The Equivariant Slice Spectral Sequence

QI ZHU

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Abstract

These are my TeX'd notes of Bert Guillou's eCHT minicourse on the equivariant slice spectral sequence. You can find the lectures, exercise sheets, and Bert's notes on his website `https://www.ms.uky.edu/~guillou/echtSlices/SlicesMinicourse.html`.

I don't entirely follow Bert's notation and left out some review on equivariant homotopy theory. Most exercises from the course along with their solutions also feature in these notes. I'm thankful to Bert for catching one typo. Comments are very welcome!

Contents

1 Introduction

Bert gave a longer review on equivariant homotopy theory, which I omitted in these notes.

1.1 Short Mackey Functor Review

Recall that for $X \in \mathbf{Sp}^G$ one obtains a **Mackey functor** $\underline{\pi}_n(X)$ which has restriction and transfer maps along with an action of $W_G H = N_G H/H \cong \mathrm{Aut}_G(G/H)$ on $\pi_n^H(X)$.[1]

Example 1.1.1. We get the Burnside ring $\underline{\pi}_0(\mathbb{S}_G) \cong \underline{A}$. For $G = C_p$ this is depicted by:

$$\begin{array}{c} \mathbb{Z}\{1, C_p\} \\ {\scriptstyle (1\ p)}\big\downarrow \big\uparrow{\scriptstyle \binom{0}{1}} \\ \mathbb{Z} \end{array}$$

There will be two Mackey functor constructions relevant to us:

Construction 1.1.2. There are two functors $F, Q : \mathbf{Mod}_{\mathbb{Z}[G]} \to \mathbf{Mack}(G)$ which on objects are given by

$$F(M)(H) = M^H \quad \text{and} \quad Q(M)(H) = M/H.$$

The restriction for F is the inclusion map and the transfer for Q is the quotient map. We will also write $F(M) = \underline{M}$.

Example 1.1.3.

(i) We get $F(\mathbb{Z}) = \underline{\mathbb{Z}}$ given by

$$\begin{array}{c} \mathbb{Z} \\ {\scriptstyle 1}\big\downarrow \big\uparrow{\scriptstyle 2} \\ \mathbb{Z} \end{array}$$

(ii) We get that $\mathbb{F}(\mathbb{Z}^\sigma)$ is concentrated in the bottom group with the sign action.

1.2 Atiyah's Real K-Theory

One can use that C_2-action on $\mathbb{C}$ to promote $\mathrm{KU} \in \mathbf{Sp}$ and obtain $\mathrm{KU}_\mathbb{R} \in \mathbf{Sp}^{C_2}$. We have $\mathrm{KU}_\mathbb{R}^e \simeq \mathrm{KU}$ and $\mathrm{KU}_\mathbb{R}^{C_2} \simeq \mathrm{KO}$.

Remark 1.2.1.

(i) By classical Bott periodicity we get a 2-periodic $\mathrm{KU}_\mathbb{R}^e$ and an 8-periodic $\mathrm{KU}_\mathbb{R}^{C_2}$. Thus, $\underline{\pi}_{n+8}\,\mathrm{KU}_\mathbb{R} \cong \underline{\pi}_n\,\mathrm{KU}_\mathbb{R}$.

(ii) There is also a Real Bott periodicity $\Sigma^\rho\,\mathrm{KU}_\mathbb{R} \simeq \mathrm{KU}_\mathbb{R}$.

Problem 1.2.2. The Real Bott periodicity of $\mathrm{KU}_\mathbb{R}$ is not detected in the Postnikov filtration.

Response. Define a new filtration for C_2-spectra.

1. It restricts to the Postnikov filtration after applying $\mathrm{Res}_e^{C_2} : \mathbf{Sp}^{C_2} \to \mathbf{Sp}$.
2. It interacts well with $\Sigma^\rho : \mathbf{Sp}^{C_2} \to \mathbf{Sp}^{C_2}$, written as $P_{k+2}^{n+2}(\Sigma^\rho X) \simeq \Sigma^\rho P_k^n(X)$. This implies the compatibility with Real Bott periodicity.

□

[1] We mod out the H-action on X, so it should have some residual G/H-action. But this doesn't make sense in general, as H need not be normal, so G/H need not be a group. The normalizer fixes this deficiency.

1.3 The (Regular) Slice Filtration ($G = C_2$)

Want fiber sequences

$$P_n X \longrightarrow X \longrightarrow P^{n-1} X$$

where P_n is $\geq n$ and P^{n-1} is $\leq n-1$. Also want

$$P^{\ell}_{k+1} X \longrightarrow P^{\ell}_{j} X \longrightarrow P^{k}_{j} X$$

for $j < k < \ell$. For example

$$P^{k+1}_{k+1} \longrightarrow P^{k+1}_{j} X \longrightarrow P^{k}_{j} X$$

giving a **$(k+1)$-slice**.

