \Omega$ is an equivalence of ∞-categories.

Proof. Consider the pullback square

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd}$$

so stability implies $\Sigma\Omega X \simeq X$. Similarly, $X \simeq \Omega\Sigma X$. □

Fact 2.6. Let $\mathscr{C}$ be a stable ∞-category. Then, $h\mathscr{C}$ has the structure of a triangulated category.

Note that stability is a property while triangulation is structure. In particular, there are potentially many possible triangulated structures that one can put on a 1-category.

What are examples of stable ∞-categories? One possibility is to consider *stabilizations* to construct stable ∞-categories.

Definition 2.7. Let $\mathscr{C}$ be an ∞-category which admits finite limits. Then, $\mathscr{C}_*$ also admits finite limits and is pointed. Its **stabilization** is

$$\mathbf{Sp}(\mathscr{C}) = \lim \left(\cdots \xrightarrow{\Omega} \mathscr{C}_* \xrightarrow{\Omega} \mathscr{C}_* \right) \in \widehat{\mathbf{Cat}}_\infty.$$

Fact 2.8. The stabilization $\mathbf{Sp}(\mathscr{C})$ is a stable ∞-category.

2.2 Spectra

We can finally define spectra.

Definition 2.9. The ∞-category

$$\mathbf{Sp} = \mathbf{Sp}(\mathcal{S}_*) = \lim\left(\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*\right) \in \widehat{\mathbf{Cat}}_\infty$$

is called **∞-category of spectra**.

Concretely, an object consists of $\{(X_n)_{n\geq 0}, \delta_n : X_n \xrightarrow{\simeq} \Omega X_{n+1}\} = X$ (and additional coherence data) and similarly,

$$\mathrm{Map}_{\mathbf{Sp}}(X,Y) \simeq \lim\left(\cdots \xrightarrow{\Omega} \mathrm{Map}_{\mathcal{S}_*}(X_n, Y_n) \xrightarrow{\Omega} \mathrm{Map}_{\mathcal{S}_*}(X_{n-1}, Y_{n-1}) \xrightarrow{\Omega} \cdots\right),$$

i.e. a map $f : X \to Y$ consists of the data $f_n : X_n \to Y_n$ together with homotopies realizing the commutativity of the diagram

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "\delta_n"'] & Y_n \arrow[d, "\delta_n'"] \\ \Omega X_{n+1} \arrow[r, "\Omega f_{n+1}"'] & \Omega Y_{n+1}. \end{tikzcd}$$

Example 2.10. For $A \in \mathbf{Ab}$ its **Eilenberg-MacLane spectrum** is given by $K(A,i)_{i\geq 1}$ with $K(A,i) \xrightarrow{\simeq} \Omega K(A, i+1)$.

Fact 2.11. The ∞-category $\mathbf{Sp}$ has all limits and colimits, in fact $\mathbf{Sp}$ is presentable. Limits and filtered colimits can be computed levelwise.[3]

Note $(\Omega X)_n = \Omega X_n \simeq X_{n-1}$, so one suggestive notation is $\Omega X = X[-1]$. Similarly, $\Sigma X = X[1]$.

Construction 2.12. There is a natural functor $\Omega^\infty : \mathbf{Sp} \to \mathcal{S}_*$, $X \mapsto X_0$ which commutes with all small limits. So it has a left adjoint $\Sigma^\infty : \mathcal{S}_* \to \mathbf{Sp}$, $X \mapsto (\mathrm{colim}_i\, \Omega^i \Sigma^{i+n} X)_n$ with structure map

$$\begin{tikzcd} \mathrm{colim}_i\, \Omega^i \Sigma^{i+n} X \arrow[r, "\simeq"] \arrow[dr, "\simeq"'] & \Omega\, \mathrm{colim}_i\, \Omega^i \Sigma^{i+n+1} X \arrow[d, dash, "\simeq"] \\ & \mathrm{colim}_i\, \Omega^{i+1} \Sigma^{i+n+1} X \end{tikzcd}$$

where the right equivalence comes from finite limits commuting with filtered colimtis and the diagonal equivalence is stability. The image of Σ^∞ consists of the **suspension spectra**.

Example 2.13. The **sphere spectrum** is $\mathbb{S} = \Sigma^\infty S^0$.

2.3 Spectra is Compactly Generated

Consider the ∞-category of pointed finite spaces $\mathcal{S}_*^{\mathrm{fin}}$.

Definition 2.14. The ∞-category of **finite spectra** is

$$\mathbf{Sp}^{\mathrm{fin}} = \mathrm{colim}\left(\mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \cdots\right) \in \mathbf{Cat}_\infty.$$

So we can view every finite spectrum as $\Sigma^{-h}\Sigma^\infty K$ for some finite space K.

[3] For limits this is always true for stabilizations. For filtered colimits this is due to filtered colimits commuting with finite limits in $\mathcal{S}_*$.

Proposition 2.15. There is an equivalence $\mathbf{Sp} \simeq \mathrm{Ind}(\mathbf{Sp}^{\mathrm{fin}})$.

Here,

$$\mathrm{Ind}(\mathscr{C}) = \mathbf{PSh}^{\mathrm{filtered}}(\mathscr{C}) \subseteq \mathbf{PSh}(\mathscr{C}) = \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathcal{S})$$

is the free ∞-category generated freely under filtered colimits. Concretely, an object in $\mathrm{Ind}(\mathscr{C})$ might suggestively be written as $\mathrm{colim}_{i\in I} F_i$ for some $F : I \to \mathscr{C}$ with some filtered I and

$$\mathrm{Map}_{\mathrm{Ind}(\mathscr{C})}\left(\operatorname*{colim}_I Fi, \operatorname*{colim}_J Gj\right) \simeq \lim_I \operatorname*{colim}_J \mathrm{Map}_{\mathscr{C}}(Fi, Gj).$$

Proof of 2.15. First, $\mathbf{Sp}^{\mathrm{fin}} \subseteq \mathbf{Sp}$ consists of the compact objects. It's a formal consequence that $\mathrm{Ind}(\mathbf{Sp}^{\mathrm{fin}}) \to \mathbf{Sp}$ is fully faithful. So we need to show essential surjectivity. Let $X \in \mathbf{Sp}$ which we wish to be written as $X \simeq \mathrm{colim}_i Y_i$ with $Y_i \in \mathbf{Sp}^{\mathrm{fin}}$. This can be accomplished: An Yoneda argument yields $X \simeq \mathrm{colim}_n \Sigma^{-n}\Sigma^\infty X_n$. □

2.4 Connective Spectra is Grouplike $\mathbb{E}_\infty$-Spaces

Being *connective* means $\pi_i = 0$ for $i < 0$.

Definition 2.16. The 1-category of finite pointed sets is $\mathbf{Fin}_*$, i.e. objects are finite pointed sets and morphisms are basepoint-preserving maps.

Definition 2.17. A **commutative monoid in spaces** (aka **$\mathbb{E}_\infty$-space**) is a functor $\underline{M} : \mathbf{Fin}_* \to \mathcal{S}$ satisfying the *Segal condition*, i.e. that the map

$$(\chi_{i,*})_i : \underline{M}(\langle n\rangle) \to \prod_{i=1}^n \underline{M}(\langle 1\rangle)$$

is an equivalence. Here,

$$\chi_i : \langle n\rangle \to \langle 1\rangle,\ j \mapsto \begin{cases} 1 & j = i, \\ * & j \neq i. \end{cases}$$

Remark 2.18. Let $\underline{M} : \mathbf{Fin}_* \to \mathcal{S}$ be an $\mathbb{E}_\infty$-space. Then, $M = \underline{M}(\langle 1\rangle) \in \mathcal{S}$ is called **underlying space** of $\underline{M}$. We obtain a multiplication

$$M \times M \xleftarrow{\simeq} \underline{M}(\langle 2\rangle) \xrightarrow{m_*} M(\langle 1\rangle) = M$$

with $m : \langle 2\rangle \to \langle 1\rangle,\ 1,2 \mapsto 1$. One may check that this gives a monoid structure which is unital, associative and commutative up to higher coherences.

Definition 2.19. An $\mathbb{E}_\infty$-space $\underline{M}$ is **group-like** if the commutative monoid structure on $\pi_0 M$ is a group. We denote the respective category by $\mathbf{CMon}^{\mathrm{gp}}(\mathcal{S})$.

Theorem 2.20 (Recognition principle)**.** There is an equivalence of ∞-categories

$$B^\infty : \mathbf{CMon}^{\mathrm{gp}}(\mathcal{S}) \rightleftarrows \mathbf{Sp}_{\geq 0} : \Omega^\infty.$$

We omit the proof but give a construction of the functors.

(i) Delooping $B^\infty : \mathbf{CMon}^{\mathrm{gp}}(\mathcal{S}) \to \mathbf{Sp}_{\geq 0}$: It's a fact that $\mathbf{CMon}(\mathcal{S})$ has a zero object and that limits and colimits are computed levelwise. In particular,

$$\Omega\underline{M} : \mathbf{Fin}_* \to \mathcal{S},\ \langle n\rangle \mapsto \Omega\underline{M}(\langle n\rangle),$$

so $\pi_i(\Omega \underline{M}) \cong \pi_{i+1}(\underline{M})$. This suggests

$$\Omega : \mathbf{CMon}^{\mathrm{gp}}(\mathcal{S})_{\geq 1} \xrightarrow{\simeq} \mathbf{CMon}^{\mathrm{gp}}(\mathcal{S}).$$

The inverse of this is B which corresponds to Σ on $\mathbf{CMon}(\mathcal{S})$. In fact,

$$B\underline{M} \simeq \operatorname*{colim}_{\Delta^{\mathrm{op}}} \left(\mathbf{Fin}_* \times \Delta^{\mathrm{op}} \to \mathbf{Fin}_* \xrightarrow{\underline{M}} \mathcal{S} \right) \simeq \operatorname{colim}\left(\cdots \underline{M} \times \underline{M} \rightrightarrows \underline{M} \to 0 \right) \in \mathbf{CMon}(\mathcal{S}).$$

Then, $B^\infty \underline{M} = \{(M, BM, B^2M, \cdots),\ B^nM \xrightarrow{\simeq} \Omega B^{n+1}M\} \in \mathbf{Sp}$ using that Ω is inverse to B. Note that $B^\infty \underline{M}$ is connective since $\pi_{-n}(B^\infty \underline{M}) \cong \pi_0(B^nM) \cong 0$.

(ii) One obtains

$$\Omega^\infty : \mathbf{Sp}_{\geq 0} \to \mathbf{CMon}^{\mathrm{gp}}(\mathcal{S}),\ E \mapsto \left(\underline{\Omega^\infty E} : \langle n \rangle \mapsto \Omega^\infty(\Sigma^\infty \langle n \rangle \otimes E) \right).$$

Note that $\Omega^\infty E$ is group-like because $\pi_0(\Omega^\infty E) = \pi_0 E$ is a group.

2.5 Examples

Let's discuss some examples.

Example 2.21. There is connective complex K-theory ku. Consider

$$\underline{\mathbf{Vect}} : \mathbf{Fin}_* \to \mathcal{S},\ \langle n \rangle \mapsto \{(V_1, \cdots, V_n) \in \mathrm{Gr}^n : V_i \perp V_j \text{ for } i \neq j\},$$

and a map $f : \langle m \rangle \to \langle n \rangle$ is sent to $\underline{\mathbf{Vect}}\langle m \rangle \to \underline{\mathbf{Vect}}\langle n \rangle$, $(V_1, \cdots, V_m) \mapsto \left(\bigoplus_{i \in f^{-1}(j)} V_i \right)_{1 \leq j \leq n}$. The Segal condition for

$$\underline{\mathbf{Vect}}(\langle n \rangle) \to \prod_{i=1}^{n} \underline{\mathbf{Vect}}(\langle 1 \rangle),\ (V_1, \cdots, V_n) \mapsto (V_1, \cdots, V_n)$$

is an equivalence because Gram-Schmidt is an homotopy equivalence. Note $\pi_0(\underline{\mathbf{Vect}}) \cong \mathbb{N}$, so we are not grouplike yet, but we can group complete to get ku $= B^\infty \underline{\mathbf{Vect}}^{\mathrm{gp}}$. In particular, $\Omega^\infty \mathrm{ku} \simeq \Omega^\infty B^\infty \underline{\mathbf{Vect}}^{\mathrm{gp}} \simeq \mathrm{BU} \times \mathbb{Z}$.

Example 2.22. For each n there is a map

$$\mathrm{BO}_n \to \mathcal{S}_* \xrightarrow{\Sigma^{-n}\Sigma^\infty} \mathbf{Sp},\ * \mapsto S^n \mapsto \mathbb{S}.$$

There is a commutative diagram

$$\begin{tikzcd} \mathrm{BO}_n \arrow[r] \arrow[d] & \mathbf{Sp} \\ \mathrm{BO}_{n+1} \arrow[ur] & \end{tikzcd}$$

which yields the **J-homomorphism** $J : \mathrm{BO} \to \mathbf{Sp}$. Then,

$$\mathrm{MO} = \operatorname{colim}(J : \mathrm{BO} \to \mathbf{Sp}) \quad \text{and} \quad \mathrm{MU} = \operatorname{colim}\left(\mathrm{BU} \to \mathrm{BO} \xrightarrow{J} \mathbf{Sp} \right)$$

are the **real/complex bordism spectra**.

3 Presentable Stable ∞-Categories and (Symmetric) Monoidal Structures (Julie Bannwart)

TALK 3
30.06.2025

3.1 Symmetric Monoidal ∞-Categories and Their Algebra Objects

A *symmetric monoidal* 1*-category* consists of the data $(\mathscr{C}, \otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}, \mathbb{1}_{\mathscr{C}} \in \mathscr{C})$ along with isomorphisms witnessing associativity, commutativity, unitality and compatibilities.

Definition 3.1. A **symmetric monoidal ∞-category** is a functor $\underline{\mathscr{C}} : \mathbf{Fin}_* \to \mathbf{Cat}_\infty$ such that

$$(\chi_i)_i : \underline{\mathscr{C}}(\langle n \rangle) \to \prod_{i=1}^{n} \underline{\mathscr{C}}(\langle 1 \rangle)$$

is an equivalence.

By unstraightening, these correspond to coCartesian fibrations $\mathscr{C}^{\otimes} \to \mathbf{Fin}_*$ together with the Segal condition. The coCartesian fibration condition is precisely to ensure some sort of functoriality on fibers of this, so for every $\langle n \rangle \to \langle m \rangle$ we get a functor $\mathscr{C}^{\otimes}_{\langle n \rangle} \to \mathscr{C}^{\otimes}_{\langle m \rangle}$.

Think that the underlying ∞-category of $\underline{\mathscr{C}}$ is $\underline{\mathscr{C}}\langle 1 \rangle \simeq \mathscr{C}^{\otimes}_{\langle 1 \rangle}$.

Proposition 3.2.

(i) Let $\mathscr{C}$ be a symmetric monoidal 1-category. Then, there exists a symmetric monoidal ∞-category $\underline{N\mathscr{C}}$ such that $\underline{N\mathscr{C}}\langle 1 \rangle \simeq N\mathscr{C}$.

(ii) Let $\underline{\mathscr{C}}$ be a symmetric monoidal ∞-category. Then, $h(\underline{\mathscr{C}}\langle 1 \rangle)$ is a symmetric monoidal 1-category.

Proposition 3.3. Let $\mathscr{C}$ be an ∞-category with finite (co-)products. Then, $\mathscr{C}$ has a symmetric monoidal structure with tensor product $\times$ (resp. $\amalg$).

We write $\mathscr{C}^{\times}$ or $\mathscr{C}^{\amalg}$.

Classically, a *lax symmetric monoidal functor* consists of a functor $F : \mathscr{C} \to \mathscr{D}$ together with maps $\lambda_{c,c'} : Fc \otimes Fc' \to F(c \otimes c')$. It is a *(strong) symmetric monoidal functor* if $\lambda_{c,c'}$ are also equivalences.

Definition 3.4. Let $\mathscr{C}^{\otimes}, \mathscr{D}^{\otimes} \to \mathbf{Fin}_*$. A **lax (resp. strong) symmetric monoidal functor** is a diagram

$$\begin{tikzcd} \mathscr{C}^{\otimes} \arrow[rr, "F"] \arrow[dr] & & \mathscr{D}^{\otimes} \arrow[dl] \\ & \mathbf{Fin}_* & \end{tikzcd}$$

such that F carries coCartesian lifts of inert[4] (resp. all) edges to coCartesian edges.

Note that a symmetric monoidal functor corresponds to a natural transformation $\underline{\mathscr{C}} \Rightarrow \underline{\mathscr{D}}$ by straightening-unstraightening but for lax ones one requires some lax natural transformation notion.

Example 3.5. A symmetric monoidal functor between (co-)Cartesian symmetric monoidal structures are exactly those preserving finite (co-)products.

This example is interesting because in general it's hard to construct a (lax) symmetric monoidal functor but in this case it's considerably easier.

Classically, an algebra object $E \in (\mathscr{C}, \otimes, \mathbb{1}_{\mathscr{C}})$ consists of a map $\mathbb{1}_{\mathscr{C}} \to E$ and a map $E \otimes E \to E$ satisfying certain axioms.

[4] An *inert edge* in $\mathbf{Fin}_*$ is an edge $f : \langle n \rangle \to \langle m \rangle$ such that $f^{-1}(i)$ is a singleton for all $1 \leq i \leq m$.

Definition 3.6. A **commutative algebra** in $p : \mathscr{C}^{\otimes} \to \mathbf{Fin}_*$ is a section of p which sends inert edges to coCartesian ones.

In other words, it's a lax symmetric monoidal functor $\mathbf{Fin}_* \to \mathscr{C}^{\otimes}$.

Definition 3.7. An **associative algebra** in $\mathscr{C}^{\otimes} \to \mathbf{Fin}_*$ is a commutative diagram

$$\begin{tikzcd} \Delta^{\mathrm{op}} \arrow[rr] \arrow[dr] & & \mathscr{C}^{\otimes} \arrow[dl] \\ & \mathbf{Fin}_* & \end{tikzcd}$$

mapping *convex*[5] edges to coCartesian edges.

Example 3.8. A commutative algebra in $\mathcal{S}^{\times}$ is exactly an $\mathbb{E}_\infty$-space as in Talk 2 (2.17).

So we introduce the notation $\mathbf{CAlg}(\mathscr{C}^{\otimes}) \subseteq \mathrm{Fun}(\mathbf{Fin}_*, \mathscr{C}^{\otimes})$ and $\mathbf{Alg}(\mathscr{C}^{\otimes}) \subseteq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathscr{C}^{\otimes})$.

Remark 3.9. For 1-categories there is an inclusion $\mathbf{CAlg}(\mathscr{C}) \subseteq \mathbf{Alg}(\mathscr{C})$ but this is false for ∞-categories. We are really dealing with extra data in this higher setup.

Proposition 3.10. Let $F : \mathscr{C}^{\otimes} \to \mathscr{D}^{\otimes}$ be a lax symmetric monoidal functor. Then, it induces a functor $\mathbf{Alg}(\mathscr{C}^{\otimes}) \to \mathbf{Alg}(\mathscr{D}^{\otimes})$ and similarly for $\mathbf{CAlg}$.

3.2 Spectra and $\mathbf{Pr}^L$

Roughly, a presentable ∞-category is one which is generated by a small amount of data.

Definition 3.11.

(i) An ∞-category $\mathscr{C}$ is **presentable** if it admits small colimits and is accessible. Being **accessible** means that it is generated under κ-filtered colimits by a small subcategory.

(ii) We denote by $\mathbf{Pr}^L \subseteq \widehat{\mathbf{Cat}}_\infty$ is the subcategory consisting of presentable ∞-categories and left adjoint functors.

(iii) We write $\mathbf{Pr}^L_{\mathrm{st}} \subseteq \mathbf{Pr}^L$ for the full subcategory of stable presentable ∞-categories.

Exercise 3.12. If $\mathscr{C}$ is presentable and $\mathscr{C}^{\mathrm{op}}$ is presentable, then $\mathscr{C}$ is a poset.

Proposition 3.13. There exists a symmetric monoidal structure on $\mathbf{Pr}^L$ such that:

(i) The inclusion $\mathbf{Pr}^{L,\otimes} \hookrightarrow \widehat{\mathbf{Cat}}_\infty^{\times}$ is lax symmetric monoidal. In other words, there is a map $\mathscr{C} \times \mathscr{D} \to \mathscr{C} \otimes_{\mathbf{Pr}^L} \mathscr{D}$ in $\widehat{\mathbf{Cat}}_\infty$.

(ii) The functor $\mathscr{C} \times \mathscr{D} \to \mathscr{C} \otimes_{\mathbf{Pr}^L} \mathscr{D}$ is initial amongst functors preserving colimits in both variables.

(iii) There is an equivalence $\mathscr{C} \otimes_{\mathbf{Pr}^L} \mathscr{D} \simeq \mathrm{RFun}(\mathscr{C}^{\mathrm{op}}, \mathscr{D})$.

(iv) The functor $- \otimes_{\mathbf{Pr}^L} -$ preserves colimits in both variables.

(v) The ∞-category $\mathbf{Pr}^L_{\mathrm{st}}$ has an induced symmetric monoidal structure.

Note moreover $\mathrm{RFun}(\mathscr{C}^{\mathrm{op}}, \mathscr{D}_*) \simeq \mathrm{RFun}(\mathscr{C}^{\mathrm{op}}, \mathscr{D})_*$.

Fact 3.14.

[5] This means the edge is injective and the image is an interval.

(i) $\mathcal{S} \in \mathbf{Pr}^L$,

(ii) $\mathscr{C} \otimes \mathcal{S}_* \simeq \mathscr{C}_*$,

(iii) $\mathscr{C} \otimes \mathbf{Sp} \simeq \mathrm{RFun}(\mathscr{C}^{\mathrm{op}}, \mathbf{Sp}) \simeq \lim \mathrm{RFun}(\mathscr{C}^{\mathrm{op}}, \mathcal{S}_*) \simeq \lim \mathscr{C}_* \simeq \mathbf{Sp}(\mathscr{C})$.

From (iii) we get $\mathbf{Sp} \otimes \mathbf{Sp} \simeq \mathbf{Sp}$, which suggests that $\mathbf{Sp} \in \mathbf{Pr}^L_{\mathrm{st}}$ is the unit.

Theorem 3.15. There exists a unique symmetric monoidal structure on $\mathbf{Sp}$ such that the unit is $\mathbb{S}$ and $- \otimes -$ preserves colimits in both variables. Such symmetric monoidal structures are called **presentably symmetric monoidal**.

Proof Idea. For uniqueness let $E \in \mathbf{Sp}$ and write $E \simeq \mathrm{colim}_m \Sigma^{-m}\Sigma^\infty_+ X_m$. If $F \in \mathbf{Sp}$, then

$$E \otimes F \simeq \operatorname*{colim}_m \Sigma^m \Sigma^\infty_+ E_m \otimes F \simeq \operatorname*{colim}_m \Sigma^m \operatorname*{colim}_{X_m} \Sigma^\infty_+ * \otimes F \simeq \operatorname*{colim}_m \Sigma^m \operatorname*{colim}_{X_m} F.$$

We got rid of $\otimes$, hence we get a uniqueness statement.

For existence, the $\mathbf{Sp} \in \mathbf{Pr}^L_{\mathrm{st}}$ being the unit implies $\mathbf{Sp} \in \mathbf{CAlg}(\mathbf{Pr}^L_{\mathrm{st}})$. This can be used to prove existence. $\square$

3.3 Ring Spectra & Examples

Objects in $\mathbf{CAlg}(\mathbf{Sp}^{\otimes})$ resp. $\mathbf{Alg}(\mathbf{Sp}^{\otimes})$ are called **$\mathbb{E}_\infty$-rings** resp. **$\mathbb{E}_1$-rings**.

Example 3.16. Consider KU. We have to adjust the **Vect** which used direct sums to

$$\underline{\widetilde{\mathbf{Vect}}}_{\mathbb{C}} \in \mathbf{CAlg}(\mathbf{CMon}^{\mathrm{gp}}(\mathcal{S}^{\times}))$$

by further adopting tensor products. This yields ku $= B^\infty \underline{\widetilde{\mathbf{Vect}}}_{\mathbb{C}}^{\mathrm{gp}} \in \mathbf{CAlg}(\mathbf{Sp}^{\otimes})$. Moreover, $\Omega^\infty \mathrm{ku} \simeq \mathbb{Z} \times \mathrm{BU}$ and there is a map

$$\mathbb{C}P^1 \simeq S^2 \to \mathbb{Z} \times \mathrm{BU} \simeq \left(\coprod_m \mathrm{BU}_m\right)^{\mathrm{gp}}$$

which corresponds to $1 - \mathcal{O}(1)$. This is the *Bott element* $\beta \in \pi_2(\mathrm{ku})$. Inverting this leads to KU. It's a non-trivial argument to see that this is still $\mathbb{E}_\infty$ and it was answered in B.6.

Example 3.17. Recall $\mathrm{MU} \simeq \mathrm{colim}\left(\mathrm{BU} \to \mathrm{BO} \xrightarrow{J} \mathbf{Sp}\right)$. Generalizing slightly yields the Thom spectrum functor

$$\mathcal{S}_{/\operatorname{Pic} E} \to \mathbf{Sp},\ (F : X \to \operatorname{Pic} E) \mapsto \operatorname*{colim}_X F.$$

This has a symmetric monoidal structure with respect to Day convolution on $\mathcal{S}_{/\operatorname{Pic} E} \simeq \mathbf{PSh}(\operatorname{Pic} E)$ which is informally given by

$$(X \to \operatorname{Pic} E) \otimes (Y \to \operatorname{Pic} E) \simeq \left(X \times Y \to \operatorname{Pic} E \times \operatorname{Pic} E \xrightarrow{\mu} \operatorname{Pic} E\right).$$

You can check that J is symmetric monoidal and so one can check that this gives a commutative algebra object in $\mathcal{S}_{/\operatorname{Pic} E}$ and by symmetric monoidality it sends to a commutative algebra object $\mathrm{MU} \in \mathbf{CAlg}(\mathbf{Sp})$.

3.4 Modules

Classically, for $E \in \mathbf{Alg}(\mathscr{C}, \otimes)$ a **module** F over E additionally has the structure of a map $E \otimes F \to F$ satisfying unitality and associativity.

Definition 3.18. Let $A : \Delta^{\mathrm{op}} \to \mathscr{C}^{\otimes}$. A **left A-module** is a functor $F : \Delta^{\mathrm{op}} \times [1] \to \mathscr{C}^{\otimes}$ such that

- The composite $\Delta^{\mathrm{op}} \times [1] \to \mathbf{Fin}_*$ has $([m], 0) \mapsto \langle m+1 \rangle$ and $([m], 1) \mapsto \langle m \rangle$.
- The restriction $\Delta^{\mathrm{op}} \times \{1\} \to \mathscr{C}^{\otimes}$ is A.
- The edges $F(\mathrm{id}_{[m]}, 0 \to 1)$ and $F(\alpha^{\mathrm{op}}, \mathrm{id}_0)$ for convex α are coCartesian edges.

Here is some intuition. The underlying module is $F([0], 0) = M \in \mathscr{C}^{\otimes}$. Using coCartesian lift arguments, one can show that $F([n], 0)$ corresponds to $(A, \cdots, A, M) \in \mathscr{C}^{n+1}$. On the other hand, $F([n], 1)$ corresponds to $(A, \cdots, A) \in \mathscr{C}^{n}$. So given $m : \langle 2 \rangle \to \langle 1 \rangle$ we obtain a map

$$(A, M) = F([1], 0) \to F([0], 0) = M$$

which by factoring over coCartesian lifts corresponds to a map $A \otimes M \to M$.

4 Operads & Koszul Duality (Markus Zetto)

4.1 Operads via Symmetric Sequences

TALK 4
30.06.2025

Fix $\mathscr{V} \in \mathbf{CAlg}(\mathbf{Pr}^{L}_{\mathrm{st}})$, i.e. a (closed) symmetric monoidal presentable stable ∞-category. Consider the chain of adjunctions

$$\mathbf{CAlg}(\mathbf{Mod}_{\mathscr{V}}(\mathbf{Cat}^{\mathrm{colim}})) \underset{\longrightarrow}{\xleftarrow{-\otimes \mathscr{V}}} \mathbf{CAlg}(\mathbf{Cat}^{\mathrm{colim}}) \underset{\longrightarrow}{\xleftarrow{\mathbf{PSh}^{\mathrm{colim}}}} \mathbf{CAlg}(\widehat{\mathbf{Cat}}) \underset{\longrightarrow}{\xleftarrow{\mathrm{Sym}}} \widehat{\mathbf{Cat}}$$

consisting of forgetful functors and the associated free functors. The composition of these left adjoints is essentially

$$\mathbf{Cat} \to \mathbf{CAlg}(\mathbf{Mod}_{\mathscr{V}}(\mathbf{Pr}^{L})), \ \mathscr{C} \mapsto \mathbf{PSh}(\mathrm{Sym}\,\mathscr{C}) \otimes \mathscr{V} \simeq \mathrm{Fun}\left(\coprod_{n \geq 0} (\mathscr{C}^{\times n}_{h\Sigma_n})^{\mathrm{op}}, \mathscr{V}\right).$$

Definition 4.1. A **$\mathscr{V}$-enriched operad** with space of colors $X \in \mathcal{S}$ is a monad on

$$\mathbf{PSh}(\mathrm{Sym}\,X) \otimes \mathscr{V} \in \mathbf{CAlg}(\mathbf{Mod}_{\mathscr{V}}(\mathbf{Pr}^{L})).$$

Explicitly, it is an algebra in the ∞-category

$$\begin{aligned} \mathrm{End}^{L,\otimes}_{\mathscr{V}}(\mathbf{PSh}(\mathrm{Sym}\,X) \otimes \mathscr{V}) &\simeq \mathrm{Fun}(X, \mathbf{PSh}(\mathrm{Sym}\,X) \otimes \mathscr{V}) \\ &\simeq \mathrm{Fun}\left(X \times \coprod_{n \geq 0} X^{\times n}_{h\Sigma_n}, \mathscr{V}\right) \\ &= \mathbf{SSeq}_X(\mathscr{V}). \end{aligned}$$

of **X-colored symmetric sequences** in $\mathscr{V}$.

Definition 4.2. We write $\mathbf{Op}(\mathscr{V}) = \mathbf{Alg}(\mathbf{SSeq}_X(\mathscr{V}))$ and $\mathbf{coOp}(\mathscr{V}) = \mathbf{coAlg}(\mathbf{SSeq}_X(\mathscr{V}))$.

So an $\mathscr{O} \in \mathbf{Op}(\mathscr{V})$ specifies a functor $\mathrm{Mul}_{\mathscr{O}} : X \times \coprod_{n \geq 0} X^{\times n}_{h\Sigma_n} \to \mathscr{V}$ and the algebra structure on $\mathscr{O}$ encodes identities and compositions on $\mathrm{Mul}_{\mathscr{O}}$.

Example 4.3. Consider $X = *$. Then,

$$\mathbf{SSeq}(\mathscr{V}) = \mathbf{SSeq}_*(\mathscr{V}) = \mathrm{Fun}\left(\coprod_{n\geq 0} B\Sigma_n, \mathscr{V}\right) = \mathrm{Fun}(B\Sigma, \mathscr{V}).$$

So we get Σ_n-action on $\mathscr{O}(n)$ for all $n \in \mathbb{N}$.

- Unit: It is given by $\mathbb{1}(n) = \begin{cases} \emptyset & n \neq 1, \\ 1_{\mathscr{V}} & n = 1. \end{cases}$
- Composition: $(\mathscr{O} \odot \mathscr{P})(n) = \coprod_{r\geq 0} \left(\mathscr{O}(r) \otimes \mathscr{P}^{\otimes r}(n)\right)_{h\Sigma_n}$.

An algebra consists of $1_{\mathscr{V}} \to \mathscr{O}(1)$ with

$$(\mathscr{O} \odot \mathscr{O})(n) = \coprod_{r\geq 0} (\mathscr{O}(r) \otimes \mathscr{O}^{\otimes r}(n))_{h\Sigma_n} \to \mathscr{O}(n)$$

where the Day convolution monoidal structure formula amounts to $\mathscr{O}^{\otimes r}(n) \simeq \mathrm{colim}_{n_1+\cdots+n_r=n} \mathscr{O}(n_i)$.

Example 4.4.

(i) The unit $\mathbb{1}$ is an operad. More generally, any category determines an operad,

(ii) $\mathbb{E}_\infty(n) = 1_{\mathscr{V}}$,

(iii) $\mathbb{E}_k(n) = \mathrm{Emb}(\{1, \cdots, n\}, \mathbb{R}^k)$ for $\mathscr{V} = \mathcal{S}$.

(iv) **Lie** for $\mathscr{V} = \mathbf{Vect}_k$ (and $\mathrm{char}\, k = 0$) freely generated by $[-,-] \in \mathbf{Lie}(2)$ with respect to relations, anti-symmetry and Jacobi. This doesn't really make sense in $\mathcal{S}$ which is why we want different enrichments.

Observation 4.5. If $X \in \mathbf{SSeq}(\mathscr{V})$ is concentrated in degree 0, then so is $\mathscr{O} \odot X$. Then, $\mathbf{SSeq}(\mathscr{V}) \circlearrowright \mathscr{V}$, or equivalently,

$$\otimes : \mathbf{SSeq}(\mathscr{V}) \to \mathrm{End}(\mathscr{V}),\ X \mapsto (\mathrm{Sym}_X : v \mapsto X \odot v[0])$$

Definition 4.6. We define $\mathbf{Alg}_{\mathscr{O}}(\mathscr{V}) = \mathbf{LMod}_{\mathrm{Sym}\,\mathscr{O}}(\mathscr{V})$ and $\mathbf{coAlg}^{\mathrm{nil,d.p}}_{\mathscr{O}}(\mathscr{V}) = \mathbf{coLMod}_{\mathrm{Sym}\,\mathscr{Q}}(\mathscr{V})$ for $\mathscr{Q} \in \mathbf{coOp}(\mathscr{V})$.[6]

Unravelled, an algebra is

$$(\mathscr{O} \odot \mathscr{V}[0])(0) \simeq \coprod_{r\geq 0} \left(\mathscr{O}(r) \otimes \mathscr{V}^{\otimes r}\right)_{h\Sigma_r} \to \mathscr{V},$$

so it makes sense that this is called algebra.

Remark 4.7. Actual coalgebras should have $A \to \prod_n (\mathscr{Q}(n) \otimes A^{\otimes n})^{h\Sigma_n}$.

4.2 Bar-Cobar Duality

Definition 4.8. Let $\mathscr{C}$ be a monoidal ∞-category $\mathscr{C}$ and $A \in \mathbf{Alg}^{\mathrm{aug}}(\mathscr{C}) = \mathbf{Alg}(\mathscr{C})_{/1_{\mathscr{C}}}$. We call $\mathrm{Bar}(A) \in \mathscr{C}$ the **bar construction** of A if for every $c \in \mathscr{C}$ we have

$$\mathrm{Map}_{\mathscr{C}}(\mathrm{Bar}\, A, c) \simeq \mathrm{Map}_{{}_A\mathbf{BiMod}_A(\mathscr{C})}(A, \rho c)$$

where $\rho : \mathscr{C} \simeq {}_1\mathbf{BiMod}_1(\mathscr{C}) \to {}_A\mathbf{BiMod}_A(\mathscr{C})$ restricts scalars along the augmentation.

[6] Here, d.p. stands for divided powers.

Proposition 4.9. If $\mathscr{C}$ admits geometric realizations, then $\mathrm{Bar}(A)$ exists and

$$\mathrm{Bar}(A) = 1 \otimes_A 1 \simeq \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}}(1 \otimes A^{\otimes n} \otimes 1)$$

Observation 4.10. Consider

$$1 \otimes_A 1 = \mathrm{Bar}(A) \to 1 \otimes_A A \otimes_A 1 \to 1 \otimes_A (1 \otimes_A 1) \otimes_A 1 \xrightarrow{\simeq} \mathrm{Bar}(A) \otimes \mathrm{Bar}(A),$$

so $\mathrm{Bar}(A)$ is a coalgebra with coaugmentation $1 \simeq 1 \otimes 1 \to 1 \otimes_A 1$. We expect to lift

$$\mathrm{Bar} : \mathbf{Alg}^{\mathrm{aug}}(\mathscr{C}) \to \mathbf{coAlg}^{\mathrm{aug}}(\mathscr{C})$$

and if $\mathscr{C}$ has totalizations we also get $\mathrm{Cobar} : \mathbf{coAlg}^{\mathrm{aug}}(\mathscr{C}) \to \mathbf{Alg}^{\mathrm{aug}}(\mathscr{C})$ which should will be a right adjoint.

Here is a proof idea.

Definition 4.11.

(i) A **pairing** of ∞-categories is a right fibration $\mathscr{M} \to \mathscr{C} \times \mathscr{D}$, i.e. a functor $\mathscr{C}^{\mathrm{op}} \times \mathscr{D}^{\mathrm{op}} \to \mathcal{S}$.

(ii) It is **left representable** if for every $c \in \mathscr{C}$ the category $\mathscr{M} \times_{\mathscr{C}} \{c\}$ has a final object, equivalently there is a factorization $\mathbb{D} : \mathscr{C}^{\mathrm{op}} \to \mathscr{D} \subseteq \mathbf{PSh}(\mathscr{D})$.

(iii) It is **right representable** if for every $d \in \mathscr{D}$ the category $\mathscr{M} \times_{\mathscr{D}} \{d\}$ has a final object and equivalently $\mathbb{D}' : \mathscr{D}^{\mathrm{op}} \to \mathscr{C} \subseteq \mathbf{PSh}(\mathscr{C})$.

If $\mathbb{D}$ is left and right representable, then $(\mathbb{D}')^{\mathrm{op}} \dashv \mathbb{D}$.

Example 4.12. An example is $\lambda : \mathbf{Tw}(\mathscr{C}) \to \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ with straightening $\mathrm{Map}_{\mathscr{C}} : \mathscr{C} \times \mathscr{C}^{\mathrm{op}} \to \mathcal{S}$. This is left and right representable, classifying the adjunction $\mathrm{id}_{\mathscr{C}} : \mathscr{C} \rightleftarrows \mathscr{C} : \mathrm{id}_{\mathscr{C}}$.

Definition 4.13. A **pairing** of $\mathbb{E}_k$-monoidal ∞-categories is an $\mathbb{E}_k$-monoidal functor $\mathscr{M} \to \mathscr{C} \times \mathscr{D}$ whose underlying functor is a right fibration.

Observe that

$$\mathbf{Alg}_{\mathbb{E}_k}(\mathscr{M}) \to \mathbf{Alg}_{\mathbb{E}_k}(\mathscr{C}) \times \mathbf{Alg}_{\mathbb{E}_k}(\mathscr{D})$$

is still a pairing.

Theorem 4.14 (Lurie)**.** Let $\mu : \mathscr{M} \to \mathscr{C} \times \mathscr{D}$ be a pairing of $\mathbb{E}_k$-monoidal ∞-categories such that $\mathscr{M} \times_{\mathscr{D}} \{1_{\mathscr{D}}\} \simeq \mathscr{C}$, it is left representable and $\mathscr{D}$ has totalizations. Then, $\mathbf{Alg}_{\mathbb{E}_k}(\mu)$ is a left representable pairing.

Example 4.15. Recall the above example (4.12). Ssuppose that $1_{\mathscr{C}}$ is final and that $\mathscr{C}$ has totalizations. Then, $\mathbf{Alg}_{\mathbb{E}_k}(\lambda) : \mathbf{Alg}_{\mathbb{E}_k}(\mathbf{Tw}(\mathscr{C})) \to \mathbf{Alg}_{\mathbb{E}_k}(\mathscr{C}) \times \mathbf{Alg}_{\mathbb{E}_k}(\mathscr{C}^{\mathrm{op}})$ is left representable. So we obtain

$$\mathrm{Bar}^{(k)} : \mathbf{Alg}_{\mathbb{E}_k}(\mathscr{C})^{\mathrm{op}} \to \mathbf{Alg}_{\mathbb{E}_k}(\mathscr{C}^{\mathrm{op}}).$$

If $\mathscr{C}$ has geometric realizations and totalizations and $1_{\mathscr{C}}$ is a zero object, then we get an adjunction[7]

$$\mathbf{Alg}_{\mathbb{E}_k}(\mathscr{C}) \underset{\mathrm{Cobar}^{(k)}}{\overset{\mathrm{Bar}^{(k)}}{\rightleftarrows}} \mathbf{coAlg}_{\mathbb{E}_k}(\mathscr{C})$$

Get rid of $1_{\mathscr{C}} = 0$ and augmentations by replacing $\mathscr{C}$ with $\mathscr{C}_{1/\!/1}$.

[7] Strictly speaking, we used $\mathbb{D}^{\mathrm{op}} \dashv (\mathbb{D}')^{\mathrm{op}}$.

Example 4.16. Let $\mathscr{C} = \mathcal{S}^\times$. In this case, $\mathbf{coAlg}_{\mathbb{E}_k}(\mathcal{S}^\times) \simeq \mathcal{S}$. We obtain

$$\mathrm{Bar}^{(k)} : \mathbf{Alg}^{\mathrm{aug}}_{\mathbb{E}_k}(\mathcal{S}) \rightleftarrows \mathcal{S}_* : \mathrm{Cobar}^{(k)}$$

which restricts to an equivalence $\mathbf{Alg}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathcal{S}) \simeq \mathcal{S}_*^{\geq k}$.

Example 4.17 (Burklund[8])**.** Let $\mathscr{E} \in \mathbf{Alg}_{\mathbb{E}_n}(\mathbf{Pr}^L_{\mathrm{st}})$ and $\mathscr{E}^{\mathrm{Gr}} = \mathrm{Fun}(\mathbb{Z}, \mathscr{E}) \in \mathbf{Alg}_{\mathbb{E}_n}(\mathbf{Pr}^L_{\mathrm{st}})$. Let $\mathscr{E}^{\mathrm{Gr}}_+ \subseteq \mathscr{E}^{\mathrm{Gr}}_{1/\!/1}$ be the full subcategory on those objects of the form $(\cdots, 0, 0, 0, 1_{\mathscr{E}}, *, *, *, \cdots)$ where $1_{\mathscr{E}}$ lies in degree 0. Then,

$$\mathrm{Bar}^{(n)} : \mathbf{Alg}_{\mathbb{E}_k}(\mathscr{E}^{\mathrm{Gr}}_+) \rightleftarrows \mathbf{coAlg}_{\mathbb{E}_k}(\mathscr{E}^{\mathrm{Gr}}_+) : \mathrm{Cobar}^{(k)}$$

restricts to an equivalence.

The last example will feature in Jordan's talk (Section 10).

5 Nilpotence & Periodicity (Henry Rice)

5.1 Self Maps

TALK 5
01.07.2025

A self-map in $\mathbf{Sp}$ is a map $\Sigma^d X \to X$. There's a few ways you can consider nilpotence in $\mathbf{Sp}$.

- Iterate the self map.
- For $f : F \to X$ consider its smash powers.
- Nilpotence in the ring $\pi_\bullet R$.

There is a nilpotence theorem for each of these.

Theorem 5.1 (Nilpotence theorem)**.** Let R be a ring spectrum.

(i) The kernel of the Hurewicz map $\pi_\bullet R \to \mathrm{MU}_\bullet R$ consists of nilpotent elements.

(ii) Consider $f : F \to X$ with a finite spectrum F. If $\mathrm{MU} \otimes f$ is nullhomotopic, then f is smash-nilpotent.

(iii) Consider a sequence

$$\cdots \longrightarrow X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots$$

with c_n-connected X_n where $c_n \geq mn + b$ for some fixed $m, b \in \mathbb{Z}$. If all

$$\mathrm{MU}_\bullet(f_n) : \mathrm{MU}_\bullet(X_n) \to \mathrm{MU}_\bullet(X_{n+1})$$

are zero, then $\mathrm{colim}_n X_n \simeq *$.

Note that (iii) has the iterate nilpotence notion as a special case. Namely, take the sequence induced by a self-map, then connectivity decreases linearly. If $\mathrm{MU}_\bullet$ of the self-map is trivial, then the colimit of this iteration is trivial, in other words the infinite composite is trivial.

[8]This was also already in Hahn-Wilson and in earlier literature as a folklore thing.

5.2 Morava K-Theory

For every prime p and every $n \in \mathbb{N}$ there is a ring spectrum $K(n)$ called **Morava K-theory** satisfying the following properties. Let v_n be the coefficient of x^{p^n} of the p-series $[p]_{\mathrm{MU}}$ of the universal formal group law. Here, $|v_n| = 2(p^n - 1)$.

Theorem 5.2. There is an isomorphism $K(n)_\bullet \cong \mathbb{F}_p[v_n^{\pm 1}]$.

Essentially, take $\mathrm{MU}_{(p)}$, invert v_n and kill all the other generators. For example use an [EKMM97, Theorem V.2.6] argument to see that $K(n)$ is a ring spectrum.[9]

Lemma* 5.3. Let $R \in \mathbf{Alg}(h\mathbf{Sp})$ and $x \in R_n$ with $\pi_{n+1}(R/x) = 0$ and $\pi_{2n+1}(R/x) = 0$. Then, R/x admits the structure of a homotopy R-ring spectrum with unit $p : R \to R/x$.

Proof Idea.* This is [EKMM97, Theorem V.2.6]. The main idea is to consider the cofiber sequence

$$\Sigma^n R/x \xrightarrow{x} R/x \xrightarrow{p\otimes \mathrm{id}} R/x \otimes_R R/x \longrightarrow \Sigma^{n+1} R/x$$

and diagram chase to construct a splitting of $\rho \otimes \mathrm{id}$ which is then the multiplication. This diagram chase abuses another cofiber sequence coming from the cofiber of multiplication by x. For more details see [EKMM97]. □

Proposition 5.4. The ring $K(n)_\bullet$ is a graded field, i.e. all modules over $K(n)_\bullet$ are free.

Corollary 5.5. There is a Künneth formula $K(n)_\bullet(X \otimes Y) \cong K(n)_\bullet X \otimes K(n)_\bullet Y$.

In particular, $K(n) \otimes X$ splits as a direct sum of shifts of $K(n)$'s.

5.3 Back to Nilpotence

Here is the detecting nilpotence lemma.

Lemma 5.6.

(i) Let R be a p-local ring spectrum, $\alpha \in \pi_\bullet R$, then α is nilpotent if and only if $K(n)_\bullet \alpha$ is nilpotent for all n.

(ii) Let F be a finite p-local spectrum. Then, $f : \Sigma^k F \to F$ is nilpotent if and only if $K(n)_\bullet f$ is nilpotent for all n.

(iii) Let F be a finite spectrum. Then, $f : F \to X$ is smash nilpotent if and only if $K(n)_\bullet f$ is nilpotent for all n.

5.4 Periodicity Theorem

This is a consequence of the thick subcategory theorem.

Definition 5.7. Let X be a p-local finite spectrum. Then, X has **type n** if it is the least integer such that $K(n)_\bullet X \neq 0$.

Proposition 5.8. Suppose that $K(n)_\bullet X \cong 0$. Then, $K(n-1)_\bullet X \cong 0$.

If X has type n, then one may ask whether X has a v_n-self map.

Definition 5.9. An **v_n-self map** is a map $f : \Sigma^d X \to X$ such that $K(n)_\bullet f$ is an isomorphism and $K(m)_\bullet f$ is nilpotent for $m \neq n$.

Note that in the case type $X > n$, one can take $0 : X \to X$ as a v_n-self map. For example, the *periodicity theorem* shows the existence of v_n-self maps in other cases. A thick subcategory theorem argument shows that the category of type $\geq n$ spectra are those admitting a v_n-self map.

[9] See also [Lur10, Lemma 22.2].

5.5 Thick Subcategory Theorem

Definition 5.10. Let $\Gamma \leq \mathbb{Z}[\![t]\!]$ be a subgroup of the form $x + b_1x^2 + b_2x^3 + \cdots$ which is a group under composition.

Consider $F(x,y) \in \mathrm{FGL}(R)$ and $g(t)$ be a strict automorphism. Then, $gF(g^{-1}x, g^{-1}y)$ is again a formal group law, so by the Yoneda Lemma one obtains a self map $L \to L$. This means that Γ acts on L.

Observation 5.11. Let X be a finite spectrum, then $\mathrm{MU}_\bullet X$ also has an Γ-action via $\gamma(\alpha x) = \gamma(\alpha)\gamma(x)$ for $\gamma \in \Gamma, \alpha \in \mathrm{MU}_\bullet, x \in \mathrm{MU}_\bullet X$.

Definition 5.12. Let $\mathbf{C\Gamma}$ denote the category of finitely presented L-modules equipped with an Γ-action compatible with its action on L.

This is equivalently also the category $\mathbf{Comod}_{\mathrm{MU}_\bullet \mathrm{MU}} \simeq \mathbf{QCoh}(\mathcal{M}^s_{\mathrm{fg}})$. The category $\mathbf{Mod}_{\mathrm{MU}_\bullet}$ is abelian, so $\mathbf{C\Gamma}$ is a subcategory of an abelian category and you can consider short exact sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

A subcategory of $\mathbf{C\Gamma}$ is thick if $B \in \mathscr{C}$ if and only if $A, C \in \mathscr{C}$.

More modernly, a thick subcategory of a stable ∞-category is one that is closed under cofibers and direct summands.

Theorem 5.13. The only thick subcategories of $\mathbf{C\Gamma}_{(p)}$ are $\mathbf{C\Gamma}_{(p)}$ and those subcategories consisting of M such that $v_{n-1}^{-1}M = 0$, also called $\mathscr{C}_{p,n}$.

Theorem 5.14 (Thick subcategory theorem)**.** The thick subcategories of $\mathbf{Sp}^{\omega}_{(p)}$ consists of $\mathbf{Sp}^{\omega}_{(p)}$ and the type $\geq n$ spectra for $n \in \mathbb{N} \cup \{\infty\}$, i.e. those X with $K(n)_\bullet X = 0$.

6 Examples of Spectra in Chromatic Homotopy Theory (Catherine Li)

In this talk, Catherine was wearing a t-shirt with Akhil's face along with a a number of fun math jokes. She wished Akhil's spirit to carry this talk. TALK 6 01.07.2025

Goal: Introduce E_n (Morava E-theory) and BP (Brown-Peterson spectrum). Moreover, construct multiplicative structures:

- $\mathbb{E}_\infty$-structure on E_n,
- $\mathbb{E}_2$-structure on BP,
- $\mathbb{E}_1$-structure on $K(n)$.

6.1 Morava E-Theory

Motivation: The moduli stack $\mathcal{M}^{=n}_{\mathrm{fg}}$ of formal groups of height exactly n has a unique geometric point, i.e. over the algebraically closed field there is one formal group of height n (up to isomorphism). This leads to $K(n)$.

The inclusion of the geometric point is not flat! So instead we want to consider a formal neighbourhood of this point which will become a Landweber exact thing. Conceptually, it is also important to study. We want to understand $\mathcal{M}_{\mathrm{fg}}$ by seeing how to glue the strata together. Here is a picture of $\mathcal{M}_{\mathrm{fg}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$:

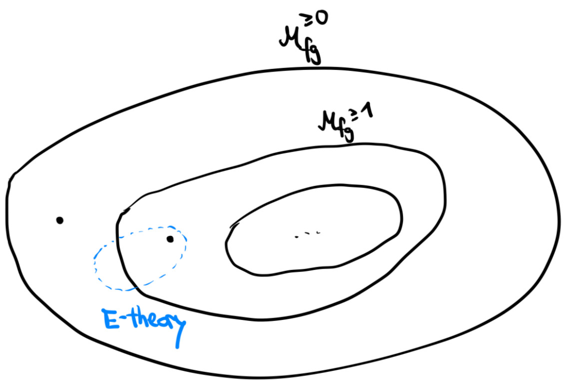

Figure 2: An imprecise picture of a stratification on $\mathcal{M}_{\mathrm{fg}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$.

A deformation of the geometric points should be Morava E-theory.

Definition 6.1. An **infinitesimal thickening** of k is a ring A with $p : A \twoheadrightarrow k$ with $\mathfrak{m} = \ker p$ such that:

(i) $\mathfrak{m}^i = 0$ for $i \gg 0$,

(ii) $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a finite-dimensional k-vector space for all i.

Example 6.2. An example is $A = k[\varepsilon]/(\varepsilon^2)$.

Definition 6.3. Let $\mathbb{G}_0 \to \operatorname{Spec} k$ be a formal group of height n. A **deformation** of $\mathbb{G}_0$ to A is a formal group $\mathbb{G} \to \operatorname{Spec} A$ such that $\mathbb{G} \times_{\operatorname{Spec} A} \operatorname{Spec} A_0 \simeq \mathbb{G}_0$.

In coordinates, say that $F_0 \in k[\![x, y]\!]$ corresponds to some formal group law $\mathbb{G}_0$ and F corresponds to some $\mathbb{G}$, then this means $F_0 \equiv F \pmod{\mathfrak{m}}$.

Example 6.4. Consider a formal group law F over $A = k[\varepsilon]/(\varepsilon^2)$ and suppose that

$$[p]_F = \varepsilon x^p + x^{p^2} + \cdots .$$

Let F_0 be the reduction of F modulo (ε), so $[p]_{F_0} = x^{p^2} + \cdots$, so we obtain a formal group law of height exactly 2. Here, F is a deformation of F_0 to A.

Notation 6.5. Let $\mathrm{Def}_{\mathbb{G}_0}(A) = \{$deformations of $\mathbb{G}_0$ to A together with isomorphisms$\}$.

Theorem 6.6. Let k be a perfect field of characteristic p and $\mathbb{G}_0 \to \operatorname{Spec} k$ be a formal group of height $n < \infty$. Then, there exists a formal group $\mathbb{G} \to \operatorname{Spf} E_0$ where $E_0 = W(k)[\![u_1, \cdots, u_{n-1}]\!]$ such that this formal group is the *universal deformation* of $\mathbb{G}_0$, i.e. it induces an equality $\operatorname{Spf}(E_0)(A) = \mathrm{Def}_{\mathbb{G}}(A)$ for all infinitesimal thickenings A of k.

In coordinates say $F_0 \leftrightsquigarrow \mathbb{G}_0$. Then, F is the universal deformation of F_0 if $v_i(F) = u_i$ and F reduces to F_0 under $E_0 \to k$. So it will look like

$$[p]_F = px + u_1 x^p + \cdots + u_{n-1} x^{p^{n-1}} + \left(\text{something like } [p]_{F_0}\right).$$

Construction 6.7. Fix k and $\mathbb{G}$. By Landweber exactness there is an even periodic, homotopy commutative, homotopy associative spectrum $E = E(k, \mathbb{G}) = E_n$ whose homology theory is given by $E_\bullet = \mathrm{MU}_\bullet \otimes_{\mathrm{MU}_\bullet} E_0[\![u^{\pm}]\!]$ with $|u| = 2$.

I'm grateful to Catherine for (re-)explaining the following to me!

Observation 6.8. In general, the only obstructions to a homotopy commutative (and associative) spectrum obtained from a commutative (co-)homology theory by Brown representability is through phantom maps.

- Hopkins: Landweber exact spectra are evenly generated [HS99, Proposition 2.12] [Lur10, Proposition 17.9].
- Tensor products of evenly generated spectra are evenly generated [HS99, Proposition 2.19].
- Strickland: Phantom maps from evenly generated spectra to even spectra are nullhomotopic [HS99, Corollary 2.15] [Lur10, Proposition 17.10].

So altogether, there are no non-trivial phantom maps $E \otimes E \to E$. Thus, we can compare the maps $\mu : E \otimes E \to E$ and $\mu \circ \tau : E \otimes E \to E$ where τ is the swap map. On homology theories these agree since $E_\bullet$ is homotopy commutative, so $\mu - \mu \circ \tau$ is a phantom map which means that it's nullhomotopic! Combining these yields that E_n is a homotopy commutative MU-algebra, as also explained with more detail in [HS99, Proposition 2.21].[10]

Here is a summary of the above:

Theorem* 6.9. Let $E_\bullet$ be a Landweber exact commutative $\mathrm{MU}_\bullet$-algebra. Then, the Landweber exactness functor yields a homotopy commutative MU-algebra.

Proof. See [HS99, Theorem 2.8]. □

6.2 Separability and $\mathbb{E}_\infty$-Structure

Let's construct an $\mathbb{E}_\infty$-structure of E_n. There is an unpublished argument from Ishan, Robert and Dustin as well (D.6). Instead, we will follow Maxime's PhD thesis.

The idea is that *separable algebras* are nice in the homotopy things lift to ∞-categorical things. Here is one slogan that you could be expecting:

Example 6.10. Say A is separable in $\mathbf{Alg}(\mathscr{C})$ where $\mathscr{C}$ is something appropriate. If A is homotopy commutative, then it extends to an $\mathbb{E}_\infty$-structure.

Problem: E_n is only *homotopy ind-separable*.

Let $(\mathscr{C}, \otimes, \mathbb{1})$ be a symmetric monoidal ∞-category.

Definition 6.11. An algebra $A \in \mathbf{Alg}(\mathscr{C})$ is called **separable** if the multiplication map

$$A \otimes A^{\mathrm{op}} \to A$$

admits a section s (as a map of $A \otimes A^{\mathrm{op}}$-left modules).

Definition 6.12. Let $\mathscr{C}$ be a compactly generated presentably symmetric monoidal stable ∞-category. Then, $A \in \mathbf{CAlg}(\mathscr{C})$ is **ind-separable** if there is a subset $S \subseteq \pi_0 \operatorname{map}(\mathbb{1}, A \otimes A)$ such that $\mu : A \otimes A \to A$ realizes $A \simeq (A \otimes A)[S^{-1}]$.

Here is the motivating example:

Example 6.13. Let $R \to A$ be a G-Galois extension for a finite group G. Then, $R \to A$ is separable and E_n is a (pro-)Galois extension of $L_{K(n)}\mathbb{S}$.

Theorem 6.14 (Ramzi)**.** Let $\mathscr{C}$ satisfy the following assumptions:

[10] See also https://mathoverflow.net/questions/387107.

(i) It is a compactly generated, stable, symmetric monoidal ∞-category in which $\otimes$ commute with colimits in each variable and $\mathscr{C}^\omega$ is closed under non-empty[11] tensor products.

(ii) Let $X, Y \in \mathscr{C}$ such that there exists an isomorphism $f : \pi_\bullet \operatorname{Map}_{\mathscr{C}}(-, X) \xrightarrow{\sim} \pi_\bullet \operatorname{Map}_{\mathscr{C}}(-, Y)$ of cohomology theories on $\mathscr{C}^\omega$, then there exists $\widetilde{f} : X \to Y$ inducing f (which is an equivalence).

Let $A \in \mathbf{CAlg}(h\mathscr{C})$ be homotopy commutative and homotopy ind-separable in $\mathscr{C}$ which receives no phantom maps from any $\otimes$-powers of A. Then, for every $1 \leq d \leq \infty$ the moduli space

$$\mathbf{Alg}_{\mathbb{E}_d}(\mathscr{C})^{\mathrm{core}} \times_{\mathbf{Alg}(h\mathscr{C})} \{A\}$$

is contractible, i.e. there exists a unique lift of A to an $\mathbb{E}_d$-algebra.

Here is the theme:

Observation 6.15. One can view separability as close to projectivity. So if A is separable, it is projective an an $A \otimes A^{\mathrm{op}}$-module. Any A-module looks like a retract of $A \otimes M$ for some M. This turns a lot of the mapping spaces discrete (or at least simply connected). This is an analogy to $\operatorname{Ext}_{A \otimes A^{\mathrm{op}}}(A, A)$.

Why does $\mathbf{Sp}_{K(n)}$ satisfy the assumptions from Maxime's theorem (6.14)?

(i) It is compactly generated by $L_{K(n)}X$ for dualizable compact X.

(ii) It ends up to suffice showing that $h(\mathscr{C}^\omega)$ is countable.

Why does E_n satisfy the hypotheses?

- To see that E_n is homotopy ind-separable, one computes

$$\pi_\bullet(L_{K(n)}E \otimes E) \cong C(\Gamma, E_\bullet)$$

  where Γ is the Morava-stablizer group. Consider

$$\mathrm{ev}_{\mathscr{C}} : \pi_\bullet(L_{K(1)}E \otimes E) \to E_\bullet \cong C(\Gamma, E_\bullet)[S^{-1}].$$

- Lack of phantom maps $E^{\otimes n} \to E$: Do things like earlier.

This concludes the proof that E_n has a unique $\mathbb{E}_d$-structure for $1 \leq d \leq \infty$.

6.3 Brown-Peterson Spectra

Recall that $\pi_\bullet(\mathrm{MU}_{(p)}) \cong \mathbb{Z}_{(p)}[v_i : |v_i| = 2p^i - 2] \otimes \mathbb{Z}_{(p)}[b_m : m \neq p^k - 1]$.

Definition 6.16. The **Brown-Peterson spectrum** is $\mathrm{BP} = \mathrm{MU}_{(p)}/(b_m : m \neq p^k - 1)$.

One obtains $\mathrm{MU}_{(p)} \simeq \bigoplus^{2m} \mathrm{BP}$.

Fact 6.17 (Quillen)**.** There is an idempotent $\varepsilon : \mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$ such that

$$\mathrm{BP} \simeq \operatorname{colim}\left(\mathrm{MU}_{(p)} \xrightarrow{\varepsilon} \mathrm{MU}_{(p)} \xrightarrow{\varepsilon} \cdots\right).$$

The idempotent is also called **Quillen idempotent**.

It turns out that BP is Landweber exact, so it is homotopy associative. Now about the $\mathbb{E}_2$-structure.

[11] So $\mathbb{1}$ doesn't need to be included.

Theorem 6.18 (Chadwick-Mandell)**.** Quillen idempotents and MU $\to$ BP admit the structure of $\mathbb{E}_2$-ring spectrum maps.

Proof. If R has an $\mathbb{E}_2$-orientation, then the idea is that we consider

$$\mathrm{Map}_{\mathbf{Alg}_{\mathbb{E}_2}(\mathbf{Sp})}(\mathrm{MU}, R) \simeq \mathrm{Map}_{\mathbf{Alg}_{\mathbb{E}_2}(\mathcal{S})}(\mathrm{BU}, \mathrm{GL}_1(\mathbb{R})) \simeq \mathrm{Map}_{\mathcal{S}}(B^2\,\mathrm{BU}, B^2\,\mathrm{GL}_1\,R)$$

Some calculations using $B^2\,\mathrm{BU} \simeq \mathrm{BSU}$ show that

$$\mathrm{Map}_{\mathcal{S}}(B^2\,\mathrm{BU}, B^2\,\mathrm{GL}_1\,R) \to \mathrm{Map}_{\mathbf{Alg}(h\mathbf{Sp})}(\mathrm{MU}(1), R)$$

is surjective on π_0 in the case of MU. So this recovers an $\mathbb{E}_2$-lift of the Quillen idempotent. □

This shows that the Quillen idempotent is an $\mathbb{E}_2$-map and that MU $\to$ BP is also $\mathbb{E}_2$. Here we are using that $\mathrm{id}_{\mathrm{MU}}$ is an $\mathbb{E}_2$-orientation on MU.

6.4 Morava K-Theory

We will now construct an $\mathbb{E}_1$-ring structure on $K(n)$ improving that homotopy associative structure stated in the previous talk (5).

Theorem 6.19 (Angeltveit)**.** Let R be an even $\mathbb{E}_\infty$-ring and I be an ideal generated by a regular sequence. Then, $A = R/I$ being homotopy associative implies that it extends to an $\mathbb{E}_1$-structure on A.

Proof Idea. It's an obstruction theory argument where the obstructions live in $\mathrm{Ext}_{A\otimes A^{\mathrm{op}}}$ which are then shown to vanish by a spectral sequence argument. Hahn-Wilson extend this to R being $\mathbb{E}_2$-among other things. □

Apply this to MU and take $I = (p, v_1, \cdots, v_{n-1}, v_{n+1}, \cdots, b_m : m \neq p^k - 1)$. This yields an $\mathbb{E}_1$-structure on $k(n)$. Moreover, we have $v_n^{-1}k(n) \simeq K(n)$ and argument that this doesn't break the $\mathbb{E}_1$-structure is for example in the appendix of [MNN17]. A different way is to try to understand the categories of modules (B.6). Some more discussion arose at the end of the talk.

- There is another paper by Angeltveit on the uniqueness of $\mathbb{E}_1$-structures on $K(n)$ but Ishan says that this is wrong. There are many homotopy associative structures on $K(n)$ [Lur10, Remark 22.3] but fixing one allows us to uniquely extend to an $\mathbb{E}_1$-structure. The above Angeltveit is correct, however.
- Gijs said that $K(n)$ can never be $\mathbb{E}_2$. This uses a Hopkins-Mahowald type result that the free $\mathbb{E}_2$-algebra with $p = 0$ is $\mathbb{F}_p$ [ACB19, Corollary 5.4]. Sil also explained a k-invariant argument to me using Hopkins-Mahowald, but I cannot quite remember how it goes.

7 Descent, Smash Product Theorem & Chromatic Convergence (Maite Carli)

The goal in this talk is to discuss the following theorem. TALK 7 01.07.2025

Theorem 7.1 (Hopkins-Ravenel, Smash Product Theorem)**.** The localization $L_{E_n} = L_n$ is *smashing*, i.e.

$$L_n X \simeq X \otimes L_n \mathbb{S}$$

for $X \in \mathbf{Sp}$.

Indeed, there is a map $X \otimes \eta : X \to X \otimes L_n\mathbb{S}$ which is an E_n-equivalence since

$$X \otimes E_n \simeq X \otimes \mathbb{S} \otimes E_n \simeq X \otimes L_n\mathbb{S} \otimes E_n$$

via η. Thus, $\mathrm{map}_{\mathrm{Sp}}(X \otimes L_n\mathbb{S}, L_nX) \to \mathrm{map}_{\mathrm{Sp}}(X, L_nX)$ is an equivalence because L_nE is E_n-local. This yields a preferred map $X \otimes L_n\mathbb{S} \to L_nX$ which is an E_n-equivalence. If we can show that $X \otimes L_n\mathbb{S}$ is E_n-local, then the theorem (7.1) follows. So our plan will be the following:

(i) Introduce descent and nilpotence.

(ii) Proof of Smash Product Theorem from it.

(iii) Proof of Chromatic Convergence.

7.1 Setup of Descendability

Definition 7.2. A subcategory of a stable ∞-category $\mathscr{C}$ is **thick** if it is closed under fibers, cofibers, contains 0 and is idempotent complete.

Let $\mathscr{C}$ be a symmetric monoidal stable idempotent-complete ∞-category and $A \in \mathbf{Alg}(\mathscr{C})$.

Definition 7.3. An object $X \in \mathscr{C}$ is **A-nilpotent** if $X \in \mathrm{Nil}_A = \mathrm{Thick}^{\otimes}(A)$.

Example 7.4. Let $\mathscr{C} = \mathcal{D}(\mathbb{Z})$ and $A = \mathbb{Z}/p\mathbb{Z}$. Then, $X \in \mathrm{Nil}_A$ if and only if there exists $n \geq 0$ such that $p^n : X \to X$ is nullhomotopic.

Question: How do we approximate $X \in \mathscr{C}$ by elements in Nil_A?

Definition 7.5. The **augmented cobar construction** $\mathrm{CB}^{\mathrm{aug}}(A)$ is the augmented cosimplicial diagram

$$\mathbb{1} \longrightarrow A \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \; A \otimes A \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \; \cdots .$$

The **cobar construction** is $\mathrm{CB}^{\bullet}(A) = \mathrm{CB}^{\mathrm{aug}}(A)|_{\Delta}$.

Remark 7.6. The augmented cobar construction $\mathrm{CB}^{\mathrm{aug}}(A)$ admits an extra degeneracy after tensoring with A, i.e. for all $F : \mathscr{C} \to \mathscr{D}$ there is an equivalence $FA \xrightarrow{\simeq} \mathrm{Tot}(F(\mathrm{CB}^{\mathrm{aug}}(A) \otimes A))$.

Proposition 7.7. Let $X \in \mathrm{Nil}_A$. Then for every stable ∞-category $\mathscr{D}$ and exact $F : \mathscr{C} \to \mathscr{D}$ the map

$$FX \to \mathrm{Tot}\,(F(X \otimes A) \rightrightarrows F(X \otimes A \otimes A) \substack{\rightarrow \\ \rightarrow \\ \rightarrow} \cdots)$$

is an equivalence.

Proof. The class for which this equivalence holds is thick, and if $X = A \otimes Y$ we are in the setting of the previous remark (7.6). □

Definition 7.8. A **tower** in $\mathscr{C}$ is a functor $\mathbb{N}^{\mathrm{op}} \to \mathscr{C}$. Towers form a stable ∞-category **Tow($\mathscr{C}$)**.