The theory was pioneered by Dugger for C_2 (and some motivic people afterwards) until HHR studied this more thoroughly and in greater generality. There is a slight variant by Ullman from his thesis which is a bit simpler and the definition we will use.

Definition 1.3.1. Let $\tau_{\geq n} \subseteq \mathbf{Sp}^{C_2}$ denote the localizing subcategory generated by

- $S^{k\rho}$ for $2k \geq n$,
- $\mathrm{Ind}_e^{C_2} S^k$ for $k \geq n$.

We write $X \geq n$ for $X \in \tau_{\geq n}$ and say that X is **slice n-connective** in that case.

Example 1.3.2. It turns out that $\tau_{\geq 0} \simeq \mathbf{Sp}^{C_2}_{\geq 0}$ and $\tau_{\geq 1} \simeq \mathbf{Sp}^{C_2}_{\geq 1}$.

Definition 1.3.3. We say $X < n$ or $X \leq n-1$ if $[W, X] = 0$ for all $W \in \tau_{\geq n}$. In that case, we say that X is **slice $(n-1)$-coconnective**.

Construction 1.3.4. We obtain a Bousfield localization $P^n \colon \mathbf{Sp}^G \to \mathbf{Sp}^G$ into $\tau_{\leq n}$. Furthermore, let

$$P_{n+1}(X) = \mathrm{fib}(X \to P^n(X)) \quad \text{and} \quad P^n_k(X) = P_k P^n X.$$

It takes some work to show $P_{n+1} X \geq n+1$.

Proposition 1.3.5 (HHR).

(i) $P^0_0 X \simeq \mathrm{H}\underline{\pi}_0 X$,

(ii) $P^{n+2}_{k+2}(\Sigma^{\rho} X) \simeq \Sigma^{\rho} P^n_k X$,

(iii) $P^1_1 X \simeq \Sigma^1 \mathrm{H}(\underline{\pi}_1 X / \ker \mathrm{res}^{C_2}_e)$,

(iv) If $X \to Y \to Z$ is a fiber sequence with $X \geq n+1$ and $Z \leq n$, then $X \simeq P_{n+1} Y$ and $Z \simeq P^n Y$.

In (iii) we force the restrictions to be injective and in (iv) the slogan is that if something looks like a slice tower, then it is a slice tower!

Exercise 1.3.6. Compute $P^1_1 \mathbb{S}^1_{C_2}$.

Proof. We use Proposition 1.3.5(iii). Note that $\underline{\pi}_1 \mathbb{S}^1_{C_2} \cong \underline{\pi}_0 \mathbb{S}_{C_2} \cong \underline{A}(C_2)$ is the Burnside Mackey functor. On the other hand, we must mod out the kernel of the restriction map which is $\langle 1 - \sigma \rangle \subseteq A(C_2)$. So we are left with $\underline{\mathbb{Z}}$. Thus, $P^1_1 \mathbb{S}^1_{C_2} \simeq \Sigma \mathrm{H}\underline{\mathbb{Z}}$. □

1.4 Back to Atiyah's Real K-Theory

The structure maps of $\mathrm{KU}_{\mathbb{R}}$ are

$$\begin{array}{c} \mathrm{KU}_{\mathbb{R}}^{C_2} \simeq \mathrm{KO} \\ c \downarrow \uparrow r \\ \mathrm{KU}_{\mathbb{R}}^{e} \simeq \mathrm{KU} \end{array}$$

given by complexification and realification. Let us try to compute the slices of $\mathrm{KU}_{\mathbb{R}}$. This was studied by Dugger [Dug05] but there is also a C_4-version by HHR [HHR17].

- Taking π_0 yields $\underline{\pi}_0 \, \mathrm{KU}_{\mathbb{R}} \cong \underline{\mathbb{Z}}$. Thus, $P_0^0 \, \mathrm{KU}_{\mathbb{R}} \simeq \mathrm{H}\underline{\mathbb{Z}}$ by Proposition 1.3.5(i).
- Taking π_1 yields