Construction 7.9. Consider $X^{\bullet} : \Delta \to \mathscr{C}$, then one obtains a tower

$$\cdots \longrightarrow \mathrm{Tot}_{\leq n}(X^{\bullet}) \longrightarrow \mathrm{Tot}_{\leq n-1}(X^{\bullet}) \longrightarrow \cdots$$

where $\mathrm{Tot}_{\leq n}(X^{\bullet}) = \lim_{\Delta_{\leq n}} X^{\bullet}$.

Fact 7.10. There is an equivalence $\mathrm{Fun}(\Delta, \mathscr{C}) \simeq \mathbf{Tow}(\mathscr{C})$.

Definition 7.11. Let $\{X_i\}_{i \geq 0}$ be a tower.

(i) The tower is **nilpotent** if there exists $N > 0$ such that for every $i \geq 0$ the map $X_{i+N} \to X_i$ is nullhomotopic.

(ii) The tower is **quickly converging** if it is in the thick subcategory generated by constant and nilpotent towers.

(iii) The corresponding $X^\bullet \in \mathrm{Fun}(\Delta, \mathscr{C})$ is **quickly converging** if the associated tower is.

Proposition 7.12. Suppose that $X^\bullet \in \mathrm{Fun}(\Delta, \mathscr{C})$ is quickly converging. Then,

(i) For every $F : \mathscr{C} \to \mathscr{D}$ and stable $\mathscr{D}$ we get that $F(X^\bullet)$ is quickly converging.

(ii) Let $\mathscr{D}$ be a stable, idempotent-complete and F be exact. Then, $F(\mathrm{Tot}(X^\bullet)) \simeq \mathrm{Tot}(F(X^\bullet))$.

(iii) We have $\mathrm{Tot}(X^\bullet) \in \mathrm{Thick}(X^i : i \in \mathbb{N})$ and $\lim(\mathrm{Tot}_{\leq i} X^\bullet)$ is a retract of some $\mathrm{Tot}_{\leq i} X^\bullet$.

Remark* 7.13. From Akhil's survey [Mat18, p. 6]: An upshot of quick convergence is that it indicates that infinite limits in a stable ∞-category behave like finite ones (up to taking retracts). For example, exact functors preserve finite limits and not totalizations in general but they do preserve quickly converging ones by the previous proposition (7.14).

Proposition 7.14. We have $X \in \mathrm{Nil}_A$ if and only if $\mathrm{CB}^\bullet(A) \otimes X$ is quickly converging with limit X.

Proof. Here are two facts:

- for $X \in \mathrm{Thick}^{\otimes}(A)$ there exists $k > 0$ such that $I^{\otimes n} \to \mathbb{1}$ is null after $- \otimes X$ where $I \to \mathbb{1} \to A$ is a fiber sequence,
- there is a cofiber sequence $(X \otimes I^{\otimes n})_{n \geq 0} \to \mathrm{const}(X) \to \mathrm{Tot}_{\leq n}(\mathrm{CB}^\bullet(A) \otimes X)$.

With those we can proceed with the proof.

$\Longrightarrow$: For $X \in \mathrm{Nil}_A = \mathrm{Thick}^{\otimes}(A)$ we get some N such that $(I^{\otimes N} \to \mathbb{1}) \otimes X \simeq 0$, so $(X \otimes I^{\otimes n})_{\geq n}$ is A-nilpotent and we use the last fact.

$\Longleftarrow$: We have $X = \mathrm{Tot}(\mathrm{CB}^\bullet(A) \otimes X) \in \mathrm{Thick}(A \otimes X, A \otimes A \otimes X, \cdots) \subseteq \mathrm{Thick}^{\otimes}(A)$.

$\square$

Definition 7.15. If $\mathbb{1} \in \mathrm{Nil}_A$, i.e. $\mathscr{C} = \mathrm{Thick}^{\otimes}(A)$, then A is called **descendable**.

We will see that E_n is descendable in the L_n-local category.

Let E be an $\mathbb{E}_1$-ring spectrum.

Proposition 7.16. Suppose that $\mathrm{CB}^\bullet(E)$ is quickly converging.

(i) There is an equivalence $\mathrm{Tot}(\mathrm{CB}^\bullet(E)) \simeq L_E\mathbb{S}$.

(ii) The E-local sphere $L_E\mathbb{S}$ is E-nilpotent.

(iii) The localization $L_E : \mathbf{Sp} \to \mathbf{Sp}$ is smashing.

Proof.

(i) By quick convergence, $E \otimes \mathrm{Tot}(\mathrm{CB}^\bullet(E)) \simeq \mathrm{Tot}(\mathrm{CB}^\bullet(E) \otimes E) \simeq E$, so $L_E\mathbb{S} \to \mathrm{Tot}(\mathrm{CB}^\bullet(E))$ is an E-equivalence and both sides are E-local (the right side is a limit of E-local objects).

(ii) This is the previous proposition (7.14).

(iii) It suffices to show that for all $X \in \mathbf{Sp}$ the spectrum $L_E\mathbb{S} \otimes X$ is E-local by the observation at the beginning of the talk, but $L_E\mathbb{S} \in \mathrm{Thick}^{\otimes}(E)$ by (ii), so $X \otimes L_E\mathbb{S} \in \mathrm{Thick}^{\otimes}(E)$. This finishes the proof: one can check by hand that the E-local spectra form a thick subcategory of $\mathbf{Sp}$. As $L_E\mathbf{Sp}$ is compactly generated by the unit $L_E\mathbb{S}$, we deduce by another thick subcategory argument that $L_E\mathbf{Sp}$ is actually a thick tensor ideal. Since E is E-local, we conclude $\mathrm{Thick}^{\otimes}(E) \subseteq L_E\mathbf{Sp}$.

□

7.2 Proofs of Big Theorems

So to prove the smash product theorem it suffices to show that $\mathrm{CB}^\bullet(E_n)$ is quickly converging!

Theorem 7.17. The cobar construction $\mathrm{CB}^{\otimes}(E_n)$ is quickly converging.

Construction 7.18. We can associate the **Bousfield-Kan spectral sequence** to the tower associated to $\mathrm{CB}^\bullet(A) \otimes X$ with signature

$$E_2^{s,t} = H^s(\pi_t(\mathrm{CB}^\bullet(A) \otimes X)) \Rightarrow \pi_{t-s}\,\mathrm{Tot}(\mathrm{CB}^\bullet(A) \otimes X),$$

also called **A-based Adams spectral sequence**.

Proposition 7.19. If $\mathrm{CB}^\bullet(A)$ is quickly converging, the associated Bousfield-Kan spectral sequence admits a *horizontal vanishing line*, i.e. there exists $N \geq 2$ and $k \geq 0$ such that $E_N^{s,t} = 0$ for all $s \geq k$.

Theorem 7.20. Let $X^\bullet$ be a cosimplicial spectrum. If there exists $s \geq 1$ such that for every finite spectrum F the Bousfield-Kan spectral sequence associated to $X^\bullet \otimes F$ vanishes at $E_s^{p,q}$ for all $p \geq s$, then $X^\bullet$ is quickly converging.

Proof of 7.17. Observe that $\mathrm{CB}^\bullet(E_n)$ is quickly converging if and only if $\mathrm{CB}^\bullet(E_n) \otimes \mathbb{S}$ is if and only if there exists a finite type 0 spectrum X such that $\mathrm{CB}^\bullet(E_n) \otimes X$ is.

We then construct X such that the Bousfield-Kan spectral sequence associated to $\mathrm{CB}^\bullet(E_n) \otimes X$ has a horizontal vanishing line on the E_2-page. It is given by the cohomology groups of the chain complex

$$(E_n)_\bullet X \longrightarrow (E_n \otimes E_n)_\bullet X \longrightarrow (E_n \otimes E_n \otimes E_n)_\bullet X \longrightarrow \cdots .$$

Recall that $((E_n)_\bullet, (E_n)_\bullet E_n)$-comodules correspond to $\mathbf{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\leq n})$, we the E_2-page will be the cohomology $H^s\left(\mathcal{M}_{\mathrm{fg}}^{\leq n}, \mathcal{F}_{\Sigma^t X}\right)$. Thus, we inductively reduce to showing

$$H^s\left(\mathcal{M}_{\mathrm{fg}}^k, F|_{\mathcal{M}_{\mathrm{fg}}^k} \otimes \mathcal{G}\right) = 0$$

for $s \gg 0$ and $\mathcal{G} \in \mathbf{QCoh}(\mathcal{M}_{\mathrm{fg}}^k)$. We thus pullback

$$\begin{tikzcd} BG_k \times \mathrm{Spec}(\overline{\mathbb{F}}_p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M}_{\mathrm{fg}}^k \arrow[d] \\ \mathrm{Spec}\,\overline{\mathbb{F}}_p \arrow[r] & \mathrm{Spec}\,\mathbb{F}_p \end{tikzcd}$$

so $H^\bullet(\mathcal{M}_{\mathrm{fg}}^k, F \otimes \mathcal{G}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \cong H^\bullet(G_k, V)$ where V is some $\overline{\mathbb{F}}_p$-vector space with a continuous G_k-action.

If $p > n+1$ then $X = \mathbb{S}_{(p)}$ because $H^\bullet(G_k, V)$ has finite cohomological dimension. Else there is more work. More details are also in Lurie's chromatic lecture notes [Lur10]. □

I'm grateful to Maite for explaining a part of the following proof again to me!

Theorem 7.21 (Chromatic Convergence)**.** Let X be a finite p-local spectrum. Then,

$$X \simeq \lim(\cdots \to L_2 X \to L_1 X \to L_0 X).$$

Proof. The conclusion of the theorem forms a thick subcategory, so it suffices to show this for $\mathbb{S}_{(p)}$ by the thick subcategory theorem. We have an Adams-Novikov filtration $\mathbb{S}_{(p)} \simeq \mathrm{Tot}(\mathrm{MU}_{(p)}^{\bullet+1})$. Here are two claims:

(i) Chromatic convergence holds for free $\mathrm{MU}_{(p)}$-modules.

(ii) The cobar construction $\mathrm{CB}^\bullet(L_n \mathrm{MU}_{(p)})$ is quickly convergent with limit $L_n\mathbb{S}$.

For the proof:

(ii) As E_n is complex oriented we have a ring map $L_n \mathrm{MU} \to E_n$, so E_n is an $L_n \mathrm{MU}$-module. Thus, $E_n \in \mathrm{Thick}^\otimes(L_n \mathrm{MU})$ and so $L_n\mathbb{S} \in \mathrm{Thick}^\otimes(L_n \mathrm{MU})$ using $L_n\mathbb{S} \in \mathrm{Thick}^\otimes(E_n)$ as in the proof of 7.14. So we proved (ii) by 7.14.

By (ii) we obtain $L_n\mathbb{S} \simeq \mathrm{Tot}(\mathrm{CB}^\bullet(L_n \mathrm{MU})) \simeq \mathrm{Tot}(L_n \mathrm{CB}^\bullet(\mathrm{MU}))$. By using the Adams-Novikov filtration as well, we then obtain

$$\mathbb{S}_{(p)} \simeq \mathrm{Tot}\left(\lim_n L_n \mathrm{CB}^\bullet(\mathrm{MU}_{(p)})\right) \simeq \lim_n \mathrm{Tot}(L_n \mathrm{CB}^\bullet(\mathrm{MU})) \simeq \lim_n L_n\mathbb{S}.$$

□

Remark* 7.22. By combining (ii) above with 7.16 we obtain

$$L_n\mathbb{S} \simeq \mathrm{Tot}(\mathrm{CB}^\bullet(L_n \mathrm{MU})) \simeq L_{L_n \mathrm{MU}}\mathbb{S}$$

which I see as a (potentially obvious?) curiosity.

8 Features and Bugs of Monochromatic Homotopy Theory (Florian Riedel)

Recall that our goal in life is to understand $\mathbb{S}$. TALK 8 01.07.2025

8.1 Recap

We might use $\mathbb{S} \to \mathrm{MU}$ which detects nilpotence by the nilpotence theorem.

Observation 8.1 (Nishida)**.** The kernel of the Hurewicz map $\pi_\bullet\mathbb{S} \to \pi_\bullet \mathrm{MU} \cong \mathbb{Z}_{(p)}[t_0, t_1, \cdots]$ consists of nilpotent elements by the nilpotence theorem. By Serre's finiteness theorem, $\pi_\bullet\mathbb{S}$ for $\bullet > 0$ consists of nilpotent elements. This recovers *Nishida's nilpotence theorem* (which he proved differently).

We should 'interpolate' between **Sp** and ($\mathrm{MU}_\bullet \to \mathrm{MU}_\bullet \mathrm{MU} \rightrightarrows \cdots$) which is close to $\mathcal{M}_{\mathrm{fg}}$. The moduli stack of formal groups $\mathcal{M}_{\mathrm{fg}}$ comes with a stratification by height

$$\mathcal{M}_{\mathrm{fg}}^{=n} \subseteq \mathcal{M}_{\mathrm{fg}}^{\leq n} \subseteq \mathcal{M}_{\mathrm{fg}}.$$

Recall also $v_0 = p$ and $v_i = t_{2^i-1}$ in $\pi_\bullet \mathrm{MU}$ which control the height.

Recall: $K(n) = \mathrm{MU}/(t_0, t_1, \cdots)[v_n^{-1}]$ and $\pi_\bullet K(n) = \mathbb{F}_p[v_n^\pm]$ and $K(0) = \mathbb{Q}$. Consider localizations $L_{K(n)} : \mathbf{Sp} \to \mathbf{Sp}_{K(n)}$ and $L_n = L_{K(0)\oplus\cdots\oplus K(n)} = L_{E_n} : \mathbf{Sp} \to \mathbf{Sp}_{E_n}$.

There is a more geometric setup, namely $\mathbf{Sp}_{K(n)} \subseteq \mathbf{Sp}_{T(n)}$ and these are the **monochromatic categories**.

Gijs said that historically, $\mathbf{Sp}_{K(n)}$ was called monochromatic, $\mathbf{Sp}_{T(n)}$ was first called monocular by Ravenel but was told by Miller not to do this.

Construction 8.2. Let V be a type n complex, i.e. for $V \in \mathcal{S}^\omega$ we have $K(m)_\bullet V = 0$ for $m < n$ and $K(n)_\bullet V \neq 0$. There exists a v_n-self map $v : \Sigma^t V \to V$ which is an $K(n)_\bullet$-equivalence and this always exists by the periodicity theorem. Then, $T(n) = \Sigma^\infty V[v^{-1}]$ or for example $\mathbb{S}/(p^{m_0}, v_1^{m_1}, \cdots, v_{n-1}^{m_{n-1}})[v_n^{-1}]$. Only its Bousfield class is well-defined! But in that regard we can form the localization

$$\begin{tikzcd} \mathbf{Sp} \arrow[r, "L_{T(n)}"] \arrow[dr, "L_{K(n)}"'] & \mathbf{Sp}_{T(n)} \\ & \mathbf{Sp}_{K(n)} \arrow[u, hook] \end{tikzcd}$$

The goal of this talk is to learn something about $\mathbf{Sp}_{T(n)}$.

8.2 Bousfield-Kuhn Functor

These monochromatic categories behave quite weirdly in some regards.

Theorem 8.3 (Bousfield-Kuhn)**.** There exists a factorization

$$\begin{tikzcd} \mathbf{Sp} \arrow[rr, "L_{T(n)}"] \arrow[dr, "\Omega^\infty"'] & & \mathbf{Sp}_{T(n)} \\ & \mathcal{S}_* \arrow[ur, "\Phi"'] & \end{tikzcd}$$

This Φ is the **Bousfield-Kuhn functor**.

Someone asked whether there is a similar story for $K(n)$ and indeed: further $K(n)$-localize.

Remark 8.4. Consider $A \in \mathcal{S}_*$ and suppose that $\Omega^t A \simeq A$. Then, A deloops to a spectrum $\{A_i\}_{i \in \mathbb{Z}}$ given by

$$\cdots, A_0 = A, A_1 = \Omega^{t-1} A, \cdots, A_{t-1} = \Omega A, A_t = A, \cdots$$

Definition 8.5. Let (V, v) be a type n space with $v : \Sigma^t V \to V$. The **Bousfield-Kuhn functor** associated to (V, v) is

$$\Phi_V(A) = \operatorname{colim}\left(\operatorname{Map}_*(V, A) \xrightarrow{v} \operatorname{Map}_*(\Sigma^t V, A) \xrightarrow{v} \cdots\right).$$

Can check that $\Omega^t \Phi_V(A) \simeq \Phi_V(A)$, so $\Phi_V(A)$ deloops to a spectrum by the previous remark (8.4).

Lemma 8.6. For every $X \in \mathbf{Sp}$ we have $\Phi_V(\Omega^\infty X) \simeq L_{T(n)} X^V \simeq L_{T(n)} X \otimes D(\Sigma^\infty V)$.

Proof. Let $A \in \mathcal{S}_*$. Then, $\Phi_V(A)$ is $T(n)$-local. Indeed, for $W \in \mathcal{S}_*^\omega$, then

$$\begin{aligned} \Phi_V(A)^W &\simeq \operatorname{colim}\left(\operatorname{Map}_*(V, A)^W \xrightarrow{v} \cdots\right) \\ &\simeq \operatorname{colim}\left(\operatorname{Map}_*(V \wedge W, A) \xrightarrow{W \wedge v} \cdots\right) \\ &\simeq \Phi_{V \wedge W}(A). \end{aligned}$$

Use that self-maps are nilpotent on complexes of the wrong type.

I don't get what and why we are trying to do here.

So with some work, the above implies $\Phi_V(\Omega^\infty X) \simeq \Phi_V(\Omega^\infty L_{T(n)} X)$. Suppose that X is $T(n)$-local. Then,

$$\begin{aligned} \Phi_V(\Omega^\infty X) &\simeq \operatorname{colim}\left(\operatorname{Map}_*(V, \Omega^\infty X) \xrightarrow{v} \cdots\right) \\ &\simeq \operatorname{colim}\left(\operatorname{Map}_{\mathbf{Sp}}(\Sigma^\infty V, X) \xrightarrow{v} \operatorname{Map}_{\mathbf{Sp}}(\Sigma^{\infty+t} V, X) \xrightarrow{v} \cdots\right) \end{aligned}$$

Note that the fiber of say the first stage is $\operatorname{Map}_{\mathbf{Sp}}(\Sigma^\infty V/v, X) \simeq 0$ which uses that X is $T(n)$-local and $\Sigma^\infty V/v$ is type $n+1$. So $\Phi_V(\Omega^\infty X) \simeq X^V$. $\square$

Contemplate the functoriality of $(V, v) \mapsto \Phi_V$ to get a refinement

$$\begin{tikzcd} \mathbf{Sp}_{\geq n}^{\mathrm{fin,op}} \arrow[r] \arrow[d, hook] & \mathrm{Fun}(\mathcal{S}_*, \mathbf{Sp}) \\ \mathbf{Sp}^{\mathrm{fin,op}} \arrow[ur, "\mathrm{Ran}_{\Phi_{(-)}}"'] & \end{tikzcd}$$

Definition 8.7. We define the **Bousfield-Kuhn functor** as $\Phi = \mathrm{Ran}_{\Phi(-)}(\mathbb{S})$.

Proof of 8.3. Compute

$$\Phi(\Omega^\infty X) \simeq \lim_{\mathbf{Sp}^{\mathrm{fin}}_{\geq n} \ni E \to \mathbb{S}} \Phi_E(\Omega^\infty X) \simeq \lim_{E \to \mathbb{S}} L_{T(n)} X^E \simeq L_{T(n)} X$$

with which we are done. The second equivalence uses the previous Lemma (8.6). □

8.3 Consequences: Higher Semiadditivity

We will talk about higher semiadditivity and the relevance to those monochromatic categories.

Definition 8.8. Let $A \in \mathcal{S}$. We call it ***m*-finite** with $m \geq -2$ if:

(i) $m = -2$: if $A = *$,

(ii) $m = -1$: if $A \in \{*, \varnothing\}$,

(iii) $m \geq 0$: if $\pi_i A$ is finite for all i and $\pi_i A = 0$ for all $i > m$.

Example 8.9. Let G be a finite group. Then, BG is 1-finite.

Definition 8.10. Let $\mathscr{C}$ be a presentable ∞-category. We call $\mathscr{C}$ ***m*-semiadditive** if it is $(m-1)$-semiadditive and have natural (specified) equivalence

$$\mathrm{Nm}_A : \operatorname*{colim}_A X \xrightarrow{\simeq} \lim_A X$$

for every $X \in \mathrm{Fun}(A, \mathscr{C})$ and m-semiadditive A.

We define it inductively since all arguments in this game are usually done inductively anyway and one needs the induction hypothesis to define the norm map. On the other hand, Ishan said that one could also have just stated that there exists an equivalence because if an equivalence exists here, then it is already the preferred equivalence.

Remark 8.11.

(i) Any $\mathscr{C} \in \mathbf{Pr}^L$ is (-2)-semiadditive.

(ii) If $\mathscr{C} \in \mathbf{Pr}^L$ is $(m-1)$-semiadditive, there is a preferred map

$$\mathrm{Nm}_A : \operatorname*{colim}_A X \to \lim_A X$$

for all $A \in \mathcal{S}^{m\text{-fin}}$. In particular, m-semiadditivity is a property of $\mathscr{C}$.

(iii) For $m = -1$ the condition is

$$\operatorname*{colim}_{\varnothing} X \simeq \varnothing_{\mathscr{C}} \xrightarrow{\simeq} *_{\mathscr{C}} \simeq \lim_{\varnothing} X,$$

i.e. $\mathscr{C}$ being pointed.

(iv) For $m = 0$ consider $A \in \mathcal{S}^{0\text{-fin}} \simeq \mathbf{Fin}$. Consider $X \in \mathscr{C}^A$ given as $\{X_a\}_{a \in A}$, then the semiadditivity condition is

$$\mathrm{Nm}_A : \operatorname*{colim}_A X \simeq \coprod_{a \in A} X_a \xrightarrow{\simeq} \prod_{a \in A} X_a \simeq \lim_{a \in A} X_a.$$

This map is essentially

$$\begin{pmatrix} \mathrm{id} & & \\ & \ddots & \\ & & \mathrm{id} \end{pmatrix} \in \prod_{(a,b) \in A \times A} \mathrm{Map}(X_a, X_b) \simeq \mathrm{Map}\left(\coprod_A X_a, \prod_A X_a\right).$$

More specifically, $\mathrm{Nm}_A^{(a,b)}$ is $\mathrm{id} : X_a \to X_a$ for $a = b$ and it is

$$\begin{tikzcd} X_a \arrow[r] \arrow[rr, bend right, "0"'] & *_{\mathscr{C}} \simeq \emptyset_{\mathscr{C}} \arrow[r] & X_b \end{tikzcd}$$

We do this all very explicitly here to generalize.

(v) Consider $m = 1$ and $A = BG$. Let $X \in \mathscr{C}^{BG}$, then informally

$$\mathrm{Nm}_{BG} : X_{hG} \to X^{hG},\ [x] \mapsto \sum_{g \in G} gx.$$

This morally says that group cohomology agrees with group homology in such categories.[12]

(vi) Check that $\mathscr{C} = \mathbf{Sp}_{\mathbb{Q}}$ is 1-semiadditive. This follows from a derived Maschke's lemma. On the other hand, $\mathbf{Mod}_{\mathbb{F}_2}$ or $\mathbf{Sp}$ is not 1-semiadditive. This already fails for C_2. Florian said that this is a good exercise to try.

(vii) The ∞-category $\mathbf{Pr}^L$ is ∞-semiadditive.

Theorem 8.12 (Hopkins-Lurie, Carmeli-Schlank-Yanovski)**.** The ∞-categories $\mathbf{Sp}_{K(n)}$ and $\mathbf{Sp}_{T(n)}$ are ∞-semiadditive.

A lot of this reduces to the case $B^n C_p$ and then you input some Morava K-theory computation of these spaces. The 1-semiadditivity case has a fun proof using the Bousfield-Kuhn functor. Very roughly, one uses that $\Omega^{\infty+1}\Sigma^\infty BG \to \Omega^{\infty+1}\mathbb{S}$ has a retract.

This says that the representation theory is a bit degenerate; cohomology agrees with homology.

9 Definition and Examples of Power Operations (Azélie Picot)

9.1 Intro

TALK 9
02.07.2025

Let $E \in \mathbf{Alg}_{\mathbb{E}_k}(\mathbf{Sp})$, then $\pi_\bullet E$ should inherit some algebraic structure from the $\mathbb{E}_k$-structure. Those will be power operations.

Application 9.1. These operations are obstructions to the existence of certain multiplicative structure.

Example 9.2.

(i) Consider $\mathbb{S} \xrightarrow{2} \mathbb{S} \to \mathbb{S}/2$, then a power operation argument shows that $\mathbb{S}/2$ is not an $\mathbb{E}_1$-ring spectrum.

(ii) With power operations one can show that $K(n)$ is not $\mathbb{E}_\infty$.

[12] In the case $\mathscr{C} = \mathbf{Sp}$ one could take $X = HM$ where M is a G-representation.

9.2 First Examples of Operations

Setup

The setup is an operad $\mathscr{O}$, e.g. $\mathscr{O} = \mathbb{E}_n$ or $\mathscr{O} = \mathbb{E}_\infty$. Let $E \in \mathbf{Alg}_{\mathbb{E}_\infty}(\mathbf{Sp})$, e.g. $E = \mathbb{S}$ or $E = H\mathbb{F}_p$. We work in $\mathbf{Alg}_{\mathscr{O}}(\mathbf{Mod}_E)$. Consider the following free-forgetful adjunctions:

$$\mathbf{Sp} \underset{\longleftarrow}{\xrightarrow{\mathrm{Sym}_{\mathscr{O}}(-)}} \mathbf{Alg}_{\mathscr{O}}(\mathbf{Sp}) \underset{\longleftarrow}{\xrightarrow{E\otimes -}} \mathbf{Alg}_{\mathscr{O}}(\mathbf{Mod}_E)$$

So the top composite is $E \otimes \mathrm{Sym}_{\mathscr{O}}(-)$. Note here that $\mathbf{Mod}_E$ is symmetric monoidal if E is $\mathbb{E}_\infty$. Moreover, recall that there is an explicit formula

$$\mathrm{Sym}_{\mathscr{O}}(X) \simeq \bigoplus_{k\geq 0} \left(\Sigma^\infty_+ \mathscr{O}(k) \otimes X^{\otimes k}\right)_{h\Sigma_k} = \bigoplus_{k \geq 0} \mathrm{Sym}^{(k)}_{\mathscr{O}}(X)$$

for the free object.

Action of $\mathscr{O}$ on Homology

We will see our first operations here, most prominently the *Browder bracket* which in some sense will obstruct the existence of higher commutative structure.

Proposition 9.3. Let E be an $\mathbb{E}_\infty$-ring spectrum and $\mathscr{O}$ be an operad.

(i) Then, $E_\bullet\mathscr{O}$ is an operad in $\mathbf{Mod}_{E_\bullet}$.

(ii) For $X \in \mathbf{Alg}_{\mathscr{O}}(\mathbf{Mod}_E)$, then $E_\bullet\mathscr{O}$ acts on $E_\bullet X$, i.e. $E_\bullet X$ is an $E_\bullet\mathscr{O}$-algebra.

Example 9.4. Let $\mathscr{O} = \mathbb{E}_n$ for $n \geq 1$ and $E = \mathbb{S}$. Recall $\mathbb{E}_n(2) = \mathrm{Conf}_2(\mathbb{R}^n) \simeq S^{n-1}$.

(i) Let $X \in \mathbf{Alg}_{\mathbb{E}_n}(\mathbf{Sp})$. Then, $\pi_0\mathbb{E}_n$ acts on $\pi_\bullet X$. In particular,

$$\pi_0\mathbb{E}_n = \begin{cases} \mathbf{Assoc} & n = 1, \\ \mathbb{C}\mathrm{omm} & n \geq 2. \end{cases}$$

This implies that $\pi_\bullet X$ is an associative resp. commutative algebra.

(ii) Browder bracket: Let $\lambda \in \pi_{n-1}S^{n-1} \cong \pi_{n-1}\mathrm{Conf}_2(\mathbb{R}^n)$ be a generator. Suppose that $\alpha_1 \in \pi_{a_1}(X)$ and $\alpha_2 \in \pi_{a_2}(X)$ are represented by maps $\mathbb{S}^{a_i} \to X$. Consider

$$\mathbb{S}^{a_1+(n-1)+a_2} \simeq \mathbb{S}^{n-1} \otimes \mathbb{S}^{a_1} \otimes \mathbb{S}^{a_2} \to \Sigma^\infty_+\mathbb{E}_n(2) \otimes X \otimes X \to X.$$

This is denoted by $[\alpha_1, \alpha_2] \in \pi_{a_1+(n-1)+a_2}(X)$ is called the **Browder bracket** of α_1 and α_2.

Proposition 9.5. For every $\mathbb{E}_n$-algebra X this generator $\lambda \in \pi_{n-1}(\mathbb{E}_n(2))$ defines a bilinear bracket

$$[-,-] : \pi_{a_1}(X) \otimes \pi_{a_2}(X) \to \pi_{a_1+(n-1)+a_2}(X).$$

It satisfies the following properties:

(i) Symmetry: $[\alpha, \beta] = (-1)^{(|\alpha|+n-1)(|\beta|+n-1)}[\beta, \alpha]$,

(ii) Leibniz: $[\alpha, \beta\gamma] = [\alpha, \beta]\gamma + (-1)^{|\beta|(|\alpha|+n-1)}\alpha[\beta, \gamma]$,

(iii) Jacobi: $0 = (-1)^?[\alpha, [\beta, \gamma]] + (-1)^?[\beta, [\gamma, \alpha]] + (-1)^?[\gamma, [\alpha, \beta]]$.

Remark 9.6. For $n = 1$ we have $[\alpha, \beta] = \alpha\beta - (-1)^{|\alpha||\beta|}\beta\alpha$.

9.3 Power Operations

Let's finally define operations.

Definition 9.7.

(i) Let $m, n \geq 0$. Then, a **homotopy operation** in $\mathbf{Alg}_{\mathscr{O}}(\mathbf{Mod}_E)$ is a natural transformation $\pi_m \Rightarrow \pi_n$.

(ii) It induces a homology operation $E_m \Rightarrow E_n$. Denote by $\mathbf{Op}^E_{\mathscr{O}}(m,n)$ the group of these operations.

(iii) There is a variation of natural transformations $\prod_{i=1}^k \pi_{m_i} \Rightarrow \pi_n$ which leads to the multiary version $\mathbf{Op}^E_{\mathscr{O}}(m_1, \cdots, m_k, n)$.

Classification

Proposition 9.8. In $h\mathbf{Alg}_{\mathscr{O}}(\mathbf{Mod}_E)$ we have natural isomorphisms

$$\pi_m(A) \cong [E \otimes \mathrm{Free}_{\mathscr{O}}(\mathbb{S}^m), A]_{\mathbf{Alg}_{\mathscr{O}}(\mathbf{Mod}_E)}.$$

Proof. This follows from the free-forgetful adjunction. □

In particular, π_m is corepresentable, so we can apply the Yoneda Lemma to compute $\mathbf{Op}^E_{\mathscr{O}}(m,n)$.

Corollary 9.9. There is an isomorphism $\mathbf{Op}^E_{\mathscr{O}}(m,n) \cong \pi_n(E \otimes \mathrm{Free}_{\mathscr{O}}(\mathbb{S}^m))$ and

$$\mathbf{Op}^E_{\mathscr{O}}(m_1, \cdots, m_k, n) \cong \pi_n \left(E \otimes \mathrm{Free}_{\mathscr{O}}(\mathbb{S}^{m_1} \oplus \cdots \oplus \mathbb{S}^{m_k})\right).$$

Proof. This follows from the Yoneda Lemma. □

Example 9.10. Browder brackets come from a class $[\iota_1, \iota_2] \in \mathbf{Op}^{\mathbb{S}}_{\mathbb{E}_n}(p, q, p + (n-1) + q)$.

Power Operations

Definition 9.11. The group of **power operations of weight** k on degree m in $\mathbf{Alg}_{\mathscr{O}}(\mathbf{Mod}_E)$ is

$$\mathrm{Pow}^E_{\mathscr{O}}(m,k) = \bigoplus_{r \in \mathbb{Z}} \pi_{m+r}\left(E \otimes (\Sigma^\infty_+ \mathscr{O}(k) \otimes (\mathbb{S}^m)^{\otimes k})_{h\Sigma_k}\right) = \bigoplus_{r \in \mathbb{Z}} \pi_{m+r}\left(E \otimes \mathrm{Sym}^{(k)}_{\mathscr{O}}(\mathbb{S}^m)\right)$$

In practice, $Q : \mathbb{S}^{m+r} \to E \otimes \mathrm{Sym}^{(k)}_{\mathscr{O}}(\mathbb{S}^m)$. Take an $\mathscr{O}$-algebra A and $x : \mathbb{S}^m \to E \otimes A$. Then,

$$Q(x) : \mathbb{S}^{m+r} \xrightarrow{Q} E \otimes \mathrm{Sym}^{(k)}_{\mathscr{O}}(\mathbb{S}^m) \xrightarrow{\overline{x}} E \otimes A$$

where $\overline{x}$ corresponds to x by adjunction.

9.4 Examples

Dyer-Lashof Operations

Let $\mathscr{O} = \mathbb{E}_n$ and $E = H\mathbb{F}_p$. These are also called *Araki-Kudo operations* for $p = 2$.

Theorem 9.12. The $\mathbb{E}_n$-algebras in $\mathbf{Mod}_{H\mathbb{F}_2}$ have Dyer-Lashof operations $Q_i : \pi_m \Rightarrow \pi_{2m+i}$ for $0 \leq i \leq n-1$ such that:

(i) Additivity: $Q_r(x+y) = Q_r(x) + Q_r(y)$ for $r < n-1$,

(ii) Square: $Q_0(x) = x^2$,

(iii) Unit: $Q_j(1) = 0$ for $j > 0$.

(iv) Cartan Formula: $Q_r(xy) = \sum_{p+q=r} Q_p(x)Q_q(y)$ for $r < n-1$.

(v) Adem relations: $Q_rQ_s(x) = \sum_j \binom{j-s-1}{2j-r-s} Q_{r+2s-2j}Q_j(x)$ for $r > 0$.

(vi) Stability: $\sigma Q_0 = 0$, $\sigma Q_r = Q_{r-1}$ for $r > 0$,

(vii) Extension: Q_r-operations for $\mathbb{E}_n$-algebras coincide with the ones for an extension to $\mathbb{E}_{n+1}$.