$$\begin{array}{c} \mathbb{Z}/2 \\ \\ 0 \end{array}$$

  which Bert calls $\mathfrak{g}$ for 'geometric' and HHR call $B(1,0)$. Thus, $P_1^1 \, \mathrm{KU}_{\mathbb{R}} \simeq \Sigma^1 \mathrm{H}\mathfrak{g} / \ker \mathrm{res} \simeq *$ by Proposition 1.3.5(iii).
- We can compute $P_2^2 \, \mathrm{KU}_{\mathbb{R}} \simeq \Sigma^\rho P_0^0 \Sigma^{-\rho} \, \mathrm{KU}_{\mathbb{R}} \simeq \Sigma^\rho P_0^0 \, \mathrm{KU}_{\mathbb{R}} \simeq \Sigma^\rho \mathrm{H}\underline{\mathbb{Z}}$ by Proposition 1.3.5(ii) and Real Bott periodicity.
- A similar computation shows $P_3^3 \, \mathrm{KU}_{\mathbb{R}} \simeq \Sigma^\rho P_1^1 \Sigma^{-\rho} \, \mathrm{KU}_{\mathbb{R}} \simeq \Sigma^\rho P_1^1 \, \mathrm{KU}_{\mathbb{R}} \simeq *$.

We deduce:

Lemma 1.4.1. Let $n \in \mathbb{Z}$. Then, $P_n^n \, \mathrm{KU}_{\mathbb{R}} \simeq \begin{cases} \Sigma^{\frac{n}{2}\rho} \mathrm{H}\underline{\mathbb{Z}} & 2 \mid n, \\ * & 2 \nmid n. \end{cases}$.

Because it is a smaller model, let us focus on the connective cover $\mathrm{ku}_{\mathbb{R}} \to \mathrm{KU}_{\mathbb{R}}$ which has $\mathrm{ku}_{\mathbb{R}}^e \simeq \mathrm{ku}$ and $\mathrm{ku}_{\mathbb{R}}^{C_2} \simeq \mathrm{ko}$. It's a fact that the connective cover is given by P_0.

Corollary 1.4.2. Let $n \in \mathbb{Z}$. Then, $P_n^n \mathrm{ku}_{\mathbb{R}} \simeq \begin{cases} \Sigma^{\frac{n}{2}\rho} \mathrm{H}\underline{\mathbb{Z}} & 2 \mid n, n \geq 0, \\ * & \text{else.} \end{cases}$.

Construction 1.4.3. From the slice filtration

$$\cdots \longrightarrow P_2 X \longrightarrow P_1 X \longrightarrow P_0 X$$

we get a spectral sequence

$$E_2^{s,t} = \underline{\pi}_{t-s} P_t^t X \Rightarrow \underline{\pi}_{t-s} X$$

with differential $d_r \colon \underline{\pi}_n P_t^t X \to \underline{\pi}_{n-1} P_{t+r-1}^{t+r-1} X$. This is the **slice spectral sequence**.

The Slice SS for $\mathrm{ku}_{\mathbb{R}}$ is depicted below:

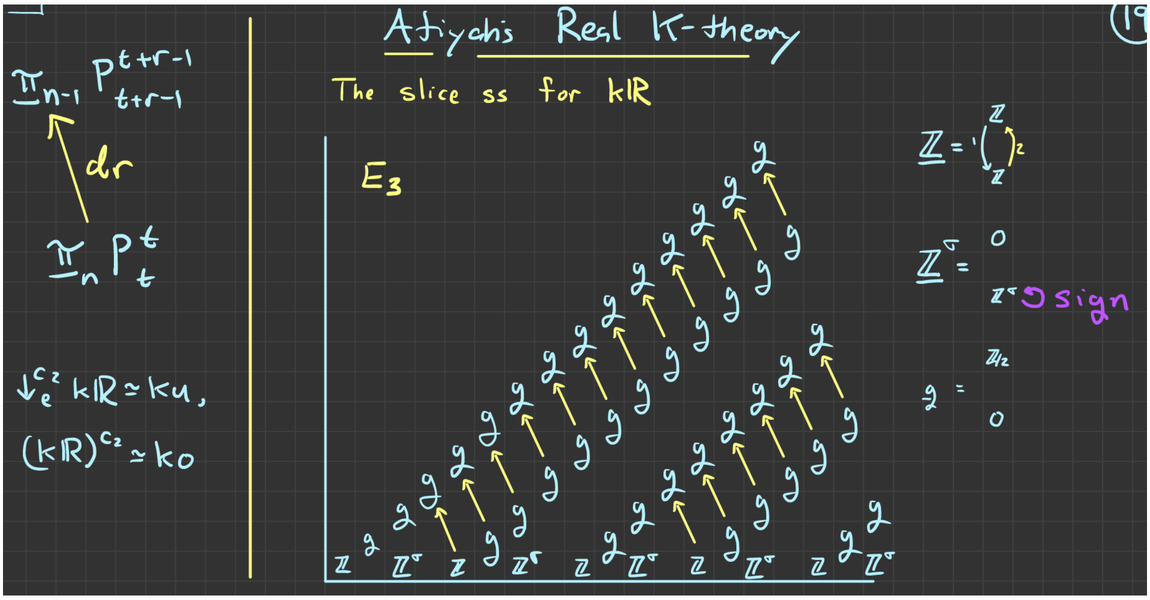


Figure 1: The E_3-page for $\mathrm{ku}_{\mathbb{R}}$.

The E_2-page looks the same, just without the differentials. We will argue that this is the spectral sequence throughout the minicourse. Note:

- For example, this bottom left antidiagonal consisting of $\mathfrak{g}$ and $\mathbb{Z}^{\sigma}$ is $\underline{\pi}_* P_2^2 \mathrm{ku}_{\mathbb{R}} \cong \underline{\pi}_* \Sigma^{\rho} \mathrm{H}\underline{\mathbb{Z}}$. We will compute these in the next lecture.
- Classically, it is known that $\pi_3 \,\mathrm{KO} = 0$, so this third $\mathfrak{g}$ from the bottom must be killed. It can only be hit by the third $\mathbb{Z}$ in the bottom row, so we obtain a differential.
- By multiplicativity this differential propagates.
- Classical Bott periodicity makes the picture repeat.

So we get the $E_4 = E_\infty$-page:

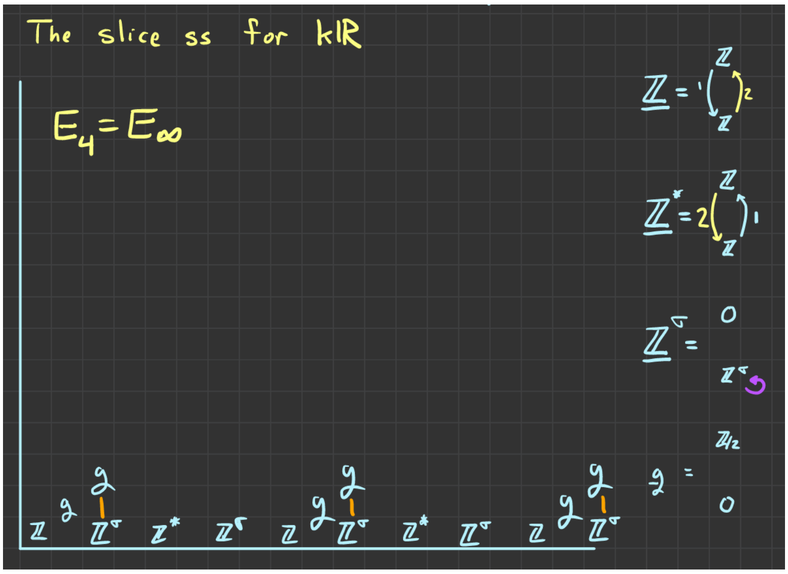


Figure 2: The E_4-page for $\mathrm{ku}_{\mathbb{R}}$.

As so often, this is not the end of a spectral sequence. We have a horizontal vanishing line and some extension problems are left (those are the orange lines).

Exercise 1.4.4. Solve the extension problem to see that $\underline{\pi}_2 \mathrm{ku}_{\mathbb{R}}$ is given by the Mackey functor

$$\begin{array}{c} \mathbb{Z}/2 \\ {\scriptstyle 0}\downarrow\uparrow \\ \mathbb{Z}^{\sigma} \end{array}$$

This is also known as $Q\mathbb{Z}^{\sigma}$.

Proof. Indeed, the extension problem is a short exact sequence

$$0 \longrightarrow \mathfrak{g} \longrightarrow \underline{\pi}_2 \mathrm{ku}_{\mathbb{R}} \longrightarrow \underline{\mathbb{Z}}^{\sigma} \longrightarrow 0$$

which written out is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \pi_2^{C_2}\mathrm{ku}_{\mathbb{R}} & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \pi_2^{e}\mathrm{ku}_{\mathbb{R}} & \longrightarrow & \mathbb{Z}^{\sigma} & \longrightarrow & 0 \end{array}$$

so we can immediately read off the groups at each level. Moreover, the only possibility for the restriction map is 0. So it remains to see if the transfer map is non-trivial. For this consider $C_2/e_+ \to S^0 \to S^{\sigma}$ and $X \in \mathbf{Sp}^{C_2}$. Tensoring yields

$$C_2/e_+ \otimes X \longrightarrow X \longrightarrow S^{\sigma} \otimes X$$

and applying $(-)^{C_2}$ then

$$X^e \longrightarrow X^{C_2} \longrightarrow (\Sigma^{\sigma})^{C_2}$$

so applying π_n we get an exact sequence

$$\pi_n^e(X) \xrightarrow{\mathrm{tr}_e^{C_2}} \pi_n^{C_2}(X) \longrightarrow \pi_n^{C_2}(\Sigma^{\sigma} X).$$