(viii) Compatibility with the Browder bracket $[-,-]$: We have

- $[x, Q_r y] = 0$ for $r < n-1$,
- $Q_{n-1}(x+y) = Q_{n-1}(x) + Q_{n-1}(y) + [x,y]$,
- $Q_{n-1}(xy) = \sum Q_p(x)Q_q(y) + x[x,y]y$,
- $[x, Q_{n-1}y] = [y,[y,x]]$,
- Can extend an $\mathbb{E}_n$-algebra to an $\mathbb{E}_{n+1}$-algebra if bracket is 0.

Azélie: *I know these are a lot!*

Remark 9.13.

(i) Dyer-Lashof operations live in weight 2.

(ii) Be careful with indexing in the literature: Q_i vs. Q^i.

$K(1)$-Local Power Operations

We work in $\mathbf{Alg}_{\mathbb{E}_\infty}(\mathbf{Sp}_{K(1)})$.

Theorem 9.14 (McClure)**.** There is an isomorphism $\pi_0 \operatorname{Free}_{\mathbb{E}_\infty}^{\mathbf{Sp}_{K(1)}}(\{x\}) \cong (\operatorname{Free}_\delta(\{x\}))_p^\wedge$.[13]

This will be discussed in Preston's talk (15).

Definition 9.15 (Joyal)**.** A **δ-ring** is a ring R together with a unary operation $\delta : R \to R$ such that the *Witt formulas*

$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y) \quad \text{and} \quad \delta(x+y) = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i}$$

and $\delta(0) = 1$ are satisfied.

This is equivalent to admitting a lift of the Frobenius:

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & R \arrow[d] \\ R/p \arrow[r, "(-)^p"'] & R/p \end{tikzcd}$$

We can put $\varphi(x) = x^p + p\delta(x)$, and the δ-ring structure gives that φ is a ring homomorphism.

10 Computing the Dyer-Lashof Algebra (Jordan Levin)

TALK 10 02.07.2025

Everytime the words Dyer-Lashof algebra falls, people reference the same book over and over again which is arguably difficult to understand. We present an argument communicated by Ishan. We want to compute the homotopy groups (rings) of $\mathrm{Free}^{\mathbb{F}_p}_{\mathbb{E}_n}(\Sigma^t\mathbb{F}_2)$. Let's make some assumptions. We put $p = 2$ (and $p > 2$ is slightly harder).

10.1 Setup

Claim 10.1. We can use pure algebra to get these rings.

How?

- Koszul duality: Consider the equivalence

$$\mathrm{Bar} : \mathbf{Alg}^{\mathrm{cn,aug}}_{\mathbb{E}_k}\left(\mathbf{Mod}^{\mathrm{gr}}_{\mathbb{F}_2}\right) \rightleftarrows \mathbf{coAlg}^{\mathrm{cn,coaug}}_{\mathbb{E}_1}\left(\mathbf{Alg}_{\mathbb{E}_{k-1}}(\mathbf{Mod}^{\mathrm{gr}}_{\mathbb{F}_2})\right) : \mathrm{Cobar}\,.$$

  Here, cn means connected, i.e. non-positively graded. (This includes the unit.)

- Spectral sequence arguments.

Fact 10.2. Let $M(n)$ be the $\mathbb{F}_2$-module M in degree n. There is an equivalence

$$\mathrm{Bar}\,\mathrm{Free}_{\mathbb{E}_k}(M(n)) \simeq \mathrm{Free}_{\mathbb{E}_{k-1}}(\Sigma M(n))$$

underlying (there is also some $\mathbb{E}_1$-coalgebra data).

Strategy: Suppose we knew $\mathrm{Free}_{\mathbb{E}_k}(\Sigma\mathbb{F}_2(1))$ with coalgebra structure. Then,

$$\mathrm{Cobar}\,\mathrm{Free}_{\mathbb{E}_k}(\Sigma\mathbb{F}_2(1)) \simeq \mathrm{Free}_{\mathbb{E}_{k+1}}(\mathbb{F}_2(1)).$$

Goal: Explain the passage from $\mathrm{Free}_{\mathbb{E}_1}(-)$ to $\mathrm{Free}_{\mathbb{E}_2}(-)$ using this sort of inductive argument. We will setup the proof to work for arbitrary $\mathbb{E}_n$ to $\mathbb{E}_{n+1}$, up to some easy modifications.

Remark 10.3. There is an equivalence[14]

$$\left(\mathrm{Free}_{\mathbb{E}_k}(\Sigma^t\mathbb{F}_2(1))\right)_w \simeq (\mathbb{E}_k(w)\otimes\Sigma^{tw}\mathbb{F}_2)_{h\Sigma_w} = \mathrm{Sym}^{\mathbb{E}_k}_w(\Sigma^t\mathbb{F}_2).$$

No information about these free algebras is lost or gained by passage to the graded setting.

10.2 Spectral Sequence Argument

Now to $\mathbb{E}_1 \rightsquigarrow \mathbb{E}_2$ and look at $\mathrm{Free}_{\mathbb{E}_2}(\mathbb{F}_2(1))$. By the bar-cobar equivalence we need to know something about

$$\mathrm{Free}_{\mathbb{E}_1}(\Sigma\mathbb{F}_2(1)) \simeq \bigoplus_{n\geq 0}\Sigma^n\mathbb{F}_2(n).$$

We think of this as a polynomial algebra. In fact, $\pi_\bullet\,\mathrm{Free}_{\mathbb{E}_1}(\Sigma\mathbb{F}_2(1)) \cong \mathbb{F}_2[x]$ with $|x| = (1,1)$ where the bidegree is (weight, topological degree). Let us consider $\mathrm{Cobar}(\mathrm{Free}_{\mathbb{E}_1}(\Sigma\mathbb{F}_2(1)))$ whose Postnikov tower gives rise to a spectral sequence

$$\mathrm{Cotor}^i_{\mathbb{F}_2[x]}(\mathbb{F}_2,\mathbb{F}_2) \Rightarrow \pi_{\bullet-i}\,\mathrm{Cobar}(\mathrm{Free}_{\mathbb{E}_1}(\Sigma\mathbb{F}_2(1)))$$

[13] Here, $\mathrm{Free}^{\mathbf{Sp}_{K(1)}}_{\mathbb{E}_\infty}(\{x\}) = \mathrm{Free}^{\mathbf{Sp}_{K(1)}}_{\mathbb{E}_\infty}(\mathbb{S})$.

[14] Note that $\Sigma^{tw}\mathbb{F}_2 \simeq (\Sigma^t\mathbb{F}_2)^{\otimes w}$ is used.

Construction 10.4 (Reminder). Recall that $M \otimes_A -$ is in general right-exact, so there exist left derived functors $\mathrm{Tor}^A(-,-)$. Similarly for comodules there is a cotensor product $M\square_A-$ which is left-exact, so there is a right derived functor $\mathrm{Cotor}_A(-,-)$.

On $\mathbb{F}_2[x] = \pi_\bullet \mathrm{Free}_{\mathbb{E}_1}(\Sigma\mathbb{F}_2(1))$ we have $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\Delta(x^n) = \Delta(x)^n$ and

$$\Delta(x^{2^n}) = x^{2^n} \otimes 1 + 1 \otimes x^{2^n}$$

by thinking about the binomial formula. As a coalgebra we can decompose

$$\mathbb{F}_2[x] = \bigotimes_{n=0}^{\infty} \mathbb{F}_2[x^{2^n}]/(x^{2^{n+1}})$$

and infinite tensor products are okay here since we are in the graded setting and every graded part is finitely generated. Given this decomposition we get

$$\mathrm{Cotor}^\bullet_{\mathbb{F}_2[x]}(\mathbb{F}_2, \mathbb{F}_2) \cong \bigotimes_{m=0}^{\infty} \mathrm{Cotor}^\bullet_{\mathbb{F}_2[x^{2^m}]/(x^{2^{m+1}})}(\mathbb{F}_2, \mathbb{F}_2) \cong \bigotimes_{m=0}^{\infty} \mathbb{F}_2[y_m].$$

So the $\mathbb{E}_2$-page of the spectral sequence is $\mathbb{F}_2[y_0, y_1, \cdots]$.

Remark 10.5. If y_m survived the spectral sequence, it would give some class in $\pi_\bullet$ Cobar in bidegree $(2^m, 2^m - 1)$.

Claim 10.6. There are no differentials in the spectral sequence, so $E_2 = E_\infty$.

Strategy: Look at what needs to appear in weight 2 and propagate using some 'naturality'.

Note that the weight 2 part of $\mathrm{Free}_{\mathbb{E}_2}(\mathbb{F}_2(1))$ is

$$(\mathbb{E}_2(2) \otimes \mathbb{F}_2)_{h\Sigma_2} \simeq \mathbb{E}_2(2)_{h\Sigma_2} \otimes \mathbb{F}_2 \simeq \mathrm{Conf}_2^{\mathrm{ord}}(\mathbb{R}^2)_{h\Sigma_2} \otimes \mathbb{F}_2 \simeq \mathbb{R}P^{2-1} \otimes \mathbb{F}_2$$

which has a copy of $\mathbb{F}_2$ in degrees 0 and 1. In the spectral sequence we can identify these with y_0^2 resp. y_1 where $y_0^2 = Q_0(y_0)$. We will think of y_1 as $Q_1(y_0)$. So essentially the E_2-page looks like $E_2 = \mathbb{F}_2[y_0, Q_1 y_0, y_2, \cdots]$. In fact, $y_i = Q_1^{(i)} y_0$. Indeed, one can 'unpack' the spectral sequence and see it. Namely, $x^2 \in \mathbb{F}_2[x]$ in the cobar complex really needed to represent $Q_1 y_0$. Essentially $(x')^2$ leads to $Q_1(-)$. We want to use a naturality argument as follows: Identify $y_2 = Q_1 Q_1 y_0$. Consider

$$\mathrm{Free}_{\mathbb{E}_2}(Q_1(x)(2)) \to \mathrm{Free}_{\mathbb{E}_2}(x(1)).$$

Apply Bar everywhere to get a comodule map which on $\pi_\bullet$ sends $Q_1 Q_1(x) \mapsto y_2$. Moreover, by naturality, y_2 cannot admit any differentials. Continue to get $y_i = Q_1^{(i)}(y_0)$. We get

$$E_2 = \mathbb{F}_2[y_0, Q_1(y_0), Q_1^{(2)}(y_0), \cdots].$$

This recovers the result:

Theorem 10.7. The algebra $\mathrm{Free}_{\mathbb{E}_2}(\mathbb{F}_2)$ is the polynomial algebra on the so-called admissible sequences $Q_1 \circ Q_1 \circ \cdots$.

Remark 10.8. More generally, $\mathrm{Free}_{\mathbb{E}_n}(\mathbb{F}_2)$ is the polynomial algebra on the admissible sequences $Q_1^{(e_1)} \circ Q_2^{(e_2)} \circ \cdots \circ Q_{n-1}^{(e_{n-1})}$.

10 ½ Multiplication on $\mathrm{BP}\langle n\rangle$ (Ryan Quinn)

Ryan: *This is very cool stuff, in particular non-formal. It feels like magic!* TALK 10.5 02.07.2025

Our magician then proceeds to ask the audience (Catherine) to take a card from his deck and leave it in front of her desk. You have to keep reading to see the trick being reveiled.

I'm also grateful to Ryan for explaining many concepts in his talk to me very patiently! His explanations worked like magic...

10 ½.1 Recap on BP

Recall that BP was built to be computable. We have

$$\pi_\bullet \mathrm{BP} = \mathbb{Z}_{(p)}[v_1, v_2, \cdots] \quad \text{and} \quad H_\bullet \mathrm{BP} = P_\bullet \subseteq \mathcal{A}_*.$$

Remark 10 ½.1. These have nice $\pi_\bullet$ and nice $H_\bullet$. That's a rarity, as usually there is a trade-off between nice homotopy vs. nice homology.

We know that there are maps $\mathrm{MU} \to \mathrm{BP} \to \mathrm{BP}\langle n\rangle$ and $\pi_\bullet(\mathrm{BP}\langle n\rangle) \cong \mathbb{Z}_p[v_1, \cdots, v_n]$.

Example 10 ½.2. These $\mathrm{BP}\langle n\rangle$ recover many familiar examples like:

(-1) $\mathrm{BP}\langle -1\rangle \simeq \mathbb{F}_p$,

(0) $\mathrm{BP}\langle 0\rangle \simeq \mathbb{Z}_{(p)}$,

(1) $\mathrm{BP}\langle 1\rangle \simeq \mathrm{ku} \simeq \mathrm{ko} \otimes C(\eta)$,[15]

(2) $\mathrm{BP}\langle 2\rangle = \mathrm{tmf}_1(3) \simeq \mathrm{tmf} \otimes DA(1)$.

The equivalence $\mathrm{ku} \simeq \mathrm{ko} \otimes C(\eta)$ is Wood's theorem and $\mathrm{tmf}_1(3) \simeq \mathrm{tmf} \otimes DA(1)$ is a tmf-analog of Wood's theorem.

Note in particular that these all have finite presentedness hypotheses, i.e. their homology are finitely presented comodules over $\mathcal{A}_*$, also called *fp spectra* due to Mahowald-Rezk.

Conjecture 10 ½.3 (Hahn-Wilson)**.** There is an equivalence $\mathrm{Thick}(\mathrm{BP}\langle n\rangle) \simeq \{\text{fp type n spectra}\}$.

10 ½.2 Multiplicative Structures

This has an interesting history. At first Kriz showed that BP has an $\mathbb{E}_\infty$-structure but it turned out to be wrong. However, it still led to interesting math.

Positive Results	Negative Results
Chadwick-Mandell: BP is $\mathbb{E}_2$.	Hu-May-Kriz: MU is not $\mathbb{E}_\infty$-BP.
Basterra-Mandell: BP is $\mathbb{E}_4$.	Lawson, Senger: $\mathrm{BP}\langle n\rangle$ is not $\mathbb{E}_{2(p^2+2)}$.
Hahn-Wilson: $\mathrm{BP}\langle n\rangle$ is $\mathbb{E}_3$-MU.	

Remember the magic trick? Guess what happened for Hahn-Wilson.

Card Reveal: 3!

Someone from the audience (Jordan?): *Where did you get 12 decks of cards?*

Here are some applications:

[15] This is for $p = 2$. In general, it is the Adams summand ℓ.

- Hahn-Wilson: Redshift at heights.
- Burklund-Hahn-Levy-Schlank: $L_{T(n)} \not\simeq L_{K(n)}$ for $n \geq 2$.

Ryan: *As I learned from the question session* $L_{T(n)} \not\simeq L_{K(n)}$.[16]

This is a pun of course. Funnily, there was a typo on the blackboard saying $L_{T(n)} \simeq L_{K(n)}$. Ishan said that it should say $\not\simeq$!

Ryan: *I didn't learn anything after all!*

10 ½.3 Prerequisites

Consider

$$k[x] \underset{\mathrm{Ext}_{\Lambda(\sigma x)}(k,k)}{\overset{\mathrm{Tor}^{k[x]}(k,k)}{\rightleftarrows}} \Lambda(\sigma x) \underset{\mathrm{Ext}_{\Gamma(\sigma^2 x)}(k,k)}{\overset{\mathrm{Tor}^{\Lambda(\sigma x)}(k,k)}{\rightleftarrows}} \Gamma(\sigma^2 x).$$

Example 10 ½.4. On the topological side we have

$$\begin{aligned} H_\bullet(\mathrm{Free}^{\mathbb{E}_1}(S^2)) &\cong k[x], \\ H_\bullet(\Sigma S^2) &\cong \Lambda(\sigma x), \\ H_\bullet(\mathbb{H}P^\infty) &\cong \Gamma(\sigma^2 x). \end{aligned}$$

Here, we use $S^3 \simeq \Omega\mathbb{H}P^\infty$ by some Hopf invariant one thing.

But a bit of black magic now happens (not Ryan's words), namely bar'ing yields

$$H_\bullet(BU) \rightsquigarrow H_\bullet(\mathrm{SU}) \rightsquigarrow H_\bullet(\mathrm{BSU})$$

which turns out to be polynomial again, so we can start again!

Remark 10 ½.5. The point here is that bar'ing on the topological side corresponds to bar'ing algebraically, namely $\mathbb{Z}[BX] \simeq \mathbb{Z} \otimes_{\mathbb{Z}[X]} \mathbb{Z}$. There is some subtlety going on here that Ryan and I are confused about; namely the statement does not seem to be true unless one takes $\otimes = \otimes_{\mathbb{Z}}$ on the right side. Is this really what one wants?

10 ½.4 Rough Sketch

The idea is to inductively build $\mathrm{BP}\langle n+1\rangle$ out of $\mathrm{BP}\langle n\rangle$. Algebraically, we have

$$\mathbb{Z}[v_1, \cdots, v_{n+1}] \cong \mathrm{Ext}_{\Lambda_{\mathrm{MU}_\bullet}(\delta v_{n+1})}(\mathrm{MU}_\bullet, \mathbb{Z}[v_1, \cdots, v_n])$$

so we are hoping for

$$\mathrm{BP}\langle n+1\rangle \simeq \mathrm{map}_{\Lambda_{\mathrm{MU}}(\delta v_{n+1})}\left(\mathrm{MU}, \mathrm{BP}\langle n\rangle\right).$$

We formulate the following goals:

- Make $\Lambda_{\mathrm{MU}}(\delta v_{n+1})$ precise: This will come from bar'ing a polynomial algebra as above.
- Show how to give $\mathbb{E}_n$-A-structures.

[16] See B.1(iii) to understand the joke.

10 ½.5 Enveloping Algebras

Let B be an $\mathbb{E}_n$-A-algebra, i.e. $B \in \mathbf{Alg}_{\mathbb{E}_n}(\mathbf{LMod}_A)$.

Definition 10 ½.6. Write $\mathcal{U}_A^{(1)}(B) = B \otimes_A B^{\mathrm{op}}$ and inductively $\mathcal{U}_A^{(n)}(B) = B \otimes_{\mathcal{U}_A^{(n-1)}(B)} B^{\mathrm{op}}$ for the **universal enveloping algebra**.

Remark* 10 ½.7.

(i) The universal enveloping algebra is the endomorphism algebra $\mathcal{U}_A^{(n)}(B) = \mathrm{end}(\mathrm{Free}^B_{\mathbb{E}_n}(A))$ [HW22, Proposition 2.2.2].

(ii) There is an equivalence $\mathbf{Mod}_B^{\mathbb{E}_n}(\mathbf{LMod}_A) \simeq \mathbf{LMod}_{\mathcal{U}_A^{(n)}(B)}$ by a Schwede-Shipley argument [Lur17, Theorem 7.1.2.1]. As such, this recovers that $\mathbf{Mod}_B^{\mathbb{E}_1}(\mathbf{LMod}_A)$ consists of bimodules.

Remark 10 ½.8. There is a formula in terms of factorization homology

$$\mathcal{U}_A^{(n)}(B) \simeq \int_{\mathbb{R}^n - \{0\}} B \simeq \int_{S^{n-1}} B.$$

This stuff might need high enough $\mathbb{E}_n$.

Example 10 ½.9. Let $n = 2$. Then, $\mathcal{U}_A^{(2)}(B) \simeq \mathrm{THH}(B/A) \simeq B \otimes_{B \otimes_A B^{\mathrm{op}}} B^{\mathrm{op}} \simeq \int_{S^1} B$.

10 ½.6 $\mathbb{E}_k$-Centers

Recall that M as an R-module classically is a ring map $R \to \mathrm{End}(M)$. Let $\mathscr{C}^{\otimes}$ be a stable presentably symmetric monoidal ∞-category.

Definition 10 ½.10. The **$\mathbb{E}_n$-A-center** of B called $\mathfrak{Z}_{\mathbb{E}_n\text{-}A}(B) \in \mathbf{Alg}_{\mathbb{E}_{n+1}}(\mathscr{C})$ is defined by the universal property that an $\mathbb{E}_{n+1}$-map $R \to \mathfrak{Z}_{\mathbb{E}_n\text{-}A}(B)$ corresponds to realizing B as an $\mathbb{E}_n$-R-algebra.

Fact 10 ½.11. Let $f : B_1 \to B_2$. Then, there exists an object $\mathfrak{Z}_{\mathbb{E}_n\text{-}A}(f)$, called **centralizer**.

(i) For $f = \mathrm{id}_B$ we have $\mathfrak{Z}_{\mathbb{E}_n\text{-}A}(\mathrm{id}_B) \simeq \mathfrak{Z}_{\mathbb{E}_n\text{-}A}(B)$.

(ii) The center $\mathfrak{Z}_{\mathbb{E}_n\text{-}A}(f)$ has underlying object $\mathrm{map}_{\mathcal{U}_A^{(n)}(B_1)}(B_1, B_2)$ in $\mathscr{C}$.

Remark 10 ½.12. While the centralizer is more general and functorial, the center still has its value. Indeed, the centralizer is in general $\mathbb{E}_n$ while the center is $\mathbb{E}_{n+1}$.

Remark* 10 ½.13. By 10 ½.7(ii) we have $\mathfrak{Z}_{\mathbb{E}_n\text{-}A}(f) \simeq \mathrm{map}_{\mathbf{Mod}_{B_1}^{\mathbb{E}_n}(\mathbf{LMod}_A)}(B_1, B_2)$.

Remark* 10 ½.14. These objects have the following universal properties:

(i) Let $f : R \to S$, then the centralizer is final in diagrams of the form

$$\begin{tikzcd} & \mathfrak{Z}_{\mathbb{E}_n\text{-}A}(f) \otimes R \arrow[dr] & \\ R \arrow[ur] \arrow[rr, "f"'] & & S \end{tikzcd}$$

in other words, $\mathfrak{Z}_{\mathbb{E}_n\text{-}A}(f) \in \mathbf{Alg}_{\mathbb{E}_n}(\mathscr{C})$ is terminal in

$$\mathbf{Alg}_{\mathbb{E}_n}(\mathscr{C}) \times_{\mathbf{Alg}_{\mathbb{E}_n}(\mathbf{LMod}_A)_{R/}} \mathbf{Alg}_{\mathbb{E}_n}(\mathbf{LMod}_A)_{R/\!/S}$$

where the left functor is given by $R \otimes -$ [Lur17, Definition 5.3.1.2].

(ii) Let $M \in \mathscr{C}$. The center $\mathfrak{Z}(M)$ is the terminal object in $\mathbf{Alg}_{\mathcal{LM}}(\mathscr{C}) \times_{\mathscr{C}} \{M\}$ [Lur17, Definition 5.3.1.6].

10 ½.7 Designer Polynomial Algebras

This has a precursor.

Fact 10 ½.15 (Lurie)**.** The free $\mathbb{E}_1$-algebra $\mathrm{Free}^{\mathbb{E}_1}(S^{2n})$ admits an $\mathbb{E}_2$-structure.

Proof Sketch. Consider

$$\mathbb{N} \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{S^{2(-)}} \mathrm{Pic}(\mathbf{Sp})$$

whose Thom spectrum is $\bigoplus_k S^{2nk}$. Lurie shows that this map is an $\mathbb{E}_2$-map, hence so is the respective Thom spectrum. He also shows that the underlying $\mathbb{E}_1$-algebra is $\mathrm{Free}^{\mathbb{E}_1}(S^{2n})$. □

Hahn-Wilson do a more structured version for BP.

Construction 10 ½.16. More specifically, we consider

$$\begin{tikzcd} & & \mathbb{Z}\times \mathrm{BU} \arrow[r, "J"] \arrow[d] & \mathrm{Pic}(\mathbf{Sp}) \\ \mathbb{N} \arrow[r] \arrow[urrr, bend right, "\mathrm{MU}[y]"'] & \mathbb{Z} \arrow[r, "n"'] & \mathbb{Z} \arrow[ur, dashed, "\mathrm{Lan}"'] & \end{tikzcd}$$

First, Lan is $\mathrm{MU}[\beta^{\pm 1}]$ with $|\beta| = 2$, a graded refinement of MUP. Restricting along $\mathbb{N} \to \mathbb{Z} \to \mathbb{Z}$ yields $\mathrm{MU}[y]$ with $|y| = 2n$. This is a refinement of the free $\mathbb{E}_1$-MU-algebra $\mathrm{MU}[y]$ and is an $\mathbb{E}_\infty$-MU-algebra since J is an $\mathbb{E}_\infty$-map.

Here are some facts from Koszul duality.

Fact 10 ½.17.

(i) There is an equivalence $\mathbb{D}^{(n)}\,\mathrm{MU}[y] \simeq \mathrm{Free}^{\mathrm{MU}}_{\mathbb{E}_{n-1}}(\Sigma^{-2k-n}\,\mathrm{MU})$ and it admits an $\mathbb{E}_n$-MU-algebra structure. It has even cells as an $\mathbb{E}_n$-algebra for even n.[17]

(ii) There is an equivalence $\mathcal{U}^{(n-1)}_{\mathbb{D}^{(n)}\,\mathrm{MU}[y]}(\mathrm{MU}) \simeq \mathrm{Bar}(\mathrm{MU}[y])$.

This $\mathbb{D}$ is essentially bar'ing and then taking duals. Indeed, bar'ing makes $\mathrm{MU}[y]$ into a coalgebra and we take duals to make it into an algebra again. That's also the reason those negative shifts appear. Since we are bar'ing a polynomial algebra, this is the exterior algebra analog that we were looking for.

Part (ii) of that fact is done a bit differently in the published version of [HW22], this argument is from version 2 on arXiv.

10 ½.8 Actual Strategy

Proceed by induction. The base case is $\mathrm{BP}\langle -1\rangle \simeq \mathbb{F}_p$. For the inductive case we assume that $\mathrm{BP}\langle n\rangle$ is an $\mathbb{E}_3$-MU-algebra.

- Compute $\pi_\bullet \mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)$. See that it is exterior on odd degrees.

[17] This $\mathbb{D}$ is Bar, then Koszul dual.

- Consider the spectral sequence

$$\mathrm{Ext}_{\pi_\bullet \mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)}\left(\mathrm{BP}\langle n\rangle_\bullet, \mathrm{BP}\langle n\rangle_\bullet\right) \Rightarrow \pi_\bullet\left(\mathfrak{Z}_{\mathbb{E}_3\text{-}\mathrm{MU}}(\mathrm{BP}\langle n\rangle)\right)$$

  and because $\pi_\bullet \mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)$ is exterior, we see that the Ext term is polynomial on even classes. Thus, the spectral sequence collapses.

- Find that there are no obstructions to being an $\mathbb{E}_3$-$\mathbb{D}^{(4)}(\mathrm{MU}[y])$-algebra.
- Check that $\mathrm{MU} \to \mathrm{BP}\langle n\rangle$ can similarly be upgraded.
- Check that $\pi_\bullet\left(\mathfrak{Z}_{\mathbb{E}_3\text{-}\mathbb{D}^{(4)}\,\mathrm{MU}[y]}(\mathrm{MU} \to \mathrm{BP}\langle n\rangle)\right) \cong \mathrm{BP}\langle n+1\rangle_\bullet$.

Remark 10 ½.18. View this backwards. Our goal is to obtain

$$\mathrm{BP}\langle n+1\rangle \simeq \mathfrak{Z}_{\mathbb{E}_3\text{-}\mathbb{D}^{(4)}\,\mathrm{MU}[y]}\left(\mathrm{MU} \to \mathrm{BP}\langle n\rangle\right)$$

but even to talk about this centralizer, we need to ensure that $\mathrm{MU} \to \mathrm{BP}\langle n\rangle$ is a map of $\mathbb{E}_3$-$\mathbb{D}^{(4)}\,\mathrm{MU}[y]$-algebras. So we should start with $\mathrm{BP}\langle n\rangle$ and we start caring about the easier $\mathbb{E}_3$-MU-structure first. So we need to understand $\mathfrak{Z}_{\mathbb{E}_3\text{-}\mathrm{MU}}(\mathrm{BP}\langle n\rangle)$ which can be understood from a spectral sequence since its underlying object is a mapping spectrum (10 ½.11).

The hard part is this computation of $\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)$ which we will focus on. This is done inductively and the base case is $\mathbb{F}_p$.

Proposition 10 ½.19. The spectrum $\mathcal{U}^{(2)}_{\mathrm{MU}}(\mathbb{F}_p) \simeq \mathrm{THH}(\mathbb{F}_p/\,\mathrm{MU})$ is polynomial on even degrees.

Proof. It's a general fact that about THH of Thom spectra that

$$\mathrm{THH}(\mathrm{MU}) \simeq \mathrm{MU} \otimes \Sigma^\infty_+ \mathrm{SU} \simeq \mathrm{MU}[\mathrm{SU}].$$

Moreover, recall Bökstedt's result $\mathrm{THH}(\mathbb{F}_p) \simeq \mathbb{F}_p[\Omega S^3]$ by which one can obtain from Hopkins-Mahowald $\mathbb{F}_p \simeq \mathrm{Th}(\Omega^2 S^2 \to \mathrm{BGL}_1(\mathbb{S}_p))$, and one can check that there is a (suitable) factorization

$$\begin{tikzcd} \mathrm{THH}(\mathrm{MU}) \arrow[rr] \arrow[dr] & & \mathrm{THH}(\mathbb{F}_p) \\ & \mathrm{MU} \arrow[ur] & \end{tikzcd}$$

This can be used to base change, so we can perform the following computation:

$$\begin{aligned} \mathrm{THH}(\mathbb{F}_p/\,\mathrm{MU}) &\simeq \mathrm{THH}(\mathbb{F}_p) \otimes_{\mathrm{THH}(\mathrm{MU})} \mathrm{MU} \\ &\simeq \mathbb{F}_p[\Omega S^3] \otimes_{\mathrm{MU}[\mathrm{SU}]} \mathrm{MU} \\ &\simeq (\mathbb{F}_p[\Omega S^3] \otimes_{\mathrm{MU}} \mathrm{MU}) \otimes_{\mathrm{MU}[\mathrm{SU}]} \mathrm{MU} \\ &\simeq \mathbb{F}_p[\Omega S^3] \otimes_{\mathrm{MU}} \mathrm{MU} \otimes (\mathbb{S} \otimes_{\mathbb{S}[\mathrm{SU}]} \mathbb{S}) \\ &\simeq \mathbb{F}_p[\Omega S^3] \otimes_{\mathrm{MU}} \mathrm{MU} \otimes \Sigma^\infty_+ \mathrm{BSU} \\ &\simeq \mathbb{F}_p[\Omega S^3 \times \mathrm{BSU}] \end{aligned}$$

This is the product of two spectra with even homotopy groups (and there is a Künneth formula for $\mathbb{F}_p$). $\square$

Remark* 10 ½.20. The base change argument is the following general higher algebra fact: Let

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \\ & C \arrow[ur] & \end{tikzcd}$$

a commutative diagram of maps of $\mathbb{E}_1$-algebras. Then, the diagram

$$\begin{tikzcd} \mathbf{RMod}_B \arrow[r, "\mathrm{Res}^B_C"] \arrow[d, "\mathrm{Res}^B_A"'] & \mathbf{RMod}_C \arrow[d, "-\otimes_C C"] \\ \mathbf{RMod}_A \arrow[r, equal] & \mathbf{RMod}_A \end{tikzcd}$$

commutes where we view $C \in {}_C\mathbf{BMod}_A$ via $A \to C$. On the other hand, I don't quite see why this is true at the moment.

Now $\mathcal{U}^{(1)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle) = \Lambda(\sigma \cdots)$ is an exterior algebra by a bar computation. It follows the following classical algebra fact:

Remark 10 ½.21. Let I be a regular sequence in R. Then, $\mathrm{Tor}_R(R/I, R/I) \cong \Lambda_{R/I}(\sigma x_i : x_i \in I)$ where σ is the shift/suspension operator. For example, $\mathrm{Tor}_{k[x,y]}(k,k) \cong \Lambda_k(\sigma x, \sigma y)$.

Thus, $\mathcal{U}^{(2)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)$ is a polynomial algebra which is even. This uses input from the $\mathbb{F}_p$-case, the $\mathbb{E}_\infty$-page is a divided power algebra as it should be but some power operations miraculously make it into a polynomial algebra. In particular, the $\mathbb{F}_p$-case is really needed throughout and does not only serve as the base step of the induction. Thus, we obtain $\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)$ which is exterior on odd degrees.

11 t-Structures, Filtered Objects, and Spectral Sequences (Lucy Grossman)

TALK 11
03.07.2025

11.1 t-Structures

We begin with t-structures which is a structure you can put on triangulated categories.

Definition 11.1. Let $\mathscr{D}$ be a triangulated category. A ***t*-structure** on $\mathscr{D}$ is a pair of full subcategories $\mathscr{D}_{\geq 0}, \mathscr{D}_{\leq 0} \subseteq \mathscr{D}$ such that:

(i) For $X \in \mathscr{D}_{\geq 0}, Y \in \mathscr{D}_{\leq 0}$ we have $\mathrm{Hom}_{\mathscr{D}}(X, Y[-1]) = 0$.

(ii) There are inclusions $\mathscr{D}_{\geq 0}[1] \subseteq \mathscr{D}_{\geq 0}$ and $\mathscr{D}_{\leq 0}[-1] \subseteq \mathscr{D}_{\leq 0}$.

(iii) For any $X \in \mathscr{D}$ there exists a cofiber sequence $X' \to X \to X''$ with $X' \in \mathscr{D}_{\geq 0}$ and $X'' \in \mathscr{D}_{\leq 0}[-1]$.

Definition 11.2. Let $\mathscr{C}$ be a stable ∞-category. A ***t*-structure** on $\mathscr{C}$ is a t-structure on $h\mathscr{C}$.

We will write $\mathscr{C}_{\geq 0} = h\mathscr{C}_{\geq 0}$ and $\mathscr{C}_{\leq 0} = h\mathscr{C}_{\leq 0}$.

Observation 11.3. There are truncation and connective cover functors sitting in adjunctions:

$$\begin{tikzcd} \mathscr{C}_{\geq 0} \arrow[r, shift left] & \mathscr{C} \arrow[l, shift left, "\tau_{\geq 0}"] \end{tikzcd} \qquad \begin{tikzcd} \mathscr{C} \arrow[r, shift left, "\tau_{\leq 0}"] & \mathscr{C} \arrow[l, shift left] \end{tikzcd}$$

Lemma 11.4. Let $m \leq n$. We write $\mathscr{C}_{[m,n]} = \mathscr{C}_{\geq m} \cap \mathscr{C}_{\leq n}$. Then, there are natural equivalences

$$\tau_{\geq m} \circ \tau_{\leq n} \simeq \tau_{\leq n} \circ \tau_{\geq m} \quad \text{and} \quad \Sigma \circ \tau_{\geq n} \simeq \tau_{\geq n+1} \circ \Sigma.$$

Definition 11.5. Let $\mathscr{C}$ hae a t-structure $(\mathscr{C}_{\geq 0}, \mathscr{C}_{\leq 0})$. The **heart** of this is $\mathscr{C}^{\heartsuit} = \mathscr{C}_{\geq 0} \cap \mathscr{C}_{\leq 0}$.

Definition 11.6. The functor $\pi_n : \mathscr{C} \to \mathscr{C}^{\heartsuit}$ is defined by

$$\Sigma^{-n} \circ \tau_{\geq n} \circ \tau_{\leq n} \simeq \Sigma^{-n} \circ \tau_{\leq n} \circ \tau_{\geq n} \simeq \tau_{\leq 0} \circ \tau_{\geq 0} \circ \Sigma^{-n} \simeq \tau_{\geq 0} \circ \tau_{\leq 0} \Sigma^{-n}.$$

Remark 11.7. If we have a fiber sequence $X \to Y \to Z$ we have an induced LES

$$\cdots \longrightarrow \pi_n X \longrightarrow \pi_n Y \longrightarrow \pi_n Z \longrightarrow \pi_{n-1} X \longrightarrow \cdots$$

11.2 Filtrations

A lot of this follows notes from the fabled Ben Antieau [Ant24]. There was a funny moment in the talk here (but that's a secret known only to the Talbot participants).

Definition 11.8. Let $\mathscr{C}$ be a stable ∞-category. A **filtered object** of $\mathscr{C}$ is a functor $X : \mathbb{Z} \to \mathscr{C}$.

Remark 11.9. The ∞-category of decreasing filtrations in $\mathscr{C}$ is $F\mathscr{C} = \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathscr{C})$ and the ∞-category of increasing filtrations in $\mathrm{Fun}(\mathbb{Z}, \mathscr{C})$.

We write a decreasing filtration $F^\bullet$ as an infinite sequence

$$\cdots \longrightarrow F^{s+1} \longrightarrow F^s \longrightarrow F^{s-1} \longrightarrow \cdots$$

A **decreasing filtration** on X is a map $F^\bullet \to X$ of filtered objects.

Definition 11.10. Let $F\mathscr{C}$ be a decreasing filtration.

(i) Suppose that it is a filtration on X, then it is **exhaustive** if $X \simeq \operatorname{colim} F^\bullet$.

(ii) It is **complete** if $F^\infty = \lim_s F^s \simeq 0$.

Remark 11.11. If $\mathscr{C}$ admits sequential colimits, then any $F^\bullet$ can be viewed as giving a filtration on $|F^\bullet| = F^{-\infty} = \operatorname{colim}_s F^s$.

Definition 11.12. Let $\mathscr{C}$ be an ∞-category with a final object and cofibers. Suppose that $F^\bullet$ is a decreasing filtration on $\mathscr{C}$. The **associated graded pieces** are $\mathrm{gr}^s_F = \mathrm{cofib}(F^{s+1} \to F^s) = F^s / F^{s+1}$.

11.3 Spectral Sequence Associated to Filtered Objects

Here is Lurie's treatment of spectral sequences associated to filtered objects [Lur17].

Definition 11.13. Let $\mathscr{C}$ be a pointed ∞-category and J be a linearly ordered set. Then, $J^{[1]}$ is the poset of pairs of elements (i, j) with the lexicographic ordering. A ***J*-complex** is a functor $F : J^{[1]} \to \mathscr{C}$ such that:

(i) for each $i \in J$ the object $F(i, i)$ is a zero object in $\mathscr{C}$,

(ii) for $i \le j \le k$ the square

$$\begin{array}{ccc} F(i,j) & \longrightarrow & F(i,k) \\ \downarrow & & \downarrow \\ F(j,j) & \longrightarrow & F(j,k) \end{array}$$

is a pushout in $\mathscr{C}$.

We write $\mathbf{Gap}(J, \mathscr{C})$ for the full subcategory of $\mathrm{Fun}(J^{[1]}, \mathscr{C})$ spanned by the J-complexes.

Remark 11.14. Let $\mathscr{C}$ be a stable ∞-category with a t-structure and let $X \in \mathbf{Gap}(\mathbb{Z}, \mathscr{C})$. For all $i \le j \le k$ there is a LES

$$\cdots \longrightarrow \pi_n(X(i,j)) \longrightarrow \pi_n(X(i,k)) \longrightarrow \pi_n(X(j,k)) \xrightarrow{\delta} \pi_{n-1}(X(i,j)) \longrightarrow \cdots$$

in $\mathscr{C}^\heartsuit$.

Definition 11.15. For every $p, q \in \mathbb{Z}$ and $r \geq 1$ we define

$$E_r^{p,q} = \operatorname{im}\left(\pi_{p+q}X(p-r,p) \to \pi_{p+q}X(p-1,p+r-1)\right)$$

with differential $E_r^{p,q} \to E_r^{p-r,q+r-1}$ by restricting the differential which leads to a commutative diagram

$$\begin{tikzcd}
\pi_{p+q}X(p-r,p) \arrow[r] \arrow[d, "\delta"'] & E_r^{p,q} \arrow[r] \arrow[d, dashed, "d_r"] & \pi_{p+q}X(p-1,p+r-1) \arrow[d, "\delta"] \\
\pi_{p+q-1}X(p-2r,p-r) \arrow[r] & E_r^{p-r,q+r-1} \arrow[r] & \pi_{p+q-r}X(p-r-1,p-1)
\end{tikzcd}$$

Lemma 11.16. Let $\mathscr{C}$ be a pointed ∞-category admitting pushouts. Then, $J = J_0 \cup \{-\infty\}$ is a linearly-ordered set containing a least element $-\infty$. So we can regard J_0 as a linearly ordered subset of $J^{[1]}$ via $i \mapsto (-\infty, i)$. Then, $\mathbf{Gap}(J, \mathscr{C}) \simeq \operatorname{Fun}(J_0, \mathscr{C})$.

Proof. Some left Kan extensions. □

Remark* 11.17. I think the idea is just that in J-complexes the diagonal is known to be 0 and all squares are pushouts. So if we know the left-most vertical line (i.e. the line at $-\infty$), we can reconstruct all other objects by pushing out. Formally, this is realized by correctly left Kan extending.

Construction 11.18. Let $\mathscr{C}$ be a stable ∞-category with a t-structure and let $X : \mathbb{Z} \to \mathscr{C}$ be a filtered object. Then, we can extend X to a complex in $\mathbf{Gap}(\mathbb{Z} \cup \{-\infty\}, \mathscr{C})$. The spectral sequence $\{E_r^{p,q}, d_r\}_{r\geq 1}$ is the spectral sequence associated to the filtered object X.

Construction 11.19. Consider $\mathrm{gr}_F^{[i,j]}$ admitting a filtration

$$\cdots \longrightarrow 0 \longrightarrow F^{j-1}/F^j \longrightarrow F^{j-2}/F^j \longrightarrow \cdots$$

where the superscript in the numerator is called **weight**. This is a complete, exhaustive filtration on $\mathrm{gr}_F^{[i,j]}$.

If $F^\bullet$ is a filtration, then the graded objects form a cochain complex with differential coming from the fiber sequence

$$\mathrm{gr}^{s+1} \longrightarrow F^s/F^{s+2} \longrightarrow \mathrm{gr}^s$$

leading to $\delta : \mathrm{gr}^s \to \mathrm{gr}^{s+1}[1]$. So we obtain a cochain complex[18]

$$\cdots \longrightarrow \mathrm{gr}_F^{-s-1}[-s-1] \longrightarrow \mathrm{gr}_F^{-s}[-s] \longrightarrow \mathrm{gr}_F^{-s+1}[-s+1] \longrightarrow \cdots.$$

Then, apply π_t to obtain a cochain complex

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_t\,\mathrm{gr}_F^{-s-1}[-s-1] \arrow[r] \arrow[d, equal] & \pi_t\,\mathrm{gr}_F^{-s}[-s] \arrow[r] \arrow[d, equal] & \pi_t\,\mathrm{gr}_F^{-s+1}[-s+1] \arrow[r] \arrow[d, equal] & \cdots. \\
& \pi_{t+s+1}\,\mathrm{gr}_F^{-s-1} & \pi_{t+s}\,\mathrm{gr}_F^{-s} & \pi_{t+s-1}\,\mathrm{gr}_F^{-s+1} &
\end{tikzcd}$$

in $\mathscr{C}^\heartsuit$.

We denote by $\mathbf{Ch}^\bullet(\mathscr{C})$ the ∞-category of coherent chain complexes on $\mathscr{C}$ [Ant24, Definition 3.19].

[18] See [Ant24, Lemma 3.18] for a verification that the composite of two maps in this sequence is nullhomotopic.

Theorem 11.20 (Ariotta's E^1-page theorem)**.** Let $\mathscr{C}$ be a stable ∞-category with sequential colimits. The associated graded functor $\mathrm{gr}^\bullet : F\mathscr{C} \to \mathrm{Gr}(\mathscr{C})$ factors through the forgetful functor $\mathbf{Ch}^\bullet(\mathscr{C}) \to \mathrm{Gr}(\mathscr{C})$ and induces an equivalence $\widehat{F\mathscr{C}} \to \mathbf{Ch}^\bullet(\mathscr{C})$ where $\widehat{F\mathscr{C}}$ is the ∞-category of complete decreasing filtrations in $\mathscr{C}$.

This is a coherently souped-up version of the above cochain complex construction [Ant24, p. 10].

Definition 11.21. The ∞-category $\mathbf{Ch}^\bullet(\mathscr{C})$ inherits a pointwise t-structure, i.e. $X^\bullet$ is connective if and only if each X^n is in $\mathscr{C}_{\geq 0}$. Use 11.20 to put a t-structure onto $\widehat{F\mathscr{C}}$. That t-structure is called the **Beilinson t-structure**.

Remark* 11.22. Comparing this with the cochain complex constructed above, we e.g. see that a connective object in the Beilinson t-structure requires $\mathrm{gr}^s_F[s]$ to be connective, i.e. $\mathrm{gr}^s_F \in \mathscr{C}_{\geq -s}$. Here is a summary [Ant24, Definition 3.24]:

- The connective objects in the Beilinson t-structure are those complete filtrations $F^\bullet$ with $\mathrm{gr}^n_F \in \mathscr{C}_{\geq n}$.
- The coconnective objects are those $F^\bullet$ with $\mathrm{gr}^n_F \in \mathscr{C}_{\leq -n}$.
- The heart is $(\widehat{F\mathscr{C}})^\heartsuit \simeq \mathbf{Ch}^\bullet(\mathscr{C}^\heartsuit)$.

Construction 11.23 (Décalage)**.** Let $\mathscr{C}$ be a stable ∞-category with sequential limits/colimits admitting a t-structure $(\mathscr{C}_{\geq 0}, \mathscr{C}_{\leq 0})$. Let $F^\bullet$ be a filtered object of $\mathscr{C}$. We write $\tau^B_{\geq \bullet}F$ as the Whitehead tower of $F^\bullet$ with respect to the Beilinson t-structure on $F\mathscr{C}$. After realization one obtains a filtered object

$$\cdots \longrightarrow |\tau^B_{\geq n+1}(F)| \longrightarrow |\tau^B_{\geq n}(F)| \longrightarrow \cdots$$

is a filtered object of $\mathscr{C}$. This is **Dec($F^\bullet$)**, the **décalage** of F.

Remark* 11.24. The décalage is responsible for a certain page-shifting. In nice enough cases and suitable indexing, there is an isomorphism $E^r(\mathbf{Dec}(F^\bullet)) \cong E^{r+1}(F^\bullet)$ [Ant24, Theorem 4.13].

Remark* 11.25. Ben gives a new proof that the two standard ways of constructing the AHSS (filter the space vs. filter the spectrum) are equivalent by means of the décalage [Ant24, Corollary 9.3].

We end by mentioning that the Adams spectral sequence $\mathrm{ASS}_E(X)$ comes from $X \otimes E^{\otimes \bullet+1}$. We will see more about this in the next talk (12).

12 Synthetic Spectra (Jonathan Pedersen)

Slogan: Synthetic spectra E are a categorification of the E-Adams spectral sequence. Talk 12 03.07.2025

12.1 Adams Spectral Sequence

Consider $(\mathscr{C}, \otimes)$, then there is an object $\mathbb{1}_{\mathscr{C}^{\mathrm{fil}}} \in \mathscr{C}^{\mathrm{fil}}$ together with a map from a shift written as $\tau : \mathbb{1}_{\mathscr{C}^{\mathrm{fil}}}(1) \to \mathbb{1}_{\mathscr{C}^{\mathrm{fil}}}$ given by

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{1}_{\mathscr{C}} \arrow[r, equal] \arrow[d, equal] & \mathbb{1}_{\mathscr{C}} \arrow[r, equal] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{1}_{\mathscr{C}} \arrow[r, equal] & \mathbb{1}_{\mathscr{C}} \arrow[r, equal] & \mathbb{1}_{\mathscr{C}} \arrow[r, equal] & \cdots
\end{tikzcd}$$

This encodes a lot of information, depicted as follows:

$$
\begin{tikzcd}
 & \mathscr{C}^{\mathrm{fil}} \arrow[dl, "\tau^{-1}"'] \arrow[dr, "-\otimes C\tau"] & \\
\mathscr{C} & & \mathscr{C}^{\mathrm{gr}}
\end{tikzcd}
$$

where τ^{-1} is essentially taking (co-)limits. In fact, $X \otimes C\tau \simeq \bigoplus_{i\in\mathbb{Z}} X_i/X_{i-1}$. Here are the heuristics:

- $-\otimes C\tau$ extracts an E_1-page,
- τ^{-1} extracts the E_∞-page.

Remark 12.1. Let E be an $\mathbb{E}_1$-ring spectrum and consider the cobar construction

$$\mathrm{CB}^\bullet_E(X) : n \mapsto X \otimes E^{\otimes n+1}.$$

We create the filtered spectrum $\mathrm{Tot}\,(\tau_{\geq *}\,\mathrm{CB}^\bullet_E(X)) \in \mathbf{Sp}^{\mathrm{fil}}$.

You can also take partial totalizations and the difference is some décalage thing, i.e. the swap between E_1 and E_2-page.

Proposition 12.2. The associated spectral sequence is the BKSS/Adams spectral sequence.

Classically, consider E, then an Adams tower is a diagram

$$
\begin{tikzcd}
X = X_0 \arrow[d, "f_0"] & X_1 \arrow[l] \arrow[d, "f_1"] & \cdots \arrow[l] \\
K_0 & K_1 &
\end{tikzcd}
$$

satisfying the following:

(i) $X_{s+1} \simeq \mathrm{fib}(f_s)$,

(ii) $E \otimes X_s$ is a retract of $E \otimes K_s$, in particular $E_\bullet(f_s)$ is mono,

(iii) K_s is the retract of $E \otimes K_s$,

(iv) There is an isomorphism

$$\mathrm{Ext}^{t,n}_{E_\bullet E}(E_\bullet, E_\bullet K_s) \cong \begin{cases} \pi_n(K_s) & t = 0, \\ 0 & \text{else} \end{cases}$$

Under nice conditions, we obtain

$$\mathrm{Ext}^{s,t}_{E_\bullet E}(E_\bullet, E_\bullet) \Rightarrow \pi_{t-s}(\mathbb{S}^\wedge_E).$$

Observation 12.3. To categorify the E-Adams spectral sequence we should think about sequences $A \to B \to C$ such that $E_\bullet A \to E_\bullet B \to E_\bullet C$ is a short exact sequence. See the above monic conditions.

12.2 The Synthetic Category

Start with a nice category (stable, presentable, symmetric monoidal, t-structure, etc.). Then, we want to force certain sequences to be exact.

To freely adjoin all colimits to an ∞-category, take $\mathbf{PSh}(\mathscr{C})$ which satisfies

$$\mathrm{Fun}^L(\mathbf{PSh}(\mathscr{C}), \mathscr{D}) \simeq \mathrm{Fun}(\mathscr{C}, \mathscr{D}).$$

If $\mathcal{K}$ is a collection of simplicial sets, one can also form $\mathbf{PSh}^{\mathcal{K}}(\mathscr{C})$ satisfying

$$\mathrm{Fun}^{\mathcal{K}}(\mathbf{PSh}^{\mathcal{K}}(\mathscr{C}), \mathscr{D}) \simeq \mathrm{Fun}(\mathscr{C}, \mathscr{D}).$$

Theorem 12.4. Let $\mathscr{C}$ be an ∞-category with finite coproducts and $\mathcal{K}$ denote the collection of filtered simplicial sets and Δ^{op}. Then, $\mathbf{PSh}^{\mathcal{K}}(\mathscr{C}) \subseteq \mathbf{PSh}(\mathscr{C})$ are exactly the finite product-preserving presheaves. We write $\mathbf{PSh}_\Sigma(\mathscr{C}) = \mathbf{PSh}^{\mathcal{K}}(\mathscr{C})$.

Start with $\mathbf{PSh}_\Sigma(\mathbf{Sp}^\omega)$ and stabilize this to get $\mathbf{PSh}_\Sigma^{\mathbf{Sp}}(\mathbf{Sp}^\omega)$. Let's now discuss symmetric monoidality. Consider fiber sequences $A \to B \to C$, then we want $E_\bullet A \to E_\bullet B \to E_\bullet C$ to be a short exact sequence. This is already a problem: It is not closed under tensor products. The map $\mathbb{S} \to \mathbb{S}/2$ is an $H\mathbb{Z}_\bullet$-surjection. It is not anymore after applying $- \otimes \mathbb{S}/2$.

Remark* 12.5. Indeed, a LES argument for $\mathbb{S}/2 \xrightarrow{2} \mathbb{S}/2 \to \mathbb{S}/2 \otimes \mathbb{S}/2$ shows

$$H\mathbb{Z}_1(\mathbb{S}/2 \otimes \mathbb{S}/2) \cong \mathbb{Z}/2$$

while $H\mathbb{Z}_1(\mathbb{S}/2) \cong 0$.

Definition 12.6. Let E be a ring spectrum, then we denote by $\mathbf{Sp}^{\mathrm{fp}} \subseteq \mathbf{Sp}$ the full subcategory on P such that $E_\bullet P$ is projective.

Claim 12.7. Here, the SES property is now closed under $\otimes$.

Definition 12.8. We define $\mathbf{Syn}_E$ as the full subcategory of $\mathbf{PSh}_\Sigma^{\mathbf{Sp}}(\mathbf{Sp}^{\mathrm{fp}})$ consisting of those $X : (\mathbf{Sp}^{\mathrm{fp}})^{\mathrm{op}} \to \mathbf{Sp}$ such that for cofiber sequences $A \to B \to C$ with $E_\bullet$ making it into an SES, then $X(C) \to X(B) \to X(A)$ is a fiber sequence.

Remark 12.9. This can also be obtained from the Grothendieck topology generated by the $E_\bullet$-surjections. Then, $\mathbf{Syn}_E$ consists of the presheaves that are sheaves.

What are examples of synthetic spectra?

Example 12.10 (Less useful)**.** Representables are examples. They don't use anything about E, so they are probably not every informative for ASS business.

Example 12.11 (More useful)**.** Consider $y : \mathbf{Sp} \to \mathbf{PSh}_\Sigma^{\mathbf{Sp}}(\mathbf{Sp}^{\mathrm{fp}})$, $X \mapsto \tau_{\geq 0}\,\mathrm{map}_{\mathbf{Sp}}(-, X)$. This is not always a synthetic spectrum! Given $A \to B \to C$, then

$$\tau_{\geq 0}\,\mathrm{map}_{\mathbf{Sp}}(C, X) \longrightarrow \tau_{\geq 0}\,\mathrm{map}_{\mathbf{Sp}}(B, X) \longrightarrow \tau_{\geq 0}\,\mathrm{map}_{\mathbf{Sp}}(A, C)$$

is a fiber sequence if and only if $[B, X] \to [A, X]$ is surjective.

Definition 12.12. The synthetic analogue $\nu : \mathbf{Sp} \to \mathbf{Syn}_E$ is the sheafification of y.

Lemma 12.13. Suppose that $A \to B \to C$ is an $E_\bullet$-SES, then $\nu A \to \nu B \to \nu C$ is a fiber sequence.

A summarizing picture is

$$\begin{tikzcd} & \mathbf{Sp} \arrow[dl, equal] \arrow[d, "\nu"] \arrow[dr, "\pi_\bullet(-\otimes E)"] & \\ \mathbf{Sp} & \mathbf{Syn}_E \arrow[l, "\tau^{-1}"] \arrow[r, "-\otimes C\tau"'] & \mathbf{Stable}_{E_\bullet E} \end{tikzcd}$$

which we will further discuss in the following.

Notation 12.14.

(i) We write $\mathbb{S}^{a,b} = \Sigma^{-b}\nu\mathbb{S}^{a+b}$.[19]

(ii) We write $\mathbb{S} = \nu\mathbb{S} = \mathbb{S}^{0,0}$.

(iii) We write $\tau : \mathbb{S}^{0,-1} = \Sigma\nu(\Sigma^{-1}\mathbb{S}) \to \nu(\mathbb{S}) = \mathbb{S}^{0,0}$.

Here are some properties of synthetic spectra.

Fact 12.15.

(i) The category $\mathbf{Syn}_E$ is stable, presentable and symmetric monoidal.

(ii) The synthetic analogue $\nu : \mathbf{Sp} \to \mathbf{Syn}_E$ is symmetric monoidal.

(iii) The cofiber $C\tau$ admits an $\mathbb{E}_\infty$-ring structure.

(iv) There is a t-structure in $\mathbf{Syn}_E$ with connective part the colimit cocompletion of $\nu(\mathbf{Sp})$.

12.3 Categorification of the Adams Spectral Sequence

Let's at least see one feature that indicates a categorification of the Adams spectral sequence.

Definition 12.16. A map $f : X \to Y$ is of **E-Adams filtration** $\geq s$ if it can be written as a composite of s maps which are trivial on $E_\bullet$.

Proposition 12.17. A map $f : X \to Y$ has Adams filtration $\geq s$ if and only if there exists a lift

$$\begin{tikzcd} & \Sigma^{0,-k}\nu(Y) \arrow[d, "\tau^k"] \\ \nu(X) \arrow[ur, dashed] \arrow[r, "\nu(f)"'] & \nu(Y) \end{tikzcd}$$

Proof. Consider

$$\Sigma^{-1}Y \xrightarrow{g} Z \xrightarrow{h} X \xrightarrow{f} Y.$$

We get

$$\begin{tikzcd} \nu(\Sigma^{-1}Y) \arrow[r] \arrow[d, "\nu(g)"'] & 0 \arrow[r] \arrow[d] & \Sigma\nu(\Sigma^{-1}Y) \arrow[r, "\tau"] \arrow[d] & \nu(Y) \arrow[d, equal] \\ \nu(Z) \arrow[r, "\nu(h)"'] & \nu(X) \arrow[r] \arrow[rr, bend right, "\nu(f)"'] & C\nu(h) \arrow[r] & \nu(Y) \end{tikzcd}$$

□

In the questions afterwards there was some discussion about geometric interpretations of deformations and generic/special fibers. Such things exist, interpreted as some sort of stack over $\mathbb{A}^1/\mathbb{G}_m$.

[19] This is the [BHS23] convention.

13 Chromatic Homotopy Theory is Algebraic when $p > n^2 + n + 1$ (Mattie Ji)

Talk 13 03.07.2025

Recall the chromatic convergence theorem (7.21), i.e. for a p-local finite spectrum there is a limit

$$X \simeq \lim \left(\cdots \to L_{E_n} X \to \cdots \to L_{E_1} X \to L_{E_0} X \right).$$

In particular, this says that $\mathbf{Sp}_{E_n}$ is pretty complicated as $n \to \infty$.

Question 13.1. What if $p \gg n$?

Piotr discusses this in his PhD thesis (which was actually not the synthetic spectra paper [Pst23]).

Theorem 13.2. Let E be a p-local Landweber exact spectrum[20] of height n. If $p > n^2 + n + 1$, then $h\mathbf{Sp}_E \simeq h\mathcal{D}(E_\bullet E)$.

Remark 13.3.

(i) If $n = 1$, then this was shown by Bousfield.

(ii) There is no equivalence $\mathbf{Sp}_E \simeq \mathcal{D}(E_\bullet E)$.

(iii) Piotr showed for $p > n^2 + n + 1 + \frac{k}{2}$ that $h_k\mathbf{Sp}_E \simeq h_k\mathcal{D}(E_\bullet E)$.

(iv) A deep result of Hovey and Strickland shows that $\mathcal{D}(E_\bullet E)$ only depends on p and n.[21] For technical reasons, Piotr works with **Johnson-Wilson theory** $E(n) = \mathrm{BP}\langle n \rangle [v_n^{-1}]$. For the rest of the talk, $E = E(n)$.

13.1 Observation 1: An Vanishing Line Results in $\mathbf{Sp}_E$

According to Piotr the following is folklore.

Theorem 13.4 (Folklore)**.** Suppose $p > n + 1$ and $M, N \in \mathbf{Comod}_{E_\bullet E}$. Then, $\mathrm{Ext}^{s,t}_{E_\bullet E}(M, N) \cong 0$ for all $s > n^2 + n$.

Lemma 13.5. Suppose $p > n + 1$, then $\mathrm{Ext}^{s,t}_{E_\bullet E}(E_\bullet, E_\bullet) \cong 0$ for $s > n^2 + n$.

Proof Idea. Note that $p - 1$ does not divide n, so the Morava stabilizer group $\mathbb{G}_n$ has no p-torsion. Thus,

$$n^2 = (\text{virtual cohomological dimension}) = (\text{cohomological dimension}).$$

Here is where the n^2 pops up. By examining the (algebraic) chromatic spectral sequence you will find the n shift. $\square$

Notation 13.6. An $N \in \mathbf{Comod}_{E_\bullet E}$ is a **good target** if $\mathrm{Ext}^{s,t}(E_\bullet, N) \cong 0$ for $s > n^2 + n$.

By 13.5 we find that $E_\bullet$ and its shifts are all good targets.

Observation 13.7. Consider an SES

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

where M is a good target. Then, N is a good target if and only if P is by the Ext LES. Thus, finitely generated $E_\bullet E$-comodules are good targets (via direct sums and quotients). Taking filtered colimits we deduce that the entire category consists of good targets.

Notation 13.8. An $M \in \mathbf{Comod}_{E_\bullet E}$ is a **good source** if $\mathrm{Ext}^{s,t}(M, N) = 0$ for $s > n^2 + n$ and all N.

From before we get that $E_\bullet$ is a good source. Moreover, good source is closed under $\oplus$ and quotients, so all of $\mathbf{Comod}_{E_\bullet E}$ are good sources. This finishes the proof of 13.4.

[20] For example E_n.

[21] Similarly for $\mathbf{Sp}_E$ which is easier though.

13.2 Observation 2: ASS collapses in ...

Proposition 13.9. Let $p > n+1$ and $2p-2 > n^2+n$. Then,

$$\mathrm{Ext}^{\bullet,\bullet}_{E_\bullet E}(E_\bullet, E_\bullet) \Rightarrow \pi_\bullet(L_E\mathbb{S})$$

collapses.

Proof. This essentially follows from the vanishing line (13.4).

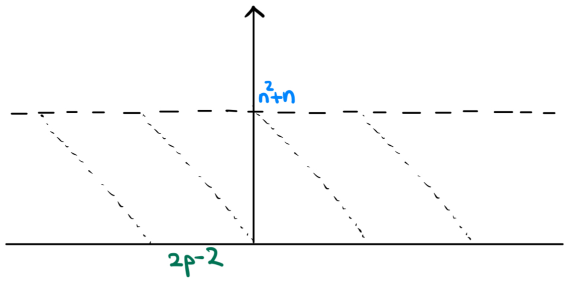

Figure 3: (Sorry Mattie, I did not live TeX this and am too lazy to TeX this.)

This uses that $\pi_\bullet E$ has elements concentrated in degree divisible by $2p-2$, as this is the case for v_1 (and higher v_n). □

Consider $\mathrm{Ext}^{\bullet,\bullet}_{E_\bullet E}(E_\bullet X, E_\bullet Y) \Rightarrow [X,Y]^\bullet$.

Definition 13.10. If the degrees of $E_\bullet X$ are concentrated in $\ell \in \mathbb{Z}/(2p-2)$ we say that $E_\bullet X$ is **pure of phase ℓ**.

Lemma 13.11. Suppose $2p-2 > n^2+n$ and let X, Y be E-local spectra such that $E_\bullet X, E_\bullet Y$ are both pure of phase $\ell \in \mathbb{Z}/(2p-2)$. Then,

$$\mathrm{map}_{\mathbf{Sp}_E}(X,Y) \to \mathrm{Hom}_{E_\bullet E}(E_\bullet X, E_\bullet Y)$$

is $(2p-2-n^2-n)$-connected.

Lemma 13.12. Let $2p > n^2+n$ and $p > n+1$ and $M \in \mathbf{Comod}_{E_\bullet E}$ and suppose that there exists X such that $E_\bullet X \cong M$. Then, X is **split** if $X \simeq \bigoplus_\ell X^\ell$ where $E_\bullet(X^\ell)$ is pure of phase ℓ.

Proof. Use the Goerss-Hopkins obstruction theory. WLOG pretend that M is pure. The obstructions to recover M as $E_\bullet X$ are in $\mathrm{Ext}^{k+2,k}_{E_\bullet E}(M,M)$ for $k \geq 1$. If $k+2 \geq (2p-2)+2 > n^2+n$ which shows that the obstructions are 0 by the vanishing result (13.4). If $k < 2p-2$, then the obstructions also vanish by pureness. □

Theorem 13.13. Let $\ell \in \mathbb{Z}/(2p-2)$ and $k = 2p-2-n^2-n$. Then, $h_k\mathbf{Sp}^\ell_E \simeq (h_k)\mathbf{Comod}^\ell_{E_\bullet E}$.

Definition 13.14. There is a **Bousfield splitting functor**

$$\beta : \mathbf{Comod}_{E_\bullet E} \to h_k\mathbf{Sp}_E, \bigoplus_{\ell\in\mathbb{Z}/(2p-2)} M^\ell \mapsto \bigoplus_{\ell\in\mathbb{Z}/(2p-2)} R^\ell(M^\ell).$$

We will use synthetic spectra to build an E-based resolution of $\mathbf{Sp}_E$ living in $\mathcal{D}(E_\bullet E)$. Here, our synthetic spectra will be hypercomplete and connective which can be phrased as being νE-local and altogether that we have spherical hypercomplete sheaves of spaces on $\mathbf{Sp}^{\mathrm{fp}}$.

Theorem 13.15. There are towers

$$\mathbf{Sp}_E = \mathscr{M}_\infty^{\mathrm{top}} \longrightarrow \cdots \longrightarrow \mathscr{M}_1^{\mathrm{top}} \longrightarrow \mathscr{M}_0^{\mathrm{top}}$$

$$\mathcal{D}(E_\bullet E) = \mathscr{M}_\infty^{\mathrm{alg}} \longrightarrow \cdots \longrightarrow \mathscr{M}_1^{\mathrm{alg}} \longrightarrow \mathscr{M}_0^{\mathrm{alg}}$$

such that the last term is $\mathbf{Comod}_{E_\bullet E}$ in both cases and the composite arrows and given by homology in both cases.

We will write $\mathscr{M}$ without the superscript when it holds for both.

Fact 13.16.

(i) Let $X \in \mathscr{M}_{\ell-1}$. Then, the obstruction to lifting X to $\widetilde{X} \in \mathscr{M}_\ell$ live in $\mathrm{Ext}_{E_\bullet E}^{\ell+2,\ell}(u_0 X, u_0 X)$ where u_0 means going to the end of the filtration.

(ii) Let $X, Y \in \mathscr{M}_\ell$ for $\ell \geq 1$. Then, there is a fiber sequence

$$\begin{tikzcd} \mathrm{map}_{\mathscr{M}_\ell}(X,Y) \arrow[r] & \mathrm{map}_{\mathscr{M}_{\ell-1}}(u_{\ell-1}X, u_{\ell-1}Y) \arrow[d] \\ & \mathrm{map}_{\mathcal{D}(\mathbf{Comod}_{E_\bullet E})}(u_0 X, \Sigma^{\ell+1} u_0 Y) \end{tikzcd}$$

Proof of 13.2. The proof is divided into the following parts:

(i) For $p > n+1$ we write $\ell = n^2 + n + k - 1$. Then, we obtain $h_k \mathbf{Sp}_E \simeq h_k \mathscr{M}_\ell^{\mathrm{top}}$ and $h_k \mathcal{D}(E_\bullet E) \simeq h_k \mathscr{M}_\ell^{\mathrm{alg}}$.

(ii) For $p > n^2 + n + 1 + \frac{k}{2}$ the Bousfield splitting functor β induces $\mathscr{M}_\ell^{\mathrm{top}} \simeq \mathscr{M}_\ell^{\mathrm{alg}}$.

Here are some proof ideas.

(i) Essential surjectivity corresponds to vanishing of obstructions for lifting which we get by the vanishing result. We show fully faithfulness by a connectivity argument with the fiber sequence.

(ii) Recall $\mathbf{Syn}^\heartsuit \simeq \mathbf{Comod}_{E_\bullet E}$ which we call *discrete objects*. Moreover, $\mathbf{Syn}$ is symmetric monoidal with unit $\mathbb{1}$. Its Postnikov filtration

$$\mathbb{1} \longrightarrow \cdots \longrightarrow \mathbb{1}_{\leq 1} \longrightarrow \mathbb{1}_{\leq 0}$$

which induces a tower

$$\mathbf{Syn} \longrightarrow \cdots \longrightarrow \mathbf{Mod}_{\mathbb{1}_{\leq 1}}(\mathbf{Syn}) \longrightarrow \mathbf{Mod}_{\mathbb{1}_{\leq 0}}(\mathbf{Syn}).$$

Then, we define $\mathscr{M}_\ell^{\mathrm{top}}$ as the ∞-category of $\mathbb{1}_{\leq \ell}$-modules X such that $\mathbb{1}_{\leq 0} \otimes_{\mathbb{1}_{\leq \ell}} X$ is discrete. By Barnes and Roitzheim there exists $P(\mathbb{1})$ such that $\mathcal{D}(E_\bullet E) \simeq \mathbf{Mod}_{P(\mathbb{1})}(\mathcal{D}(\mathbf{Comod}_{E_\bullet E}))$. We define $P = P(\mathbb{1})_{\geq 0}$ and we define $\mathscr{M}_\ell^{\mathrm{alg}}$ consists of $P - \ell = P \otimes \mathbb{1}_{\leq \ell}$-modules M such that $P_{\leq 0} \otimes_{P_{\leq \ell}} M$ is discrete.

Proposition 13.17. The functor β induces a monadic adjunction

$$\mathcal{D}(\mathbf{Comod}_{E_\bullet E}) \underset{\beta_*}{\overset{\beta^*}{\rightleftarrows}} \mathbf{Mod}_{\mathbb{1}_{\leq \ell}}(\mathbf{Syn}).$$

It extends to an adjoint equivalence

$$\mathbf{Mod}_{\beta_*\mathbb{1}_{\leq \ell}} \underset{\gamma_*}{\overset{\gamma^*}{\rightleftarrows}} \mathbf{Mod}_{\mathbb{1}_{\leq \ell}}(\mathbf{Syn})$$

We obtain a square

$$\begin{tikzcd} \mathbf{Mod}_{\mathbb{1}_{\leq \ell}}(\mathbf{Syn}) \arrow[r, "\gamma_*"] \arrow[d] & \mathbf{Mod}_{\beta_*(\mathbb{1}_{\leq t})}(\mathcal{D}(\mathbf{Comod}_{E_\bullet E})_{\geq 0}) \arrow[d] \\ \mathbf{Mod}_{\mathbb{1}_{\leq 0}}(\mathbf{Syn}) \arrow[r] & \mathbf{Mod}_{(\beta_*(\mathbb{1}_{\leq \ell}))_{\leq 0}}(\mathcal{D}(\mathbf{Comod}_{E_\bullet E})_{\geq 0}) \end{tikzcd}$$

Now $\beta_*\mathbb{1}_{\leq \ell}$ and $P_{\leq \ell}$ are the same thing.

□

13.3 Applications

With this β map Piotr discussed the algebraity of Pic.

Theorem 13.18 (Pstragowski)**.** The map

$$\mathrm{Pic}(\mathbf{Sp}_{K(n)}) \to \mathrm{Pic}(E^\vee_\bullet E)$$

is an equivalence for $2p - 2 > n^2 + n$ where $E^\vee_\bullet E \cong \pi_\bullet L_{K(n)}(E \otimes E)$ with Morava E-theory E.

Apparently, people just use PVK to speak about Paul VanKoughnett PVK because it's a bit unclear how his last name is pronounced. His name came up in a question about the (abstract) Goerss-Hopkins obstruction theory paper [PV22].

14 Multiplicative Structure on Quotient Ring Spectra (Emma Brink)

Let $R \in \mathbf{CAlg}(\mathbf{Sp})$. The goal of this talk is: TALK 14 03.07.2025

Theorem 14.1 (Burklund)**.** Let $R \in \mathbf{CAlg}(\mathbf{Sp})$ with $x \in R$ such that R/x admits a left unital multiplication. Then, R/x^{n+1} admits an $\mathbb{E}_n$-algebra structure for $n \geq 1$.

We will prove something slightly stronger.

Theorem 14.2. Let $\mathscr{C} \in \mathbf{CAlg}(\mathbf{Pr}^L)$ and $\mathscr{C} = \mathrm{Ind}(\mathscr{C}^\omega)$ where $\mathscr{C}^\omega \subseteq \mathscr{C}$ is a monoidal subcategory. Consider $x : X \to \mathbb{1}_\mathscr{C}$ in $\mathscr{C}^\omega$ such that $\mathbb{1}_\mathscr{C}/x$ has a left unital multiplication. Then, $\mathbb{1}_\mathscr{C}/x^{n+1}$ admits an $\mathbb{E}_n$-algebra structure for $n \geq 1$.

We will be looking at maps $\mathbb{1}_\mathscr{C}/x^{n+1} \to \mathbb{1}_\mathscr{C}/^{\mathbb{E}_n} x^{n+1}$.

14.1 Obstruction Theory

Let $\mathscr{C} \in \mathbf{CAlg}(\mathbf{Pr}^L_{\mathrm{st}})$ and $x : X \to \mathbb{1}_\mathscr{C}$ with $n \in \mathbb{N}$. Consider

$$\begin{tikzcd} & \mathscr{C}^{\mathrm{fil}} \arrow[dl] \arrow[dr, "(-)^{\tau=1}"] & \\ \mathscr{C}^{\mathrm{gr}} & & \mathscr{C} \end{tikzcd}$$

Consider $\mathbb{1}_\mathscr{C}\{\Sigma X(1)\} \to \mathbb{1}_\mathscr{C} \oplus \Sigma X(1)$. Suppose that

$$R^1 \longrightarrow R^2 \longrightarrow \cdots \longrightarrow R^{k-1}$$

in $\mathbf{Alg}_{\mathbb{E}_n}(\mathscr{C}^{\mathrm{gr}})/(\mathbb{1} \oplus \Sigma X(1))$ such that $R^{k-1} \to \mathbb{1} \oplus \Sigma X(1)$ is an equivalence in degrees $\leq k-1$. Write

$$X_k = \mathrm{fib}\left((R^{k-1})_k \to (\mathbb{1}_{\mathscr{C}^{\mathrm{gr}}} \oplus \Sigma X(1))_k\right)$$

which leads to an $\mathbb{E}_1$-pushout

$$\begin{tikzcd} \mathbb{1}\{X_n(k)\} \arrow[r] \arrow[d] & R^{k-1} \arrow[d] \\ \mathbb{1} \arrow[r] & R^k \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Consider a map $\mathrm{colim}_k R^k \to \mathbb{1} \oplus \Sigma X(\mathbb{1})$. We would like to have

$$\widetilde{R}^1 = \mathbb{1}_{\mathscr{C}}^{\mathrm{fil}_{\mathbb{E}_n}}/\tau x \longrightarrow \widetilde{R}^2 \longrightarrow \cdots$$

such that for all k the square

$$\begin{tikzcd} \mathbb{1}_{\mathscr{C}^{\mathrm{fil}}}\{X_k(k)\} \arrow[r, "\widetilde{s}_k"] \arrow[d] & \widetilde{R}^{k-1} \arrow[d] \\ \mathbb{1}_{\mathscr{C}^{\mathrm{fil}}} \arrow[r] & \widetilde{R}^k \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout square in the $\mathbb{E}_n$-category, such that $\mathrm{gr}_x(p) = \overline{p}$. Consider $\widetilde{R}^1 \to \widetilde{R}^{k-1}$. Consider

$$\begin{tikzcd} X_k \arrow[r, dashed, "\widetilde{s}_k"] \arrow[dr, "s_k"] \arrow[ddr, bend right, "0"'] & \widetilde{R}^{k-1}_k \arrow[d] \\ & R^{k-1}_k = \widetilde{R}^{k-1}_k/\widetilde{R}^{k-1}_{k-1} \arrow[d, "\partial"] \\ & \Sigma\widetilde{R}^{k-1}_{k-1} \end{tikzcd}$$

This lower square is the obstruction to building the dashed arrow. We have $\partial s_k \in [X_k, \Sigma\widetilde{R}^{k-1}_{k-1}]$.

- We have $\mathrm{gr}_\bullet \widetilde{R}^{k-1} = R^{k-1}_\bullet \cong (\mathbb{1} \oplus \Sigma X(\mathbb{1}))_\bullet \cong 0$ for $\bullet \leq 2 \leq k-1$. This implies

$$\widetilde{R}^{k-1}_{k-1} \simeq \widetilde{R}^{k-1}_1 \simeq \mathbb{1}_{\mathscr{C}}/x.$$

- Put $X_k = \Omega^{1+n} D^n_k(\Sigma^{n+1}X)$. Here,

$$D^n_k : \mathscr{C} \longrightarrow \mathrm{Fun}(\mathbb{E}_n(k) /\!\!/ \Sigma_k, \mathscr{C}) \xrightarrow{\lim} \mathscr{C}.$$

For $\mathbb{E}_n(k) = \mathbb{E}_n \times_{\mathbf{Fin}_*} \{k\}$ we consider

$$\mathbb{E}_n(k) \longrightarrow \mathrm{Fun}(\mathscr{C}^k, \mathscr{C}) \xrightarrow{\Delta^*} \mathrm{Fun}(\mathscr{C}, \mathscr{C})$$

$$\langle \alpha \rangle \longmapsto (\mathscr{C}^k \simeq \mathscr{C}_{\langle \alpha \rangle} \to \mathscr{C})$$

This induces a map $\mathbb{E}_n(k) /\!\!/ \Sigma_k \to \mathrm{Fun}(\mathscr{C}, \mathscr{C})$.

So we are interested in $[\Omega^{1+n} D_k^n(\Sigma^{n+1} X), \Sigma \mathbb{1}_{\mathscr{C}}/x]$.

Remark 14.3. Recall that $\mathbb{E}_n(k) /\!\!/ \Sigma_k$ is the unordered configuration space of k points in $\mathbb{R}^n$. It has a *Fox-Neuwirth* cell structure which is finite and has cells in dimensions between 0 and $t = (n-1)(k-1)$.

Let $c \in \mathscr{C}$. Then, we get a map $c^{\otimes \bullet} : E = \mathbb{E}_n(k)/\Sigma_k \to \mathscr{C}$. This yields a diagram

$$\lim_E c^{\otimes \bullet} \longrightarrow \lim_{E_{\leq t-1}} c^{\otimes \bullet} \longrightarrow \cdots \longrightarrow \lim_{E_0 \mathscr{C}^{\otimes \bullet}} .$$

where the fiber at the $t-s$ term is $\prod_{(t-s) \text{ cells}} \Omega^{t-s} c^{\otimes k}$. Apply $\pi_0 \operatorname{map}(\Omega^{n+1}(-), \Sigma \mathbb{1}/c)$ to the fiber with $c = \Sigma^{n+1} X$ to obtain

$$\pi_0 \operatorname{map}\left(\Omega^{n+1} \prod_{(t-s) \text{ cells}} \Omega^{t-s} (\Sigma^{n+1} X)^{\otimes k}, \Sigma \mathbb{1}/X\right)$$

so if these are all zero, we get surjections

$$[\Omega^{1+n} D_k^n(\Sigma^{n+1} X), \Sigma \mathbb{1}_{\mathscr{C}}/x] \twoheadleftarrow \bigoplus_{\text{cells}} [\Omega^{n+1}(\Sigma^{n+1} X)^{\otimes k}, \Sigma \mathbb{1}_X].$$

The upshot is that if

$$\left[\Omega^{n+ks}(\Sigma^{n+1} X)^{\otimes k}, \Sigma \mathbb{1}/x\right] = 0$$

for $0 \leq s \leq (k-1)(n-1)$, then $\mathbb{1}/x$ admits an $\mathbb{E}_n$-algebra structure.

14.2 Deforming $\mathscr{C}$

Theorem 14.4. Let $\mathscr{C} \in \mathbf{CAlg}(\mathbf{Pr}^L_{\mathrm{st}})$ and $\mathscr{C} = \mathrm{Ind}(\mathscr{C}^\omega)$ such that $\mathscr{C}^\omega \subseteq \mathscr{C}$ admits a monoidal subcategory where $x : X \to \mathbb{1}_{\mathscr{C}}$ in $\mathscr{C}^\omega$ admits a left unital mutliplication. If there exists a deformation

$$\begin{tikzcd} & \mathrm{Def}(\mathscr{C}, x) \arrow[dr, "(-)^{\tau=1}"] & \\ \mathscr{C} \arrow[ur, "\nu"] \arrow[rr, equal] & & \mathscr{C} \end{tikzcd}$$

in $\mathbf{CAlg}(\mathscr{C})$ then, $\nu(\mathbb{1}/x^q)$ admits an $\mathbb{E}_n$-algebra structure for $q > n$.

Consider

$$\begin{tikzcd} \mathbf{Sp}\left(\mathbf{PSh}_\Sigma^X(\mathscr{C}^\omega)\right) \arrow[drrr, bend left] & & & \\ \mathbf{PSh}_\Sigma^X(\mathscr{C}^\omega) \arrow[u] \arrow[rr, hook] & & \mathbf{PSh}(\mathscr{C}^\omega) \arrow[dr] & \\ & \mathscr{C}^\omega \arrow[ul] \arrow[ur] \arrow[rr, hook] & & \mathscr{C} \end{tikzcd}$$

where $\mathbf{PSh}_\Sigma^X(\mathscr{C}^\omega)$ consists of those $F : \mathscr{C}^{\omega,\mathrm{op}} \to \mathcal{S}$ such that F preserves finite products and for every $a \to b$ in $\mathscr{C}^\omega$ such that $p \otimes \mathbb{1}/x$ splits we get $F(b) \simeq \lim_\Delta F(\check{\mathscr{C}}(a))$.

We have

$$\begin{tikzcd} & \mathbf{Sp}(\mathbf{PSh}_\Sigma^X(\mathscr{C}^\omega)) \arrow[dr, bend left, "(-)^{\tau=1}"] & \\ \mathscr{C} \arrow[ur, bend left, "\nu"] & \mathscr{C}^\omega \arrow[l] \arrow[r] \arrow[u] & \mathscr{C} \arrow[ll, bend left, equal] \end{tikzcd}$$

If $a \to b \xrightarrow{p} c$ is a cofiber sequence in $\mathscr{C}^\omega$ such that $p \otimes \mathbb{1}/x$ splits, then $\nu a \to \nu b \to \nu c$ is a cofiber sequence.

So we get a cofiber sequence

$$\nu \mathbb{1}_{\mathscr{C}} \longrightarrow \nu \mathbb{1}_{\mathscr{C}}/x \longrightarrow \nu \Sigma X$$

Consider $\widetilde{x} : \widetilde{X} = \Omega \nu \Sigma X \to \nu \mathbb{1}_{\mathscr{C}}$. Then, $(\mathbb{1}/\widetilde{x}^q)\tau^{-1} = \mathbb{1}/x^q$.

Claim 14.5. For $q > n$ the object $\mathbb{1}_{\mathrm{Def}}/\widetilde{q}^q$ admits an $\mathbb{E}_n$-algebra structure

Proof. We need to consider

$$\left[\Omega^{2+n+r}(\Sigma^{n+1}(\Omega \nu \Sigma X)^{\otimes q})^{\otimes k}, \mathbb{1}_{\mathscr{C}}/\widetilde{x}^q\right]$$

and for $0 \leq r \leq (n-1)(k-1)$ the set

$$\left[\Omega^{2+n+r}\Sigma^{n+1-q}\nu(\Sigma X)^{\otimes q}, \mathbb{1}_{\mathscr{C}}/\widetilde{x}^q\right].$$

We claim that for $y \in \mathscr{C}^\omega$ and $s \geq q$ there is an isomorphism $[\Omega^s \nu Y, \mathbb{1}_{\mathscr{C}}/\widetilde{x}^q] \cong 0$.

Consider $p : a \to b$ in $\mathscr{C}^\omega$ such that $p \otimes \mathbb{1}/x$ admits a section. For all $T \in \mathscr{C}^\omega$ we have a surjection

$$p^*[-, \mathbb{1}/x \otimes T]_{\mathscr{C}^\omega} \twoheadrightarrow [-, \mathbb{1}/x \otimes T]_{\mathscr{C}^\omega}.$$

So for all $Y, T \in \mathscr{C}^\omega$ and $s > 0$ we get $[\Omega^s \nu Y, \nu(\mathbb{1}/x) \otimes \nu T] = 0$. There is a cofiber sequence

$$(\Omega \nu \Sigma X)^{\otimes q-1} \otimes \nu \mathbb{1}/\widetilde{x} \longrightarrow \mathbb{1}/\widetilde{x}^q \longrightarrow \mathbb{1}/\widetilde{x}^{q-1}.$$

□

15 McClure's Theorem (Preston Cranford)

Today was independence day and we were awaiting the american bbq team for their big show this evening. TALK 15 04.07.2025

Preston: *Happy fourth of July.*

15.1 Description of Theorem

Here is our main theorem.

Theorem 15.1 (McClure)**.** The homotopy groups of spectrum $\mathrm{Free}^{\mathbb{E}_\infty}_{\mathrm{KU}_p^\wedge}(x)_p^\wedge$ is a free p-complete δ-ring[22] where x is a singleton set.

Preston: *There are not many proofs in the literature, so Ishan told me about the argument in this talk. ...All mistakes are due to Ishan.*

[22] See 9.15.

Fact 15.2. Let R be a p-torsionfree ring. Then, there is a bijection

$$\{\delta\text{-structures}\} \cong \{\varphi : R \to R \text{ ring homomorphism lifting the Frobenius } R/p \to R/p\}.$$

Here, $\varphi(x) = x^p + p\delta(x)$.

Claim 15.3. The ring $\mathbb{Z}[x, \delta(x), \delta^2(x), \cdots]_p^\wedge$ is $\mathrm{Free}_\delta(x)_p^\wedge$.

Think $x_0 = x, x_1 = \delta(x), x_2 = \delta^2(x), \cdots$.

Let us compute

$$\mathrm{Free}^{\mathbb{E}_\infty}_{\mathrm{KU}_p^\wedge}(x) \simeq \bigoplus_{n\geq 0} \left((\mathrm{KU}_p^\wedge)^{\otimes_{\mathrm{KU}_p^\wedge} n}_{h\Sigma_n} \right)_p^\wedge \simeq \bigoplus_{n\geq 0} \left(\mathrm{KU}_p^\wedge \otimes B\Sigma_n \right)_p^\wedge$$

where we note that this is the relative tensor product, so only one copy of $\mathrm{KU}_p^\wedge$ remains in the second equivalence which is why Σ_p acts trivially leading to $B\Sigma_n$.

Remark 15.4. Here is a reminder for group cohomology: For $H \leq G$ there are two maps

$$H^\bullet(BG; M) \underset{\mathrm{Tr}}{\overset{\varepsilon}{\rightleftarrows}} H^\bullet(BH; M)$$

then $\mathrm{Tr} \circ \varepsilon = [G : H]$ by the double coset formula.

15.2 Case 1: $n < p$

For $n < p$ let us discuss $(\mathrm{KU}/p)^\bullet(B\Sigma_n)$. Note also $(\mathrm{KU}/p)^\bullet(Be) \cong (\mathrm{KU}/p)^\bullet$. Note that homology and cohomology are the same here because KU/p is a field.

Fact 15.5. We have $\mathrm{rk}_{\mathbb{F}_p}((\mathrm{KU}/p)_\bullet(B\Sigma_n)) = \mathrm{rk}_{\mathbb{Z}_p}((\mathrm{KU}_p^\wedge)_\bullet(B\Sigma_n))$.

Proof. Consider $\mathrm{KU}_p^\wedge \xrightarrow{p} \mathrm{KU}_p^\wedge \to \mathrm{KU}/p$ and use that $(\mathrm{KU}/p)_\bullet(B\Sigma_n)$ is even. □

15.3 Case 2: $n = p$

Proposition 15.6. There is an isomorphism

$$(\varepsilon, \mathrm{tr}^e_{\Sigma_p}) : (\mathrm{KU}/p)^\bullet(B\Sigma_p) \xrightarrow{\sim} (\mathrm{KU}/p)^\bullet \oplus (\mathrm{KU}/p)^\bullet$$

which is generated by elements called x^p resp. δx.

Pause. Dog entered.

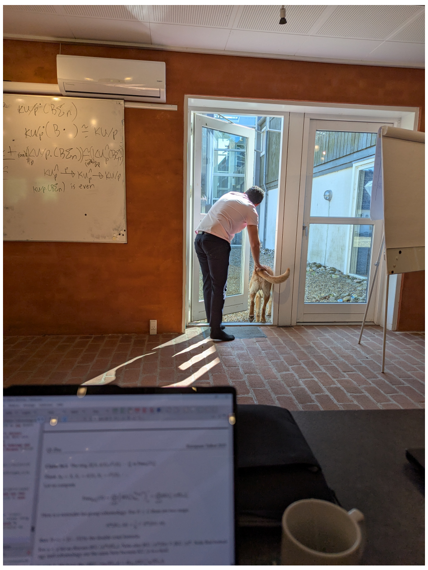

Figure 4: The dog entered!

Dog left.

Proof. Consider $BC_p \subseteq B\Sigma_p$ where C_p is a p-Sylow subgroup of Σ_p.

> **Fact.** Let $H \leq G$ with $[G : H]$ coprime to p and let $M \in \mathbf{Mod}_{\mathbb{F}_p}$. Then,
>
> $$H^\bullet(G; M) \cong H^\bullet(H; M)^{G/H} \hookrightarrow H^\bullet(H, M).$$

Here is another claim:

> **Lemma.** There is an isomorphism $(\mathrm{KU}/p)^\bullet(BC_p) \cong \mathbb{F}_p[t]/t^p \otimes \mathbb{F}_p[\beta^\pm]$ for $|t| = 2$ and $|\beta| = 2$.
>
> *Proof.* Consider the AHSS
>
> $$H^p\left(BC_p; \pi_q(\mathrm{KU}/p)\right) \Rightarrow (\mathrm{KU}/p)^{p+q}(BC_p).$$
>
> We compute
> $$H^\bullet(BC_p; \mathbb{F}_p) \cong \mathbb{F}_p[t] \otimes \Lambda_{\mathbb{F}_p}(e)$$
> with $|t| = 2$ and $|e| = 1$ with $\mathrm{d}e = [p]_F t = v_1 t^p = \beta^{p-1} t^p$. This essentially follows from some Gysin sequence argument on
>
> $$S^1 \longrightarrow BC_p \longrightarrow BS^1 \xrightarrow{p} BS^1.$$
>
> Play with the spectral sequence to conclude. □
>
> **Remark*.** Let R be any complex oriented cohomology theory. Then, there is an isomorphism $R^\bullet(BC_p) \cong R[\![t]\!]/[p]t$.

Consider $B(C_p \rtimes \mathrm{Aut}(C_p)) \subseteq B\Sigma_p$.

> **Claim.** The ring $(\mathrm{KU}/p)^\bullet\left(B(C_p \rtimes \mathrm{Aut}(C_p))\right)$ is of rank 2.
>
> *Proof.* Consider
>
> $$0 \longrightarrow C_p \longrightarrow C_p \rtimes \mathrm{Aut}(C_p) \longrightarrow \mathrm{Aut}(C_p) \longrightarrow 0.$$
>
> There is an isomorphism
>
> $$(\mathrm{KU}/p)^\bullet(B(C_p \rtimes \mathrm{Aut}(C_p))) \cong (\mathrm{KU}/p)^\bullet(BC_p)^{h\,\mathrm{Aut}(C_p)}$$
>
> by some Atiyah-Hirzebruch spectral sequence argument. The C_p-action sends
>
> $$1 \mapsto 1, t \mapsto \mu t, t^2 \mapsto \mu^2 t^2, \cdots$$
>
> with $\mathrm{ord}(\mu) = n - 1$. So only 1 and t^{p-1} are fixed by this. □

I'm thankful to Preston for explaining the following to me:

Since $[\Sigma_p : C_p \rtimes \mathrm{Aut}(C_p)]$ is coprime to p we conclude that there is an injection

$$(\mathrm{KU}/p)^\bullet(B\Sigma_p) \hookrightarrow (\mathrm{KU}/p)^\bullet(B(C_p \rtimes \mathrm{Aut}(C_p)))$$

by the double coset formula (15.4) or (equivalently) the first fact in this proof and an AHSS argument. Thus, $(\mathrm{KU}/p)^\bullet(B\Sigma_p)$ is an $\mathbb{F}_p$-vector space of dimension ≤ 2.

We show that it is ≥ 2. To do so, we wish to identify two of its summands. Indeed, there are maps

$$\mathrm{tr} : (\mathrm{KU}/p)_{h\Sigma_p} \to \mathrm{KU}/p \quad \text{and} \quad \varepsilon : \mathrm{KU}/p \to (\mathrm{KU}/p)^{h\Sigma_p}.$$

On the other hand, $(\mathrm{KU}/p)_{h\Sigma_p} \simeq (\mathrm{KU}/p)^{h\Sigma_p}$ by Tate vanishing (8.12). Then, we can compute that $\mathrm{tr} \circ \varepsilon$ is $p! = 0$ on KU/p by the double coset formula (15.4). □

So up until now we know the first $(p+1)$-degrees to be $x, x^2, \cdots, x^p, \delta(x)$. We put $\varphi(\varepsilon) = 1$ and $\varphi(\mathrm{tr}) = 0$.

Lemma 15.7. This φ is additive: $\varphi(x+y) = \varphi(x) + \varphi(y)$.

Proof. Consider $\mathrm{Free}_{\mathrm{KU}_p^\wedge}(z) \to \mathrm{Free}_{\mathrm{KU}_p^\wedge}(x,y)$, $z \mapsto x+y$. Get

$$\mathrm{Free}_{\mathrm{KU}_p^\wedge}(x,y) \cong \bigoplus_{n>0} \left(\mathrm{KU}_p^\wedge \oplus \mathrm{KU}_p^\wedge\right)_{h\Sigma_n}^{\otimes_{\mathrm{KU}_p^\wedge} n}.$$

The p-ary part is essentially indexed by $\mathrm{KU}_p^\wedge \times \{x,y\}^p$. We have some terms $x^p\,\mathrm{KU}_p^\wedge(B\Sigma_p)$ as well as $x^i y^j\,\mathrm{KU}_p^\wedge(B(\Sigma_i \times \Sigma_j))$ for $i+j=p$ and $y^p\,\mathrm{KU}_p^\wedge(B\Sigma_p)$. Consider

$$\mathrm{Free}_{\mathrm{KU}_p^\wedge}(z) \longrightarrow \mathrm{Free}_{\mathrm{KU}_p^\wedge}(x,y) \longrightarrow \mathrm{Free}_{\mathrm{KU}_p^\wedge}(w)$$

where the second arrow is $x \mapsto w$, $y \mapsto 0$. □

Lemma 15.8. This φ is multiplicative $\varphi(xy) = \varphi(x)\varphi(y)$.

Claim 15.9. The ring $\mathrm{KU}_p^\wedge(B\Sigma_{2p})$ consists of three copies consisting of $(\delta x)^2, x^p\delta x, x^{2p}$.

Proof. We will look at $C_p \rtimes \mathrm{Aut}(C_p) \times C_p \rtimes \mathrm{Aut}(C_p) \rtimes C_2$. □

Claim 15.10. There is a surjection $\mathrm{Free}_\delta(x) \twoheadrightarrow \mathrm{Free}_{\mathrm{KU}_p^\wedge}(y)$.

Proof Sketch. Consider $\mathrm{KU}_p^\wedge(B\Sigma_p)$ and $n = \sum_{i=1}^k a_i p^i$ with $a_i < p$. Consider $BC_p^{\rtimes a_0}$. □

Claim 15.11. The map is also injective.

Proof. Consider

$$\begin{tikzcd} & \mathbf{Sp} \\ \mathrm{Free}_\delta(X) \arrow[r] \arrow[ur] & \mathrm{Free}_{\mathrm{KU}_p^\wedge}(w) \arrow[u] \end{tikzcd}$$

and $(\Sigma_+^\infty \mathbb{N}[1/p])^{\otimes\infty} \otimes \mathrm{KU}$ leading to $\mathbb{F}_p[x_0^{1/p^\infty}, x_1^{1/p^\infty}, \cdots]$. □

Some a priori reason of this result is the Atiyah-Segal completion theorem which allows you to compute the K-theory of classifying spaces. A slogan is that p-complete λ-rings are essentially δ-rings. There is some nice explanation about some operation in [CSY22]. For general heights there is some notion of a $\mathbb{T}$-algebra due to Rezk [Rez09].

Remark 15.12. In this talk we worked in $\mathbf{LMod}_{\mathrm{KU}_p^\wedge}$ but the results are also true in $\mathbf{Sp}_{K(1)}$.

16 The Chromatic Nullstellensatz (Max Blans)

Suppose that L is an algebraically closed field. TALK 16 04.07.2025

Theorem 16.1 (Nullstellensatz, Hilbert)**.** If $(f_1, \cdots, f_k) \subsetneq L[x_1, \cdots, x_n]$, then the L-algebra $L[x_1, \cdots, x_n]/(f_1, \cdots, f_k)$ admits a map to L.

A map $L[x_1, \cdots, x_n]/(f_1, \cdots, f_k)$ is the choice of n elements on which the polynomials $f_1, \cdots, f_k$ vanish; hence Nullstellensatz.

We want to move this theorem to chromatic homotopy theory for which we need a more categorical formulation of the result.

Observation 16.2. These $L[x_1, \cdots, x_n]/(f_1, \cdots, f_k)$ are precisely the compact, non-terminal objects in $\mathbf{Alg}_L$.

Max: *The following theorem is harder. ...It required more people.*

Theorem 16.3 (Chromatic Nullstellensatz, Burklund-Schlank-Yuan)**.** Let $E(L)$ be Morava E-theory. Every compact, non-terminal object in $\mathbf{CAlg}^{\wedge}_{E(L)} = \mathbf{CAlg}_{E(L)}(\mathbf{Sp}_{T(n)})$ admits a map to $E(L)$.

Remark* 16.4. In some sense, Hilbert's Nullstellensatz is characterizes algebraically closed fields. In this interpretation, Morava E-theory should be viewed as the algebraically closed fields in chromatic homotopy theory.

The hard thing is to construct maps to Morava E-theory. This is a consequence of:

Theorem 16.5. Let $0 \not\simeq R \in \mathbf{CAlg}(\mathbf{Sp}_{T(n)})$. Then, there exists a map $R \to E(L)$.

The goal of this talk is to give a proof at height 1. The idea is essentially the same but there are more power operations at higher heights.

16.1 Morava E-Theory & Tilting

Recall that an $\mathbb{F}_p$-algebra is *perfect* if the Frobenius is an isomorphism.

Theorem 16.6 (Goerss-Hopkins-Miller, Lurie)**.** Let A be a perfect $\mathbb{F}_p$-algebra and $\mathbb{H}_0$ be a formal group over A of height n. Then, there exists a 2-periodic $E(A; \mathbb{H}_0) \in \mathbf{CAlg}(\mathbf{Sp}_{K(n)})$ with

$$\pi_\bullet E(A; \mathbb{H}_0) \cong W(A)[\![u_1, \cdots, u_{n-1}]\!][u^{\pm 1}]$$

with $|u| = 2$. This is functorial in $(A; \mathbb{H}_0)$.

We will work at height 1 over $E(\mathbb{F}_p; \mathbb{G}_m) \simeq \mathrm{KU}^{\wedge}_p$.

Theorem 16.7. There is an adjunction

$$\mathbf{Perf}_{\mathbb{F}_p} \underset{(-)^\flat}{\overset{E(-)}{\rightleftarrows}} \mathbf{CAlg}^{\wedge}_{\mathrm{KU}^{\wedge}_p}$$

such that:

(i) $E(A) = E(A; \mathbb{G}_m)$,

(ii) $E(-)$ is fully faithful, i.e. the adjunction is a colocalization.

(iii) There is an isomorphism $R^\flat \cong ((\pi_0 R)/p)^\flat = \lim \left(\pi_0 R/p \xleftarrow{(-)^p} \pi_0 R/p \xleftarrow{(-)^p} \cdots \right)$.

Since $\mathbf{Perf}_{\mathbb{F}_p}$ is a 1-category, this means that it is easy to map out of Morava E-theory.

16.2 The Proof in a Nutshell

Suppose $0 \not\simeq R \in \mathbf{CAlg}(\mathbf{Sp}_{T(1)})$. We want a map $R \to E(L)$ for some algebraically closed field L.

Observation 16.8. Here are two reductions.

(i) We have $R \to R \otimes \mathrm{KU}_p^\wedge \not\simeq 0$ where the latter uses that $\mathrm{KU}_p^\wedge \otimes -$ is conservative.[23] So we can assume that R is a $\mathrm{KU}_p^\wedge$-algebra by replacing R with $R \otimes \mathrm{KU}_p^\wedge$.

(ii) It suffices to give a map $R \to E(A)$ for some $A \not\simeq 0$ because we can then postcompose by $E(A) \to E(L)$ via $A \to A/\mathfrak{m} \to \overline{A/\mathfrak{m}}$.

Let us formulate the proof strategy.

Proof Idea. We have the counit map $E(R^\flat) \to R$ from 16.7 and want to modify R until this becomes an equivalence. $\square$

Example 16.9. Suppose that $\alpha \in \pi_1 R$. Consider the pushout

$$\begin{tikzcd} \mathrm{KU}_p^\wedge\{z^1\} \arrow[r, "z' \mapsto \alpha"] \arrow[d, "z' \mapsto 0"'] & R \arrow[d] \\ \mathrm{KU}_p^\wedge \arrow[r] & R' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where the upper-left corner is the free $\mathrm{KU}_p^\wedge$-algebra on a generator. So $\alpha = 0$ in R'.

Definition 16.10. Let $R \in \mathbf{CAlg}(\mathbf{Sp}_{T(1)})$.

(i) A map $f : M \to N$ in $\mathbf{Mod}_R^\omega(\mathbf{Sp}_{T(1)})$ is **nilpotent** if $f^{\otimes_R k} \simeq 0$ for $k \gg 0$.

(ii) A map $R \to S$ in $\mathbf{CAlg}(\mathbf{Sp}_{T(1)})$ **detects nilpotence** if $f : M \to N$ is nilpotent if and only if $f \otimes_R S$ is nilpotent.

Example 16.11. If $R \to S$ is conservative (i.e. $- \otimes_R S$ is conservative), then it detects nilpotence.

Observation 16.12. Suppose that $0 \not\simeq R \to S$ detects nilpotence, then $S \not\simeq 0$.

Proof.* Pick $f = \mathrm{id}$. $\square$

Proposition 16.13. Nilpotence detecting maps are closed under base change, retracts, transfinite compositions (weakly saturated).

Here is our strategy: We will construct

(i) $f : E(\mathbb{F}_p[t^{1/p^\infty}]) \to E(\mathbb{F}_p[t^{\pm 1/p^\infty}]) \times E(\mathbb{F}_p)$,

(ii) $g : \mathrm{KU}_p^\wedge\{z^0\} \to E(A)$ with A to be determined,

(iii) $h : \mathrm{KU}_p^\wedge\{z^1\} \to \mathrm{KU}_p^\wedge$.

We show that if R has the right lifting property with respect to these maps, then $R \simeq E(R^\flat)$ plus these maps are nilpotence detecting.

Lemma* 16.14 (Small Object Argument)**.** Let $\mathscr{C} \in \mathbf{Pr}^L$ and S be a set of morphisms in $\mathscr{C}$. Then, every map $f : X \to Z$ admits a factorization

[23] I think this strictly speaking uses the height 1 telescope conjecture so $E(1) \simeq \mathrm{KU}_p^\wedge$ sees all of $\mathbf{Sp}_{T(1)}$.

$$X \xrightarrow{f'} Y \xrightarrow{f''} Z$$

where $f' \in \overline{S}$ lies in the weak saturated closed of S and $S \perp f''$.

Proof.* See [Lur11, Proposition 1.4.7]. I think [Lan21, Proposition 1.3.9] explains this quite well (modulo some modifications to make this work for ∞-land). I forgot about the small object argument, so let me try to sketch the proof including the ∞-modifications.

We consider the collection of all squares

$$\begin{tikzcd} U_s \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ V_s \arrow[r] & Z \end{tikzcd}$$

where $U_s \to V_s$ is a map in S. Indexing coproducts over all such squares, we can take a pushout

$$\begin{tikzcd} \coprod U_s \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, bend left, "f"] & \\ \coprod V_s \arrow[r] \arrow[drr, bend right] & E^1(f) \arrow[ul, phantom, "\ulcorner"] \arrow[dr, dashed] & \\ & & Z \end{tikzcd}$$

Repeat this for $X \to E^1(f)$ many times for a big enough ordinal to obtain a tower

$$X \longrightarrow E^1(f) \longrightarrow E^2(f) \longrightarrow \cdots \longrightarrow Z.$$

For that big enough ordinal α we put $E^\alpha(f) = \operatorname{colim}_{k\le\alpha} E^k(f)$ and so we obtain a factorization

$$X \longrightarrow E^\alpha(f) \longrightarrow Z$$

The first map lies in $\overline{S}$ by the closure properties of weak saturation. The second map has the RLP with respect to S by a compactness argument. Since $\mathscr{C}$ is presentable, we can choose some cardinal κ such that all U_i are κ-compact and so having chosen α big enough, we obtain factorizations

$$\begin{tikzcd} U_s \arrow[r, dashed] \arrow[d] & E^\beta(f) \arrow[r] \arrow[d] & E^\alpha(f) \arrow[d] \\ V_s \arrow[r, dotted] & E^{\beta+1}(f) \arrow[r] \arrow[ur] & Z \end{tikzcd}$$

by compactness which allows us to solve our lifting problem. □

This yields the version stated in [BSY22, Proposition 4.35].

Corollary* 16.15 ([BSY22, Proposition 4.35])**.** Let $\mathscr{C} \in \mathbf{Pr}^L$ and S be a weakly saturated class of morphisms in $\mathscr{C}$ with $S_0 \subseteq S$ be a set. Let $A \in \mathscr{C}$. Then, there exists a map $A \to B$ in $\mathscr{C}$ such that

(i) The map $A \to B$ is in S.

(ii) For every $f \in S_0$ we have $f \perp B$.

Proof.* Apply the small object argument (16.14) to $A \to *_{\mathscr{C}}$. □

Theorem* 16.16 ([BSY22, Theorem 4.36])**.** The collection of nilpotence detecting maps is weakly saturated in $\mathbf{CAlg}(\mathbf{Sp}_{T(1)})$.

Some of the following is also discussed in this talk. But for clarity of the main argument let me already include the result at this point.

Proposition* 16.17 ([BSY22, Theorem 4.40, Proposition 5.15, 5.17])**.** The maps f, g, h detect nilpotence.

With 16.16 and 16.17 we can run a small object argument (16.15) to produce a nilpotence detecting map $R \to S$ with $f, g, h \perp S$. This allows us to show deduce $S \simeq E(A)$ for some perfect A of Krull dimension 0 [BSY22, Proposition 5.11]. The main ingredients are:

- Use f to show that $E(S^\flat) \to S$ is injective on π_0.
- Use g to show that $E(S^\flat) \to S$ is surjective on π_0.
- Use h to show that $\pi_1 S = 0$.

See also [BSY22, Proposition 5.9].

16.3 The Map h

It is clear that $h \perp R$ implies $\pi_1 R = 0$.

$$\begin{tikzcd} \mathrm{KU}_p^\wedge\{z^1\} \arrow[r] \arrow[d, "z'\mapsto 0"'] & R \\ \mathrm{KU}_p^\wedge \arrow[ur, dashed] & \end{tikzcd}$$

Example 16.18. One computes $\pi_\bullet \mathbb{F}_p^{tC_p} \cong \mathbb{F}_p[t_2^{\pm 1}] \otimes \Lambda(\alpha_{-1})$ with $\beta\alpha_{-1} = t_2^{-1}$.

Example 16.19. Consider $\mathbb{Q}\{z^1\} = \mathbb{Q} \oplus \mathbb{Q}[1] \to \mathbb{Q}$.

With the bar spectral sequence you can compute $\pi_\bullet \mathrm{KU}_p^\wedge\{z^1\} \cong \Lambda(z^1, \psi(z^1), \psi^2(z^1), \cdots)$. In the end, you show that h is nilpotence detecting.

16.4 The Map g

We want to construct $g : \mathrm{KU}_p^\wedge\{z^0\} \to E(A)$. Last talk we saw

$$\pi_0 \mathrm{KU}_p^\wedge\{z^0\} = \mathbb{Z}_p[z^0, \delta(z^0), \cdots]_p^\wedge = \mathrm{Free}_\delta(z^0)_p^\wedge.$$

Moreover, $\pi_0 E(A) = W(A)$ and it turns out that this has a unique δ-ring structure.

Theorem 16.20 (Joyal)**.** Let $A \in \mathbf{CRing}$, then $W(A)$ is the cofree δ-ring on A.

This is part of the story that gets more complicated at higher heights. This cofreeness statement is still true at higher heights for the $\mathbb{T}$-algebra structure, but it is much harder to prove and maybe the most difficult part of the paper.

Proposition 16.21. Let $A \in \mathbf{Perf}_{\mathbb{F}_p}$. The map

$$(-/p)^\# : \pi_0 \operatorname{Map}_{\mathbf{CAlg}_{\mathrm{KU}_p^\wedge}^\wedge}\left(\mathrm{KU}_p^\wedge\{z^0\}, E(A)\right) \to \operatorname{Hom}_{\mathbf{Perf}_{\mathbb{F}_p}}\left((\mathrm{Free}_\delta(z^0)/p)^\#, A\right)$$

is a bijection. Here, the *colimit perfection* is given by $B^\# = \operatorname{colim}\left(B \xrightarrow{(-)^p} B \xrightarrow{(-)^p} \cdots\right)$.

Proof. String together some adjunctions:

$$\begin{aligned}\pi_0 \operatorname{Map}(\mathrm{KU}_p^\wedge\{z^0\}, E(A)) &\cong \pi_0\Omega^\infty E(A)\\ &\cong W(A)\\ &\cong \operatorname{Hom}_\delta(\operatorname{Free}_\delta(z^0), W(A))\\ &\cong \operatorname{Hom}_{\mathbf{CRing}}(\operatorname{Free}_\delta(z^0), A)\\ &\cong \operatorname{Hom}_{\mathbf{Perf}_{\mathbb{F}_p}}((\operatorname{Free}_\delta(z^0)/p)^\#, A).\end{aligned}$$

□

Definition 16.22. Let $g : \mathrm{KU}_p^\wedge\{z^0\} \to E((\operatorname{Free}_\delta(z^0)/p)^\#)$ correspond to $\mathrm{id}_{(\operatorname{Free}_\delta(z^0)/p)^\#}$ under the bijection of 16.21.

Proposition 16.23. If $g \perp R$, then $E(R^\flat) \to E$ is surjective on π_0.

Proof. Let $x \in \pi_0 R$. Consider

$$\begin{tikzcd} \mathrm{KU}_p^\wedge\{z^0\} \arrow[r, "z^0\mapsto x"] \arrow[d, "g"] & R & E(R^\flat) \arrow[l] \\ E(A) \arrow[ur, dashed] \arrow[urr, dashed, bend right] & & \end{tikzcd}$$

□

Proposition 16.24. The map g detects nilpotence.

Proof. Since $- \otimes_{\mathrm{KU}_p^\wedge} \mathrm{KU}/p : \mathbf{Mod}^\wedge_{\mathrm{KU}_p^\wedge} \to \mathbf{Mod}^\wedge_{\mathrm{KU}_p^\wedge}$ is conservative, it suffices to check this after modding out by p. On π_0 the map g/p is given by

$$\operatorname{Free}_\delta(z^0)/p = \mathbb{F}_p\left[z^0, \delta(z^0), \cdots\right] \hookrightarrow \mathbb{F}_p\left[(z_0)^{1/p^\infty}, \cdots\right] = (\operatorname{Free}_\delta(z^0)/p)^\#.$$

This is a faithfully flat map. By the Tor spectral sequence we obtain conservativity. □

16.5 The Map f

Consider the map $f : E(\mathbb{F}_p[t^{1/p^\infty}]) \to E(\mathbb{F}_p[t^{\pm 1/p^\infty}]) \times E(\mathbb{F}_p)$ induced by $t \mapsto (t, 0)$.

Proposition 16.25. We have $f \perp R$ if and only if $R^\flat = (\pi_0 R/p)^\flat$ is of Krull dimension 0.

Proof. The condition $f \perp R$ is equivalent to $(\mathbb{F}_p[t] \to \mathbb{F}_p[t^\pm] \times \mathbb{F}_p) \perp R^\flat$ by applying adjunctions. This is equivalent to $R^\flat$ being reduced and having Krull dimension 0 by some commutative algebra fact. The reducedness is automatic since $R^\flat$ is perfect. □

Proposition 16.26. Suppose that $R^\flat$ is of Krull dimension 0. Then, $E(R^\flat) \to R$ is injective on π_0.

Proof. We claim that it suffices to check this on $\pi_0(-)/p$. Indeed, consider

$$\begin{tikzcd} \pi_0(E(R^\flat)) \arrow[r] \arrow[d] \arrow[rr, equal, bend left] & W(\pi_0 E(R^\flat)) \arrow[r, "W(-/p)"] \arrow[d] & W(\pi_0 E(R^\flat)/p) \arrow[d] \\ \pi_0 R \arrow[r] & W(\pi_0 R) \arrow[r, "W(-/p)"'] & W(\pi_0 R/p) \end{tikzcd}$$

The right vertical map is injective because $W(-)$ preserve injective maps and we assumed that we know this on $\pi_0(-)/p$. But then going around the rectangle the other way is also injective and hence in particular $\pi_0(E(R^\flat)) \to \pi_0 R$.

Now to check it. Consider $\pi_0(E(R^\flat))/p \to \pi_0(R)/p$ where

$$R^\flat \cong ((\pi_0 R)/p)^\flat = \lim \left(\pi_0 R/p \xleftarrow{(-)^p} \pi_0 R/p \xleftarrow{(-)^p} \cdots \right).$$

Suppose that $y \in R^\flat$ is in the kernel. Since $R^\flat$ is of Krull dimension 0, we get that (y) is generated by the idempotent e. All components of e in this limit are idempotent and nilpotent. It follows that $e = y = 0$. □

Applications will follow in the next talk!

17 Applications of Chromatic Nullstellensatz (Vignesh Subramanian)

Vignesh: *It was a long time ago that we were interested in spaces.* TALK 17 04.07.2025

17.1 The Result and Precursors

Consider the Goodwillie tower of ∞-categories (in the sense of Gijs)

$$\begin{tikzcd} & \mathcal{S} \arrow[dl] \arrow[d] \arrow[dr] & \\ \mathbf{Sp} \simeq P_1\mathcal{S} & P_2\mathcal{S} \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

Here is a first evidence for such results:

Theorem 17.1 (Quillen, Sullivan)**.** There is a diagram

$$\begin{tikzcd} & \mathbf{cdgA} \arrow[dd, leftrightarrow, "\simeq"] \\ \mathcal{S}^{\geq 1}_{\mathbb{Q}} \arrow[ur, hook] \arrow[dr, hook] & \\ & \mathbf{dgLie} \end{tikzcd}$$

of fully faithful functors.

In the chromatic tower

$$\mathbb{Q} \longrightarrow K(1) \longrightarrow K(2) \longrightarrow \cdots \longrightarrow \mathbb{F}_p$$

we considered the first point $\mathbb{Q}$. One could also consider the point at ∞, namely $\mathbb{F}_p$, leading to:

Theorem 17.2 (Mandell)**.** There is a fully faithful functor $\mathcal{S}^{\geq 1}_{p\text{-cpl,ft}} \hookrightarrow \mathbf{CAlg}_{\overline{\mathbb{F}}_p}$, $X \mapsto \overline{\mathbb{F}}_p^X$.

But we have so many chromatic points in between. That will be our goal:

Theorem 17.3 (Hopkins-Lurie, Burklund-Schlank-Yuan)**.** There is a fully faithful functor

$$(\mathbf{Sp}^{\leq n}_{p\text{-fin}})^{\mathrm{op}} \hookrightarrow \mathbf{CAlg}_{E_n}(\mathbf{Sp}_{K(n)}) = \mathbf{CAlg}^{\wedge}_{E_n}, \; X \mapsto E_n^X.$$

17.2 Setting Up Hopkins-Lurie Equivalence

Here, $E_n = E_n(L)$ for an algebraically closed field L.

Let A be an $\mathbb{E}_\infty$-$K(n)$-local ring and consider the global sections functor

$$\mathbf{Mod}_A(\mathbf{Sp}_{K(n)})^X \to \mathbf{Mod}_A(\mathbf{Sp}_{K(n)}),\ F \mapsto C^\bullet(X;F) = \lim_X F$$

where $C^\bullet(X;F)$ has an action of $C^\bullet(X;A)$. So we obtain a factorization

$$\begin{tikzcd} \mathbf{Mod}_A(\mathbf{Sp}_{K(n)})^X \arrow[rr] \arrow[dr] & & \mathbf{Mod}_A(\mathbf{Sp}_{K(n)}) \\ & \mathbf{Mod}_{C^\bullet(X;A)}(\mathbf{Sp}_{K(n)}) \arrow[ur] & \end{tikzcd}$$

Question 17.4. Is the left arrow is an equivalence?

Recall by ambidexterity (8.12) that for a p-finite space X and a functor $F : X \to \mathbf{Sp}_{K(n)}$ there is a map $\mathrm{Nm}_X : \mathrm{colim}_X F \to \lim_X F$ which is an equivalence.

Theorem 17.5. Let $f : X \to Y$ be a map of spaces and let $A \in \mathbf{Alg}\left(\mathbf{Sp}^X_{K(n)}\right)$.

(i) Assume that fib f is m-truncated and p-finite. Then,

$$G : \mathbf{LMod}_A\left(\mathbf{Sp}^X_{K(n)}\right) \to \mathbf{LMod}_{f_*A}\left(\mathbf{Sp}^Y_{K(n)}\right)$$

has a fully faithful left adjoint.[24]

(ii) If fib f is p-finite and n-truncated, then G is an equivalence.

Remark* 17.6. Taking $f : X \to *$ answers the question (17.4).

Proposition 17.7 (Push-Pull)**.** Let $f : X \to Y$ be m-truncated and π-finite and $A \in \mathbf{Alg}(\mathbf{Sp}^Y_{K(n)})$. Let $M \in \mathbf{RMod}_{f^*A}(\mathbf{Sp}^X_{K(n)})$ and $N \in \mathbf{LMod}_A(\mathbf{Sp}^Y_{K(n)})$. Then map

$$\beta_{M,N} : f_*M \otimes_A N \to f_*(M \otimes_{f^*A} f^*N)$$

is an equivalence.

Proof. It's enough to do the case $Y \simeq *$. It's a standard good object argument that Lurie uses all the time. Fix M and consider $\mathscr{C} \subseteq \mathbf{LMod}_A(\mathbf{Sp}_{K(n)})$ consisting of N such that $\beta_{M,N}$ is an equivalence.

Observe $A \in \mathscr{C}$ and that $\mathscr{C}$ is closed under colimits since f_* preserves colimits by an ambidexterity argument (8.12). Thus, $\mathscr{C} \simeq \mathbf{LMod}_A(\mathbf{Sp}_{K(n)})$. □

Remark 17.8. Thus, there is an equivalence $C^\bullet(X;M) \simeq C^\bullet(X;A) \otimes M$ for $M \in \mathbf{LMod}_A(\mathbf{Sp}_{K(n)})$. This is a surprising statement.

Theorem 17.9. Let A be a $K(n)$-local $\mathbb{E}_\infty$-ring. Consider the pullback

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

[24] Here, f_*A is given by right Kan extension.

in $\mathcal{S}$ which with $C^\bullet(-;A)$ leads to

$$\begin{tikzcd} A^Y \arrow[r] \arrow[d] & A^X \arrow[d] \\ A^{Y'} \arrow[r] & A^{X'} \end{tikzcd}$$

If Y is n-truncated and p-finite, X is m-type, then the above diagram is a pushout in $\mathbf{CAlg}_A^\wedge$.

We will prove 17.3 through the following equivalent statements.

Proposition 17.10. Let A be an $\mathbb{E}_\infty$-$K(n)$-local ring. The following are equivalent.

(i) The functor $C^\bullet(-;A) : \left(\mathcal{S}_{\le n}^{p\text{-fin}}\right)^{\mathrm{op}} \to \mathbf{CAlg}_A^\wedge$ is fully faithful.

(ii) For all $X \in \mathcal{S}_{\le n}^{p\text{-fin}}$ the unit map $X \to \mathrm{Map}_{\mathbf{CAlg}_A}(A^X, A)$ is an equivalence.

(iii) Check (ii) for $X = K(\mathbb{Z}/p, n)$.

Proof. Note that fully faithfulness is $\mathrm{Map}(Y, X) \to \mathrm{Map}(A^X, A^Y)$. So (i) implies (ii) by putting $Y = *$. Moreover, (ii) $\implies$ (iii) is clear.

We check (ii) $\implies$ (i)'. Need

$$\mathrm{Map}_{\mathcal{S}}(Y, X) \xrightarrow{\simeq} \mathrm{Map}_{\mathbf{CAlg}_A}(A^X, A^Y)$$

for p-finite n-truncated X.

For (iii) $\implies$ (ii) let $F : \mathcal{S} \to \mathcal{S},\ X \mapsto \mathrm{Map}_{\mathbf{CAlg}_A}(A^X, A)$ and $\alpha : \mathrm{id} \Rightarrow F$. Then,

$$\mathscr{C} = \{X : \alpha \text{ is equivalence}\} \subseteq \mathcal{S}_{\le n}^{p\text{-fin}}.$$

By (iii) we get $K(\mathbb{Z}/p, n) \in \mathscr{C}$, then an Eilenberg-Moore argument implies that $\mathscr{C}$ is closed under finite limits. So $K(\mathbb{Z}/p, m) \in \mathscr{C}$ for all $m \le n$. For $m = 0$ we get that finite sets are in $\mathscr{C}$. Can then write

$$\begin{tikzcd} 0 \arrow[r] & G' \arrow[r] \arrow[d, equal] & G \arrow[r] & G'' \arrow[r] & 0 \\ & \mathbb{Z}/p & & & \end{tikzcd}$$

leading to a fiber sequence

$$K(G', m) \longrightarrow K(G, m) \longrightarrow K(G'', m)$$

which lets you recover all n-truncated p-finite spaces. $\square$

17.3 Orientation Theory after Burklund-Schlank-Yuan Strict

Set $A = E_n$ in 17.10. Recall the *Fourier transform* $L_{K(n)}E_n[\mathbb{Z}/p] \simeq E_n^{K(\mathbb{Z}/p,n)}$.[25] Then,

$$\mathrm{Map}_{\mathbf{CAlg}_{E_n}^\wedge}(E_n^{K(\mathbb{Z}/p,n)}, E_n) \simeq \mathrm{Map}_{\mathbf{Sp}_{\ge 0}}(\mathbb{Z}/p, \mathrm{gl}_1 E_n)$$

is the space of **strict units** which we wish to understand.

Definition 17.11.

[25] Recall $K(\mathbb{Z}/p, n) \simeq \Omega^\infty \mathbb{Z}/p[n]$.

(i) The **strict elements** are elements in $\mathrm{Map}_{\mathbf{Sp}_{\geq 0}}(\mathbb{S}[\mathbb{N}], R)$.

(ii) The **$\mathbb{Z}/p$-strict units** are $\mathrm{Map}_{\mathbf{Sp}_{\geq 0}}(\mathbb{Z}/p, \mathrm{gl}_1 R)$.

Remark 17.12. Quotienting out by strict elements preserves the $\mathbb{E}_n$-structure. This is because $\mathbb{N}$ compared to Free$(*)$ has no power operation data and so quotienting by it doesn't kill power operations which is why it is well-behaved with respect to multiplicative structures. In discussion we've Ishan, we've concluded that the only strict elements in $\mathbb{S}$ should be 0 and 1.

Let L be algebraically closed and $0 \not\simeq R \in \mathbf{CAlg}^{\wedge}_{E(L)}$. By the chromatic Nullstellensatz (16.3) there is a map $R \to E(L)$. There is a special property for such maps:

Theorem 17.13. Let R be as above, then the map $\mathrm{pic}\left(\mathbf{Mod}^{\wedge}_{E(L)}\right) \to \mathrm{pic}\left(\mathbf{Mod}^{\wedge}_{R}\right)$ has a retract.

*Proof**. See [BSY22, Theorem 8.1]. This is a non-trivial result which depends on the chromatic Nullstellensatz (16.3). □

Recall that $(-)^{\wedge}$ means that the tensor product is in the $K(n)$-local category.

With this, we return to the strict unit computation.

Proposition 17.14. Let $\mathscr{C} \in \mathbf{CAlg}(\mathbf{Pr}^L)$ and $f : X \to \mathrm{pic}(\mathscr{C})$ be a map in $\mathbf{Sp}_{\geq 0}$, then consider its Thom spectrum $Mf \in \mathbf{CAlg}(\mathscr{C})$. We have the following equivalent statements:

(i) The map f is nullhomotopic.

(ii) There exists a map $Mf \to \mathbb{1}_{\mathscr{C}}$,

(iii) The map $\mathrm{pic}(\mathscr{C}) \to \mathrm{pic}(\mathbf{Mod}_{Mf}(\mathscr{C}))$ has a retract.

*Proof**.

(i) $\implies$ (ii): By (i) we get $Mf \simeq X \otimes \mathbb{1} \to *_{\mathscr{C}} \otimes \mathbb{1} \simeq \mathbb{1}$.

(ii) $\implies$ (iii): This comes from $\mathbb{1}_{\mathscr{C}} \to Mf \to \mathbb{1}_{\mathscr{C}}$.

(iii) $\implies$ (i): By (iii) it suffices to show that the composite

$$X \xrightarrow{f} \mathrm{pic}(\mathscr{C}) \longrightarrow \mathrm{pic}\left(\mathbf{Mod}_{Mf}(\mathscr{C})\right)$$

is nullhomotopic. This is the same thing as an Mf-orientation which corresponds to a map $Mf \to Mf$ by [ACB19, Lemma 3.15]. Choose id_{Mf}.

□

Corollary 17.15. Let L be an algebraically closed field an $f : X \to \mathrm{pic}\left(\mathbf{Mod}^{\wedge}_{E(L)}\right)$ be a map in $\mathbf{Sp}_{\geq 0}$. Then, the following are equivalent:

(i) $Mf \not\simeq 0$,

(ii) There is a map $Mf \to E(L)$ in $\mathbf{CAlg}^{\wedge}_{E(L)}$.

(iii) The map f is nullhomotopic.

(iv) $Mf \simeq E(L)[X] \in \mathbf{CAlg}^{\wedge}_{E(L)}$.

*Proof**. The directions (iii) $\implies$ (iv) $\implies$ (i) are fine. For (ii) $\implies$ (iii) see 17.14. Besides all these formal parts, the implication (i) $\implies$ (ii) is really a consequence of the Chromatic Nullstellensatz. Namely, 17.14(iii) holds by 17.13. □

Remark* 17.16. There is a $K(n)$-local version as a direct consequence of the above [BSY22, Corollary 8.13].

Proposition 17.17. Let L be an algebraically closed field and H be a p-torsion abelian group. Then,

$$\operatorname{map}_{\mathbf{Sp}_{\geq 0}}\left(H, \operatorname{Pic}(\mathbf{Mod}^{\wedge}_{E(L)})\right) \simeq \Sigma^{n+1} H^*$$

where H^* stands for the Pontryagin dual.

Proof. By some reduction arguments it's enough to prove for H finite.

Let

$$f \in \pi_m\left(\operatorname{map}_{\mathbf{Sp}_{\geq 0}}(H, \operatorname{pic}(\mathbf{Mod}^{\wedge}_{E(L)})\right) \cong \pi_0\left(\operatorname{map}_{\mathbf{Sp}_{\geq 0}}(\Sigma^m H, \operatorname{pic}(\mathbf{Mod}^{\wedge}_{E(L)}))\right),$$

i.e. we are looking at $f : B^m H \to \operatorname{Pic}(\mathbf{Mod}^{\wedge}_{E(L)})$, so $Mf \simeq \lim_{B^m H} f$ by ambidexterity (8.12). For $m \leq n$ and $m \geq n+2$ it holds that

$$\lim_X : \operatorname{Fun}\left(B^m H, \mathbf{Mod}^{\wedge}_{E(L)}\right) \to \mathbf{Mod}^{\wedge}_{E(L)}$$

is conservative [HL13, Corollary 5.4.4, 5.4.5(2)]. So in these cases $Mf \not\simeq 0$ implying that f is nullhomotopic by orientation theory (17.15). So we are reduced to the functor

$$H \mapsto \left[\Sigma^{n+1} H, \operatorname{pic}(\mathbf{Mod}^{\wedge}_{E(L)})\right]$$

which can be checked to correspond to $H \mapsto H^*$ [BSY22, Proposition 8.14]. □

The case $H = C_p$ computes the strict units as desired.

Remark 17.18. An application of this material is a redshift result: Let $R \in \mathbf{CAlg}^{\wedge}_{T(n)}$, then $L_{T(n+1)} K(R) \not\simeq 0$.

18 Conclusion & Outlook (Gijs Heuts)

Gijs will try to touch on the following: TALK 18 04.07.2025

- Interaction between different heights,
- Interaction between chromatic homotopy theory & geometry & homological stability.

18.1 Interaction between Different Heights

For classical algebra there is already a thick subcategory theorem in $\mathcal{D}(\mathbb{Z})^\omega$. These are pretty simple though. A picture for $\operatorname{Spec} \mathbb{Z}$ is

$$\begin{tikzcd}
\mathbb{F}_2 \arrow[drr, no head] & \mathbb{F}_3 \arrow[dr, no head] & \mathbb{F}_5 \arrow[d, no head] & \mathbb{F}_7 \arrow[dl, no head] & \cdots \arrow[dll, no head] \\
 & & \mathbb{Q} & &
\end{tikzcd}$$

In higher algebra, consider $\mathbf{Sp}^\omega = \mathcal{D}(\mathbb{S})^\omega$ and now $\operatorname{Spec} \mathbb{S}$ is precisely described by the thick subcategory theorem:

$$\begin{tikzcd}
\mathbb{F}_2 \arrow[d, dash] & \mathbb{F}_3 \arrow[d, dash] & \mathbb{F}_5 \arrow[d, dash] & \mathbb{F}_7 \arrow[d, dash] & \cdots \\
\vdots \arrow[d, dash] & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] & \\
K(2)_2 \arrow[d, dash] & K(2)_3 \arrow[d, dash] & K(2)_5 \arrow[d, dash] & K(2)_7 \arrow[d, dash] & \cdots \\
K(1)_2 & K(1)_3 & K(1)_5 & K(1)_7 & \cdots \\
 & & \mathbb{Q} \arrow[llu, dash] \arrow[lu, dash] \arrow[u, dash] \arrow[ru, dash] \arrow[rru, dash] & &
\end{tikzcd}$$

Then, homotopy theory breaks into two parts:

(i) Understand the *monochromatic pieces*, i.e. $\mathbf{Sp}_{K(n)}$ or $\mathbf{Sp}_{T(n)}$. These are the *local* parts.

(ii) Glue together pieces into a global picture of $\mathbf{Sp}$.

This week we mostly focused on (i) and not so much about (ii). But we did a little bit, e.g. chromatic convergence. Let us start by mentioning some problems for (ii).

(a) Blueshift & redshift: What's up with blueshift?

Theorem 18.1 (Kuhn)**.** Let $X \in L_n^f\mathbf{Sp}$ and G be a finite group acting on X. Then,

$$X^{tG} = \mathrm{cofib}(X_{hG} \to X^{hG})$$

is L_{n-1}^f-local.

Redshift comes out of the study of algebraic K-theory which is of course a hot topic these days. Since there has already been a lot of workshops on recently, Gijs and Ishan decided not to go in that direction for this Talbot.

Observation 18.2 (Ausoni-Rognes)**.** Taking K-theory increases height by 1.

This was an informal thing but there are some ways of making this into a theorem.

Theorem 18.3 (Many people)**.** Let $R \in \mathbf{CAlg}(\mathbf{Sp}_{T(n)})$. Then, $L_{T(n+1)}K(R) \not\simeq 0$.

Quantitative versions of redshift: There are some calculations that one could do.

- What is $L_{T(n+1)}K(E_n)$?
- What is $L_{T(2)}K(\mathbb{S}_{K(1)})$?

We have seen this week that we have good ring spectra approximating $\mathbb{S}_{K(n)}$, namely E_n. We don't have this for $\mathbb{S}_{T(n)}$ when $n \geq 2$ (yet). You could hope that something like $L_{T(2)}K(\mathbb{S}_{K(1)})$ is a good approximation for $\mathbb{S}_{T(2)}$.

The challenge is to find good ring spectra over $\mathbb{S}_{T(n)}$.

(b) Chromatic splitting conjecture (Hopkins): Gijs gave an entire speech that in our field there should be more conjectures! People need to write up their work and need to state their questions. It's a lot cooler to prove Levy's Conjecture than to say that you have computed some obscure thing.

Lemma 18.4 (Arithmetic fracture square)**.** Let $X \in \mathbf{Sp}_{(p)}$. Then, there is a pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_p^\wedge \arrow[d] \\ X_\mathbb{Q} \arrow[r] & (X_p^\wedge)_\mathbb{Q} \end{tikzcd}$$

Similarly chromatically:

Lemma 18.5 (Chromatic fracture square)**.** There is a pullback square

$$\begin{tikzcd} L_n X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_{K(n)} X \arrow[d] \\ L_{n-1} X \arrow[r] & L_{n-1} L_{K(n)} X \end{tikzcd}$$

for $n \geq 1$.

The chromatic splitting conjecture says that it's supposed to be easier than gluing successively.

Conjecture 18.6 (Chromatic splitting conjecture, Hopkins)**.** The map $L_{n-1}X \to L_{n-1}L_{K(n)}X$ is the inclusion of a summand.

If this were true, it would have a lot of funny consequences.

Corollary 18.7. If X is the p-completion of a finite spectrum, then $X \to \prod_{n\geq 0} L_{K(n)}X$ is the inclusion of a summand.

So one doesn't need to know so much about the gluing maps to understand X. The conjecture is known for $n \leq 2$ by brute force calculation. These calculations is the reason that Mike made this conjecture. But it seems like there is no conceptual understanding for this.

(c) Unstable Homotopy Theory: We saw this week that there are those Bousfield-Kuhn functors $\Phi_n : \mathcal{S}_* \to \mathbf{Sp}_{T(n)}$ for $n \geq 0$ which in some sense record the 'v_n-periodic homotopy types' of spaces.

Question 18.8. How do the different Φ_n relate?

Here is a more refined question.

Lemma 18.9. The functor Φ_n lifts to an equivalence $\mathcal{S}_{v_n} \xrightarrow{\simeq} \mathbf{Lie}(\mathbf{Sp}_{T(n)})$.

The challenge is to come up with a theory of 'transchromatic Lie algebras' $\widetilde{\mathbf{Lie}}_{\leq n}$. These should combine Φ_i and $\mathbf{Lie}(\mathbf{Sp}_{T(i)})$ for $i \leq n$ into one object.

Warning 18.10. The category $\mathbf{Lie}(L_n^f\mathbf{Sp})$ is the wrong answer.

The moral reason is that $\mathcal{S}_*$ becomes 'algebraic' when localized at a single height, but this can never be algebraic when localized at more than one height.

Gijs said something slightly vague but after his speech about conjectures, we pressed him to state a more precise conjecture.

Conjecture 18.11 (Heuts' European Talbot Conjecture)**.** The adjunction

$$\mathbf{coSp}(L_n^f \mathcal{S}_*^{p^\infty\text{-torsion}}) \underset{\Phi}{\overset{\Theta}{\rightleftarrows}} L_n^f \mathcal{S}_*^{p^\infty\text{-torsion}}$$

is monadic, $\Phi\Theta$ preserves sifted colimits and $\mathbf{coSp}(L_n^f \mathcal{S}_*^{p^\infty\text{-torsion}})$ admits a nice description (in terms of some recollement).

Gijs: *This is vague but I blame mistakes to the note takers.*

This is related to a theorem from Yuqing in the monochromatic setting. Classically these **coSp** for spaces are 0.

18.2 Connections to Geometry/Homological Stability

This week we constructed a lot of funky cohomology theories while the cohomology theories we started out with had a good geometric interpretation.

(i) Are there geometric descriptions of cochains for E_n or $K(n)$? Is there a geometric description for tmf (Stolz-Teichner program)? This is supposed to be about 2D field theories with a million adjectives. What adjectives to put... nobody knows.

Gijs: *The spectrum* tmf *comes from physics and they hold themselves to a higher standard and can prove many things we cannot.*

A cool thing for K-theory is index theory, like the Atiyah-Singer index theorem which connects geometry with analysis. Witten's original idea is that tmf should play a similar role with indices on free loop spaces.

(ii) Manifolds: In the past years there was lots of progress of computing $H^\bullet(-;\mathbb{Q})$ and $\pi_\bullet(-)\otimes\mathbb{Q}$ of BDiff(M). The methods are quite robust! Why not K-theory? From the point of view of this week you should think that K-theory is much easier than $\mathbb{F}_p$, e.g. in terms of power operations there is just the δ while on $\mathbb{F}_p$ there are all these Q_i's.

(iii) Homological stability: Randall-Williams has been speaking a lot about this and it sounds like things related to the periodicity theorem. Say R is a graded $\mathbb{E}_2$-algebra over k. Consider for example $R = \{C_\bullet(\mathrm{Conf}_d(\mathbb{R}^2);k)\}_{d\geq 0}$ or $R = \{C_\bullet(G_d;k)\}_{d\geq 0}$ where $\coprod_{d\geq 0} BG_d$ should be a braided monoidal groupoid. Then, we can talk about its bigraded homotopy groups $\pi_{n,d}R \cong \pi_n R(d)$.

Stability says that we have some $\sigma \in \pi_{0,1}R = \pi_0 R(1)$ such that the cofiber R/σ has a *vanishing line*, i.e. $\pi_{n,d}(R/\sigma) = 0$ for some $d < An + B$. In other words, σ is a homology isomorphism in some range.

Observation 18.12 (GKRW)**.** Let R be the $\mathbb{E}_2$-algebra of $C_\bullet(\mathrm{MCG}_\delta(S))$ of some surface S.[26] Then,

(i) We have $\pi_{n,d}(R/\sigma) = 0$ for $d < \frac{2n}{3}$.

(ii) There are maps $\varphi : \mathbb{S}^{3,2} \otimes R/\sigma \to R/\sigma$ such that $\pi_{n,d}(R/(\sigma,\varphi)) = 0$ for $d < \frac{3n}{4}$. Equivalently,
$$H_{d-2}(G_{n-3}, G_{n-4}) \xrightarrow{\sim} H_d(G_n, G_{n-1}).$$

This is related to metastability.

[26]Here, S is a two holed torus with a disc cut out. Gijs: *This will really test the skills of the note takes.*

Theorem 18.13 (Periodicity Theorem)**.** Let R be a good enough graded $\mathbb{E}_2$-ring over $\mathbb{F}_p$. Then, there are self-maps

$$\begin{aligned}
&\alpha_1 : \mathbb{S}^{n_1,d_1} \otimes R \to R, \\
&\alpha_2 : \mathbb{S}^{n_2,d_2} \otimes R/\alpha_1 \to R/\alpha_1, \\
&\vdots, \\
&\alpha_i : \mathbb{S}^{n_i,d_i} \otimes R/(\alpha_1, \cdots, \alpha_{i-1}) \to R/(\alpha_1, \cdots, \alpha_{i-1})
\end{aligned}$$

such that:

(i) each α_i is not nilpotent,

(ii) $R/(\alpha_1, \cdots, \alpha_i)$ has a vanishing line of slope < 1.

A Complex-Oriented Cohomology Theories

Here is an interactive and improvised part by the mentors.

A.1 Complex Orientations

Definition A.1. Let E be a cohomology theory. Then, E is **complex orientable** if

$$\widetilde{E}^2(\mathbb{C}P^\infty) \to \widetilde{E}^2(\mathbb{C}P^1) \cong \widetilde{E}^0(S^0)$$

is surjective. An lift of $1 \in \widetilde{E}^0(S^0)$ is a **complex orientation** of E.

In this case, we have a good notion of Chern classes.

Fact A.2. There is an isomorphism $E^\bullet(\mathbb{C}P^\infty) \cong E^\bullet[\![t]\!]$.

This t should be thought of as the first Chern class on the universal line bundle on $\mathbb{C}P^\infty$.

Quillen thought $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$ is correct in general and arrived at all sorts of ridiculous contradictions. So he was debating what was actually the correct formula.

There is a map $m : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ inducing

$$m^* : E^\bullet[\![t]\!] \to E^\bullet[\![x,y]\!],\ t \mapsto f(x,y).$$

Then, this $f(x,y)$ is a **formal group law** over $E^\bullet$, i.e. it satisfies the formulas

(i) $f(x,y) = f(y,x)$,

(ii) $f(x,0) = x, f(0,y) = y$,

(iii) $f(f(x,y),z) = f(x, f(y,z))$.

In other words, applying Spf, this is really the structure of a formal group (with the choice of a coordinate).

Example A.3. For example $H\mathbb{Z}$, KU, MU. In fact, any even ring spectrum is complex orientable. Do obstruction theory! You want to construct a lift

$$\begin{tikzcd}
\mathbb{C}P^\infty \arrow[dr, dashed] & \\
\mathbb{C}P^1 \arrow[u] \arrow[r] & \Omega^{\infty-2}E
\end{tikzcd}$$

and the obstructions are all in odd classes.

Proof.* Here are more details. Proceed inductively. We obtain $\mathbb{C}P^{n+1}$ from $\mathbb{C}P^n$ by attaching an $(2n+2)$-cell, i.e. there is a cofiber sequence $S^{2n+1} \to \mathbb{C}P^n \to \mathbb{C}P^{n+1}$, so we are considering the diagram

$$\begin{tikzcd}
\mathbb{C}P^{n+1} \arrow[dr, dashed] & \\
\mathbb{C}P^n \arrow[u] \arrow[r] & \Omega^{\infty-2}E \\
S^{2n+1} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

where by the universal property of cofibers[27] the dashed arrow exists if and only if the composite $S^{2n+1} \to \mathbb{C}P^n \to \Omega^{\infty+2}E$ is nullhomotopic. But this is the case by evenness of E! Thus, we can lift inductively. An $\lim^1$ argument (using evenness again) shows that this extends to the limit $\mathbb{C}P^\infty$. $\square$

Note that the functor $\mathbf{CRing} \to \mathbf{Set}$, $R \mapsto \mathrm{FGL}(R)$ is corepresentable. More specifically, it is corepresentable by the **Lazard ring** L.

Theorem A.4 (Lazard)**.** There is an isomorphism $L \cong \mathbb{Z}[a_1, a_2, \cdots]$.

Can also put everything in a graded setting where we put a formal group law $f = \sum_{i,j} b_{ij} x^i y^j$, put $|x| = |y| = -2$ and $|b_{ij}| = 2(i+j-1)$. In Lazard's theorem then $|a_i| = 2i$ (which corepresents $\mathbf{grCRing} \to \mathbf{Set}$).

Since MU is complex orientable, there is a map $L \to \mathrm{MU}_\bullet$.

Theorem A.5 (Quillen)**.** This map $L \to \mathrm{MU}_\bullet$ is an isomorphism.

A different (but actually much easier) result is that complex orientations on E correspond to maps $\mathrm{MU} \to E$ in $\mathbf{CAlg}(h\mathbf{Sp})$.

A.2 Moduli Stack of Formal Groups

To speak about stacks, we better encode automorphisms.

Observation A.6. Suppose that E is complex oriented, then

$$\pi_\bullet(E \otimes \mathrm{MU}) \cong E_\bullet \mathrm{MU} \cong E_\bullet[b_1, b_2, b_3, \cdots].$$

Now we have two complex orientations

$$(E \otimes \mathrm{MU})[\![t_E]\!] = (E \otimes \mathrm{MU})^{\mathbb{C}P^\infty} = (E \otimes \mathrm{MU})[\![t_{\mathrm{MU}}]\!]$$

where $g = t + b_1 t^2 + b_2 t^3 + \cdots$ is a change of base of these two coordinates. Setting $E = \mathrm{MU}$ we learn that $\mathrm{MU}_\bullet \mathrm{MU}$ parametrizes universal graded formal group laws and strict automorphisms of formal group laws. In particular, there are now maps

$$\begin{tikzcd}
\mathrm{MU} \arrow[r, shift left=2, "\eta_L"] \arrow[r, shift right=2, "\eta_R"'] & \mathrm{MU} \otimes \mathrm{MU} \arrow[l, "\mu" description]
\end{tikzcd}$$

The left side corepresents formal group laws while the right side corepresents strict isomorphisms of formal group laws. Here, η_L, η_R are source and target and μ is the identity. One could continue this diagram.

[27] An equivalent argument is via the cofiber LES on E-cohomology.

This now allows us to understand the Adams-Novikov spectral sequence. There is an augmented cosimplicial object

$$\mathbb{S} \longrightarrow \mathrm{MU} \rightrightarrows \mathrm{MU}\otimes\mathrm{MU} \, \substack{\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow} \cdots$$

We get $\mathrm{MU}^{\otimes \bullet+1} \in \mathrm{Fun}(\Delta, \mathbf{Sp})$. One obtains an associated filtered spectrum and then associated to it the Adams-Novikov spectral sequence.

In particular, we obtain a diagram

$$\begin{tikzcd} & & \mathbf{Grpd} \arrow[d, hook] \\ \mathbf{CRing} \arrow[r] \arrow[urr, dashed, bend left, "\mathcal{M}^s_{\mathrm{fg}}"] & \mathbf{sSet} \arrow[r] \arrow[ur] & \mathcal{S} \end{tikzcd}$$

where the bottom left map is defined as $R \mapsto \mathrm{Hom}_{\mathbf{CRing}}(\pi_\bullet \mathrm{MU}^{\otimes\bullet+1}, R)$. One can extract a groupoid out of this since $(\mathrm{MU}_\bullet, \mathrm{MU}_\bullet \mathrm{MU})$ is a Hopf algebroid. This is the functor that defines $\mathcal{M}^s_{\mathrm{fg}}$. It's the so-called **moduli stack of formal groups (with strict isomorphisms)**. I can't make sense of this factorization though;[28] more explicitly, what is $\mathbf{sSet} \to \mathbf{Grpd}$?

Let's classify formal groups over algebraically closed fields up to isomorphism.

Fact A.7. There is one formal group up to isomorphism to characteristic 0 (this is in height 0). There are infinitely many formal group laws up to isomorphism characterized by the height $n \in \mathbb{N}_{\geq 0} \cup \{\infty\}$.

To understand heights, recall that $[p]t = t +_F \cdots +_F t$ where it's doing this p times and where we write $F(x, y) = x +_F y$. Here, we have

$$f([p]x, [p]y) = [p](f(x, y)).$$

Then, one can write $[p]t = g(t^{p^h})$ such that $\frac{\mathrm{d}g}{\mathrm{d}t}(0) \neq 0$ and here h is the **height** of the formal group law which is an invariant of the formal group and in fact the only invariant for algebraically closed fields (**A.7**).

There was some discussion about MO but this spectrum is chromatically not nearly as interesting as MU. There was also some interest about $\mathbb{E}_\infty$-complex orientations. These are in general hard to understand and there are relations to power operations and such.[29]

B Question Session 1

There were some announcements. For instance, Daniel brought beer and suggested a drinking game:

Daniel: *Every time you hear the word local you have to take a sip.*

B.1 Localizations

Gijs: *Just wanted to start by briefly saying something about... localizations.*

We discussed some basics about Bousfield/reflective localizations which I did not type. But here are some relevant answers to questions that arose.

[28] Sorry, I was almost falling asleep, a cracking bunk bed was responsible for a long night.
[29] My PhD project focuses on such objects...

(i) There is an formula for $K(n)$-localization: $L_{K(n)}X \simeq \left((X \otimes E_n)^{\wedge}_{p,v_1,\cdots,v_{n-1}}\right)^{h\mathbb{G}_n}$.

(ii) The functor $\mathbf{Sp} \to L_n^f\mathbf{Sp}$ is the localization with respect to $V_{n+1} \to 0$ of $\mathbf{Sp}$.

(iii) Ryan asked about which localizations are easy to compute. The answer is that they are all hard. Ryan asked about L_n MU. The answer is that this is easy. Let X be an MU-module, then there is a pullback

$$\begin{tikzcd} L_1X \arrow[r] \arrow[d] & X[v_1^{-1}] \arrow[d] \\ X[p^{-1}] \arrow[r] & X[p^{-1}, v_1^{-1}] \end{tikzcd}$$

and for L_2X is some sort of cube.

Ryan asked if there is some easy example showing that $L_{K(2)} \not\simeq L_{T(2)}$. Ishan was a bit flabbergasted. #telescopeconjecture

(iv) I asked what's the easiest way to see that $L_{K(n)}$ is not smashing. Here is Ishan's answer.

Consider the colimit diagram

$$\mathbb{S}/p \longrightarrow \mathbb{S}/p^2 \longrightarrow \cdots \longrightarrow \mathbb{Q}/\mathbb{S}_{(p)}$$

If you $K(1)$-localize, you get

$$\cdots \longrightarrow \cdots \longrightarrow \cdots \longrightarrow \Sigma L_{K(1)}\mathbb{S}$$

and everything is rationally trivial but the last term is a copy of $\mathbb{Q}_p$. This is because with the formula in (i) we have $L_{K(1)}\mathbb{S} = (\mathrm{KU}_p^{\wedge})^{h\mathbb{Z}_p^\times}$ with action by Adams operations. Let $\ell \in \mathbb{Z}_p^\times$ be a generator. Then, we get a fiber sequence

$$L_{K(1)}\mathbb{S} \longrightarrow \mathrm{KU}_p^{\wedge} \xrightarrow{1-\psi^\ell} \mathrm{KU}_p^{\wedge}$$

where on homotopy groups the second map is

$$\mathbb{Z}_p[\beta^{\pm}] \to \mathbb{Z}_p[\beta^{\pm}],\ \beta^n \mapsto \beta^n(1-\ell^n).$$

We compute $\pi_0 = \mathbb{Z}_p$ and $\pi_{-1} = \mathbb{Z}_p$. Moreover,

$$\pi_{2n-1} = \mathbb{Z}_p/(1-\ell^n) = \begin{cases} 0 & p-1 \nmid n, \\ \mathbb{Z}_p/p^{v_p(n)} & \text{else.} \end{cases}$$

It goes through $(\mathbb{Z}_p, \mathbb{Z}_p, \cdots, \mathbb{Z}/p, 0, \cdots, 0, \mathbb{Z}/p, 0, \cdots)$ where we got $\mathbb{Z}/p$ in $2p-3$ and $4p-5$ and so on. This first $\mathbb{Z}/p$ gives α_1.

B.2 How canonical are the v_n?

The answer is that they are really not canonical, but v_n is canonical in $(p, v_1, \cdots, v_{n-1})$. Moreover, v_n-self maps are asymptotically periodic and one can always find a self-map of the form $v_k^{p^n}$.

There is a v_1-self map $\mathbb{S}/2 \otimes \mathbb{S}/\eta \to \mathbb{S}/2 \otimes \mathbb{S}/\eta$.

B.3 Is there an unstable thick subcategory theorem

Yes, and it's not so different from the stable one:

Theorem B.1 (Bousfield)**.** Let $X, Y \in \mathcal{S}_*^{\mathrm{fin}}$. Then, $\langle \Sigma X \rangle \leq \langle \Sigma Y \rangle$ if and only if the following are satisfied:

(i) type $Y \leq$ type X,

(ii) conn $Y \leq$ conn X.

So you only need one more condition. The only weird thing is that you need these Σ. Without, it is an open problem, i.e. figuring out Bousfield classes in $\mathcal{S}_*^{\mathrm{fin}}$. Here is another open problem:

Problem B.2. What is the lowest connectivity of a finite space of type n?

It is known that it is $\geq n$. I asked where this can be found. Gijs said that this is in some paper from Bousfield... but he doesn't know where. Maite then retorted that the precise reference can be found in a survey paper by Gijs himself [Heu20]!

B.4 What's the role of BP?

Have $\pi_\bullet \mathrm{BP} \cong \mathbb{Z}_{(p)}[v_1, v_2, \cdots]$. This is a smaller version of MU that is maybe more useful in calculations. It classifies p-typical formal group laws. The p-series has a nice formula, namely $[p]t = pt +_F v_1 t^p +_F v_2 t^{p^2} + \cdots$.

B.5 Can you say something about Koszul duality?

Gijs asked about which version of Koszul duality he should speak about. We settled on the following:

Consider $(\mathscr{C}, \otimes)$. Then, there is an adjunction

$$\mathbf{Alg}_{\mathbb{E}_1}^{\mathrm{aug}}(\mathscr{C}) \underset{\mathrm{Cobar}}{\overset{\mathrm{Bar}}{\rightleftarrows}} \mathbf{coAlg}_{\mathbb{E}_1}^{\mathrm{aug}}(\mathscr{C})$$

where $\mathrm{Bar}(A \to \mathbb{1}) \simeq \mathbb{1} \otimes_A \mathbb{1}$.

Example B.3. Let $\mathscr{C} = \mathbf{Mod}_k$ and $R = k[x_1, \cdots, x_n]$. Then,

$$\pi_\bullet(k \otimes_R k) \cong \mathrm{Tor}_\bullet^R(k, k) \cong \Lambda(\sigma x_1, \cdots, \sigma x_n).$$

The duality is that you can go back via Cobar.

This will also feature in Ryan's talk (**10 ½**).

B.6 How does localization of multiplicative objects work?

The question is essentially about the following result.

Proposition B.4. Let R be an $\mathbb{E}_\infty$-ring and $r \in \pi_n R$. Then, $R[r^{-1}]$ is an $\mathbb{E}_\infty$-ring.

Proof. Consider $\mathbf{Mod}_R$ which is symmetric monoidal with unit R. In a symmetric monoidal ∞-category $(\mathscr{C}, \otimes)$ we always have that $\mathrm{End}(\mathbb{1})$ is an $\mathbb{E}_\infty$-ring spectrum. Consider $\langle R/r \rangle \subseteq \mathbf{Mod}_R$ and consider the localization that kills that object, so we get $\mathbf{Mod}_R[r^{-1}]$. So these are the modules with the property that there are no maps from R/r. In other words, that's precisely the property that multiplication by r is an equivalence. We now observe that this localization $\mathbf{Mod}_R \to \mathbf{Mod}_R[r^{-1}]$ is a symmetric monoidal localization using that $\langle R/r \rangle$ is a thick tensor ideal. Then, $\mathrm{End}(\mathbb{1}) = R[r^{-1}]$. □

Remark B.5. This argument was for $\mathbb{E}_\infty$-objects. For lower commutativity there are some relations with the Ore condition. This $\mathrm{End}(\mathbb{1})$ is still $\mathbb{E}_n$ for $\mathbb{E}_n$-rings R even if $\mathbf{Mod}(R)$ is only $\mathbb{E}_{n-1}$.

We defined $\mathrm{KU} = \mathrm{ku}[\beta^{-1}]$ and this gives an $\mathbb{E}_\infty$-structure to KU.

Remark* B.6. Another formulation of this is in [HW21, Proposition III.4]. The formal idea is essentially that $R \to R[r^{-1}]$ is a $\otimes$-idempotent in $\mathbf{Mod}_R$, so localizing with respect to it is a symmetric monoidal Bousfield localization.

C Chili Gong Show

On Wednesday evening we had a *chili gong show* which I think was mainly initiated by Gijs. The cooking team went to the bazar in the afternoon and fetched a variety of different chilis and the gist was that the group (everyone voluntarily) ate one of these chilis and then some of the people gave (not too serious) mathematical gong show talks.

Figure 5: Aftermath of the chili.

Gijs: *We also have some ice-cream, yoghurt and milk... but nothing really helps!*

Some people were definitely hit harder than others, as we had also seen from some of the talks... Unfortunately, I didn't take notes of any of these talks (sorry!) but let me say that my talk borrowed a bit of Ryan's magic.

D Question Session 2

D.1 Hopf Algebroid vs. Stacks

The main example is $(\mathrm{MU}_\bullet, \mathrm{MU}_\bullet \mathrm{MU})$ and then you get

$$\begin{tikzcd} \mathrm{MU}_\bullet \arrow[r, shift left=2, "\eta_L"] \arrow[r, shift right=2, "\eta_R"'] & \mathrm{MU}_\bullet \mathrm{MU} \arrow[l] \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & \cdots \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] \end{tikzcd}$$

One has $\mathbf{HopfAlgbr} \simeq \mathbf{coGrpd}(\mathbf{Ring})$. In other words, the Hopf algebroid $(\mathrm{MU}_\bullet, \mathrm{MU}_\bullet \mathrm{MU})$ corepresents a functor $\mathbf{CRing} \to \mathbf{Grpd}$. This is really just a linguistic device and the language of stacks was popularized by Mike's COCTALOS lectures. It is also useful to describe the

Landweber exact functor theorem in stacky language (although in practice one still sits down and tries to work with regular sequences).

One uses the fpqc topology which is like the coarsest topology you can reasonably use.

D.2 Vanishing Curve on ANSS

Recall that MU detects nilpotence by the nilpotence theorem. Recall that the ANSS has signature

$$E_2 = H^\bullet\left(\mathcal{M}_{\mathrm{fg}}; \omega^{\otimes \bullet}\right) \Rightarrow \pi_\bullet \mathbb{S}.$$

Recall that descendability says that the ANSS has a horizontal vanishing line. Nilpotence says that we have a vanishing curve of sublinear growth.

Since $\pi_\bullet \mathbb{S}$ is torsion for $\bullet > 0$, non-trivial elements have to be of positive filtration. But then taking successive multiplications is on a line which eventually overtakes the sublinear curve, so it becomes 0.

D.3 Non-Existence of Multiplicative Structures on Moore Spectra

The spectrum $\mathbb{S}/2$ doesn't even admit a left unital multiplication, so it admits essentially no multiplicative structure. The reason is that $\mathrm{id}_{\mathbb{S}/2}$ has order 4. Suppose it has a left unital multiplication, then we can consider

$$\mathbb{S}/2 \longrightarrow \mathbb{S}/2 \otimes \mathbb{S}/2 \longrightarrow \mathbb{S}/2$$

which yields a splitting but this cannot happen because it doesn't split as $\mathbb{S}/2 \oplus \Sigma\mathbb{S}/2$. This can be checked on $H^\bullet(-; \mathbb{F}_2)$ for which one only needs the Bockstein and the Cartan formula.

The spectra $\mathbb{S}/p^k$ are never $\mathbb{E}_\infty$ and there must be a million ways to see this but the usual way is via power operations. For example, KU/p^k would be $\mathbb{E}_\infty$ for which there is some δ-ring structure obstruction. This δ has the effect of making elements less p-divisible.

Ryan: *Is there any non-trivial element whose cofiber is $\mathbb{E}_\infty$?*

Gijs: *I don't think so.*

Ishan: 0.

Ishan: 1.

Ishan: -1!

Gijs: *I give up.*

Ishan says that this seems not to be known.

D.4 Examples of Filtrations and Spectral Sequences

Let's try to compute $M \otimes_R N$. What can we do? Take the Postnikov tower: $(\tau_{\geq \bullet} M) \otimes_{\tau_{\geq \bullet}} (\tau_{\geq \bullet} N)$ whose associated graded is

$$\Sigma^\bullet \pi_\bullet M \otimes_{\Sigma^\bullet \pi_\bullet R} \Sigma^\bullet \pi_\bullet N \cong \mathrm{Tor}_{\pi_\bullet R}(\pi_\bullet M, \pi_\bullet N).$$

This converges to $\pi_\bullet(M \otimes_R N)$.

Bhavna asked if the Adams spectral sequence can come from a Postnikov filtration. Ishan answers that the Postnikov tower gives the $\mathbb{S}$-based Adams spectral sequence.

For the Grothendieck spectral sequence consider

$$\begin{tikzcd} \mathcal{D}(\mathscr{A}) \arrow[r, "LF"'] \arrow[rr, bend left, "L(G\circ F)"] & \mathcal{D}(\mathscr{B}) \arrow[r, "LG"'] & \mathcal{D}(\mathscr{C}). \end{tikzcd}$$

Take $LG(\tau_{\geq \bullet} LF)$, then its associated graded is $LG(\pi_\bullet LF)$ which converges to $L(G \circ F)$.

D.5 Operads

The question is about naturally appearing operads not in spaces. Gijs' favourite example is the spectral Lie operad.

(i) Goodwillie derivatives of the identity functor on $\mathcal{S}_*$. Recall that rationally $\mathcal{S}_*$ can be modelled by dg Lie algebras by Quillen. This phenomenon always appears by spectrally enriching.

(ii) Koszul Duality: There is an equivalence $\mathbf{Lie} \simeq \mathrm{Bar}(\mathbb{C}\mathrm{omm})^\vee$.[30]

(iii) One can explicitly describe $\mathbf{Lie}(n) \simeq (\mathbf{Part}^{\pm}(n)^\diamond)^\vee$ where $\mathbf{Part}^{\pm}$ is the partition poset of $\{1, \cdots, n\}$.

There is a functor $\mathrm{TAQ} : \mathbf{CAlg}^{\mathrm{aug}}(\mathscr{C}) \to \mathscr{C}$ such that $\mathrm{TAQ}^\vee(R)$ is naturally a spectral Lie algebra. That's why Lie algebras are important in deformation theory.

I inquired about what happens if we replace **Part** by the linear partition complex. Gijs said that for the Tits building and the linear partition complex this will not be an operad.

I also asked about examples other than the Lie operad. There are many operads, e.g. there is something called the *gravity operad*. But in Gijs' honest (politically correct opinion) his library of operads are just the $\mathbb{E}_n$-operad and the Lie operad. Gijs comments that some French mathematicians care about operads on chain complexes. I will not say more.

D.6 Fancy Way of $\mathbb{E}_\infty$-Structure on E_n

Here is a quick way to see why an $\mathbb{E}_\infty$-structure on E_n exists. The construction is due to Ishan, Robert and Dustin. Consider

$$\mathbf{Sp} \xleftarrow{\tau^{-1}} \mathbf{Syn}_{\mathrm{MU}}^{\mathrm{even}} \xrightarrow{-\otimes C\tau} \mathbf{IndCoh}(\mathcal{M}_{\mathrm{fg}})$$

We will try to construct an $\mathbb{E}_\infty$-object in $\mathbf{Syn}_{\mathrm{MU}}^{\mathrm{even}}$. Consider a stack

$$\mathbf{Alg}_{\mathbb{E}_\infty}(\mathbf{Sp}) \to \mathcal{S}, R \mapsto \mathrm{Fun}^{\otimes}(\mathbf{Syn}_{\mathrm{MU}}, \mathbf{Mod}_R).$$

Look at the connective cover of the descent spectral sequence on MUP via

$$\tau_{\geq \bullet} \mathrm{MUP} \rightleftarrows \tau_{\geq \bullet} \mathrm{MUP} \otimes \mathrm{MUP} \cdots$$

So this gives rise to a stack which we call $\widetilde{\mathcal{M}}_{\mathrm{fg}}$. There is some line bundle $\mathbb{S}^{0,-2}$ on the stack and some τ, and moreover, $\widetilde{\mathcal{M}}_{\mathrm{fg}}^{\leq n}/\tau = \mathcal{M}_{\mathrm{fg}}^{\leq n}$. Now, $\mathcal{M}_{\mathrm{fg}}^{=n} = \mathcal{M}_{\mathrm{fg}}^{\leq n}/(p, v_1, \cdots, v_{n-1})$. Consider a formal group $\mathbb{G} : \mathrm{Spec}\, \overline{\mathbb{F}}_p \to \mathcal{M}_{\mathrm{fg}}^{=n}$. The main thing to observe is that this is formally étale. This is because $\mathcal{M}_{\mathrm{fg}}^{=n}$ is very much like the classifying stack of a group, and so this is very much like a Galois extension and those are formally étale.

If there is some deformation of $\mathcal{M}_{\mathrm{fg}}^{=n}$, then it can always be lifted to a deformation of $\mathrm{Spec}\, \overline{\mathbb{F}}_p$. There is a pullback

[30] There is an equivalence $\mathrm{Bar}(\Sigma^\infty_+ \mathbb{E}_n) \simeq s^n \mathbb{E}_n^\vee$, so we don't gain anything new for $\mathbb{E}_n$-operads with $n < \infty$.

$$\begin{tikzcd} \operatorname{Spec} \overline{\mathbb{F}}_p \arrow[r, "\mathbb{G}"] \arrow[d, hook, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M}_{\mathrm{fg}}^{=n} \arrow[d, hook] \\ \operatorname{Spec} R \arrow[r] & \widetilde{\mathcal{M}}_{\mathrm{fg}}/(\tau^i, p^{a_0}, v_1^{a_1}, \cdots, v_{n-1}^{a_{n-1}}) \end{tikzcd}$$

Take the colimit of this stack thing on the right as $i, a_0, \cdots, a_{m-1} \to \infty$. Then, the deformation on the left is Spec $\tau_{\geq \bullet} E_n$. We get $W(\overline{\mathbb{F}}_p)[\![v_1, \cdots, v_{n-1}]\!][\beta]$ and without τ, p this is without the β and the W. With p we get W. With τ, we also get β. So this recovers connective Morava K-theory.

Taking some global equivariant version of $\mathbf{Syn}_{\mathrm{MU}}^{\mathrm{even}}$, one can get some p-divisible group shenanigans.

D.7 Generalized Chromatic Homotopy Theory

More precisely, the question is about chromatic homotopy theory when adding words like equivariant, motivic, synthetic, and so on.

Most work has been done in the equivariant setting. An essential part of the chromatic story is the thick subcategory theorem which was mostly dealt with by Balmer-Sanders for finite groups who described the Balmer spectrum as a set and some parts of the topology but not all. Others later resolved the question about the topology for finite abelian groups and so on. The finite group case is still open... or is it? Ishan interrupted and asked whether unpublished work counts. The finite group case for the thick subcategory theorem has been resolved in unpublished work by Ishan, Robert, Markus and Lennart.

About motivic and synthetic... Ishan, Robert and Piotr have unpublished work on the motivic thick subcategory theorem and periodicity theorem via some the cellular synthetic categories

$$\left(\mathbf{Syn}_{\mathrm{MU}}^{\mathrm{even}}\right)_p^\wedge \simeq \left(\mathcal{SH}(\mathbb{C})^{\mathrm{cell}}\right)_p^\wedge .$$

In particular, the (cellular) motivic and synthetic questions are dealt with at the same time.

D.8 Less Serious Questions

Someone asked whether Gijs and Ishan could answer some of the less serious questions.

Gijs: *...They are on the level of... 'What's the best time to get married?' 'How do you stay happy?'*

I'm still waiting for my answers.

E Non-Mathematical Parts

Certainly, I wasn't part of every activity at Talbot but here are some of the more global events.

E.1 Cooking

As per Talbot tradition, we split into cooking (and cleaning) teams to take turns making dinner.

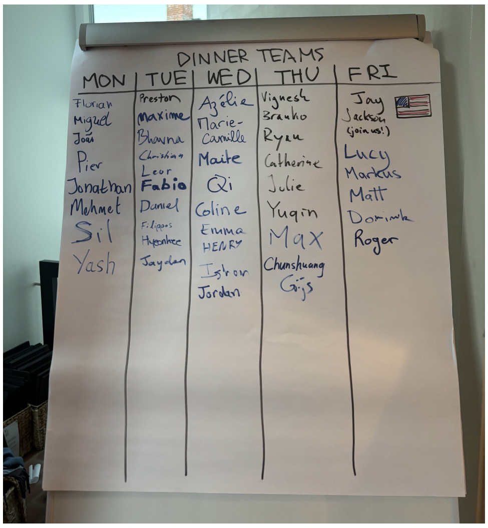

Figure 6: Dinner Teams

A lot of yummy dishes were created and the last day was rounded off by a special occassion: 4th of July. The Americans prepared a barbeque evening!

Figure 7: The American evening.

Team USA delivered.

E.2 Fun in the Water

Some of us discovered kayaks on the first day, and so I believe it was then that the idea was born that excursion day would be spent on the water. We went kayaking and canoeing!

Figure 8: Talbot group on the middle of the waters.

There were no casualities.

E.3 Karaoke Night

The entire event was brought to an end with a final karaoke night. I don't have any fitting pictures but this certainly seems like a fitting end to an amazing week.

References

[ACB19] Omar Antolín-Camarena and Tobias Barthel. A simple universal property of Thom ring spectra. *J. Topol.*, 12(1):56–78, 2019. (Cited on pages 24 and 67.)

[Ant24] Benjamin Antieau. Spectral sequences, décalage, and the beilinson t-structure, 2024. (Cited on pages 43, 44, and 45.)

[BHS23] Robert Burklund, Jeremy Hahn, and Andrew Senger. On the boundaries of highly connected, almost closed manifolds. *Acta Math.*, 231(2):205–344, 2023. (Cited on page 48.)

[BSY22] Robert Burklund, Tomer M. Schlank, and Allen Yuan. The chromatic nullstellensatz, 2022. (Cited on pages 61, 62, 67, and 68.)

[CSY22] Shachar Carmeli, Tomer M. Schlank, and Lior Yanovski. Ambidexterity in chromatic homotopy theory. *Invent. Math.*, 228(3):1145–1254, 2022. (Cited on page 58.)

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole. (Cited on page 19.)

[Heu20] Gijs Heuts. Lie algebra models for unstable homotopy theory. In *Handbook of homotopy theory*, CRC Press/Chapman Hall Handb. Math. Ser., pages 657–698. CRC Press, Boca Raton, FL, [2020] ©2020. (Cited on page 76.)

[HL13] Michael Hopkins and Jacob Lurie. Ambidexterity in $k(n)$-local stable homotopy theory. 2013. Unpublished Notes. (Cited on page 68.)

[HS99] Mark Hovey and Neil P. Strickland. Morava K-theories and localisation. *Mem. Amer. Math. Soc.*, 139(666):viii+100, 1999. (Cited on page 22.)

[HW21] Fabian Hebestreit and Ferdinand Wagner. Algebraic and hermitian k-theory. 2021. Lecture Notes. (Cited on page 77.)

[HW22] Jeremy Hahn and Dylan Wilson. Redshift and multiplication for truncated Brown-Peterson spectra. *Ann. of Math. (2)*, 196(3):1277–1351, 2022. (Cited on pages 39 and 40.)

[Lan21] Markus Land. *Introduction to infinity-categories*. Compact Textbooks in Mathematics. Birkhäuser/Springer, Cham, 2021. (Cited on page 61.)

[Lur10] Jacob Lurie. Chromatic homotopy theory. 2010. Lecture Notes. (Cited on pages 19, 22, 24, and 27.)

[Lur11] Jacob Lurie. Derived algebraic geometry x: Formal moduli problems. 2011. Unpublished Notes. (Cited on page 61.)

[Lur17] Jacob Lurie. Higher algebra, September 2017. Unpublished notes. (Cited on pages 39 and 43.)

[Mat18] Akhil Mathew. Examples of descent up to nilpotence. In *Geometric and topological aspects of the representation theory of finite groups*, volume 242 of *Springer Proc. Math. Stat.*, pages 269–311. Springer, Cham, 2018. (Cited on page 26.)

[MNN17] Akhil Mathew, Niko Naumann, and Justin Noel. Nilpotence and descent in equivariant stable homotopy theory. *Adv. Math.*, 305:994–1084, 2017. (Cited on page 24.)

[Pst23] Piotr Pstragowski. Synthetic spectra and the cellular motivic category. *Invent. Math.*, 232(2):553–681, 2023. (Cited on page 49.)

[PV22] Piotr Pstragowski and Paul VanKoughnett. Abstract Goerss-Hopkins theory. *Adv. Math.*, 395:Paper No. 108098, 51, 2022. (Cited on page 52.)

[Rez09] Charles Rezk. The congruence criterion for power operations in Morava E-theory. *Homology Homotopy Appl.*, 11(2):327–379, 2009. (Cited on page 58.)