We plug in $X = \mathrm{ku}_{\mathbb{R}}$ and $n = 2$; we are interested in $\mathrm{tr}_e^{C_2}$. On the other hand,

$$\pi_2^{C_2}(\Sigma^{\sigma}\mathrm{ku}_{\mathbb{R}}) \cong \pi_2^{C_2}(\Omega \mathrm{ku}_{\mathbb{R}}) \cong \pi_3^{C_2}(\mathrm{ku}_{\mathbb{R}}) \cong \pi_3(\mathrm{ko}) \cong 0$$

by Bott periodicity. So $\mathrm{tr}_e^{C_2}$ is the unique non-trivial map. □

2 Bredon Homology Computations

2.1 Bredon Homology

Recall that we arrived at the slice spectral sequence $\underline{\pi}_n P_t^t \mathrm{ku}_{\mathbb{R}} \cong \underline{\pi}_n \Sigma^{\frac{t}{2}\rho} \mathrm{H}\underline{\mathbb{Z}} \Rightarrow \underline{\pi}_n \mathrm{ku}_{\mathbb{R}}$ last time (Corollary 1.4.2). So we need to compute these slices which are given by Bredon homology.

Definition 2.1.1. Let $X \in \mathcal{S}_*^G$ and $\underline{M} \in \mathbf{Mack}(G)$. Then, $\widetilde{\underline{H}}_n(X; \underline{M}) = \underline{\pi}_n(X \otimes \mathrm{H}\underline{M})$ is **Bredon homology**.

Example 2.1.2. Last time for $G = C_2$ we claimed that

$$\underline{\pi}_n(\Sigma^{\rho}\mathrm{H}\underline{\mathbb{Z}}) \cong \begin{cases} \underline{\mathbb{Z}}^{\sigma} & n = 2, \\ \mathfrak{g} & n = 1, \\ 0 & \text{else.} \end{cases}$$

Let us verify this now.

Proof. Since $S^\rho \simeq S^1 \wedge S^\sigma$ and S^1 only shifts $\underline{\pi}_n$, it suffices to understand $\Sigma^\sigma H\underline{\mathbb{Z}}$. To compute this we use the C_2-CW structure on S^σ which can be described as the cofiber sequence

$$C_{2+} \wedge S^0 \longrightarrow S^0 \longrightarrow S^\sigma,$$

i.e. attach a single free C_2-cell. Tensoring with $H\underline{\mathbb{Z}}$ yields

$$C_{2+} \otimes H\underline{\mathbb{Z}} \longrightarrow H\underline{\mathbb{Z}} \longrightarrow S^\sigma \otimes H\underline{\mathbb{Z}}$$

and the last term is the one we want to understand! We note the following:

(i) Always, $C_{2+} \otimes X \simeq \operatorname{Ind}_e^{C_2} \operatorname{Res}_e^{C_2} X$.

(ii) We obtain $\underline{\pi}_n(C_{2+} \otimes X) \cong \operatorname{ind}_e^{C_2} \pi_n(\operatorname{Res}_e^{C_2} X)$, an **induced Mackey functor**.[2] for $M \in \mathbf{Ab}$ the Mackey functor $\operatorname{ind}_e^{C_2} M$ is given by

$$\begin{array}{c} M \\ {\scriptstyle 1+\gamma}\downarrow\uparrow{\scriptstyle \nabla} \\ \mathbb{Z}[C_2] \otimes_{\mathbb{Z}} M \end{array}$$

Applying $\underline{\pi}_*$ to the aforementioned cofiber sequence thus yields a LES

$$0 \longrightarrow \underline{\widetilde{H}}_1(S^\sigma;\mathbb{Z}) \longrightarrow \operatorname{ind}_e^{C_2} \mathbb{Z} \longrightarrow \underline{\mathbb{Z}} \longrightarrow \underline{\widetilde{H}}_0(S^\sigma;\mathbb{Z}) \longrightarrow 0.$$

So we need to compute the kernel and cokernel of $\operatorname{ind}_e^{C_2} \mathbb{Z} \to \underline{\mathbb{Z}}$ which is:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \cdots \\ & & {\scriptstyle 1+\gamma}\downarrow\uparrow & & {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 2} & & \\ \cdots & \longrightarrow & \mathbb{Z}[C_2] & \longrightarrow & \mathbb{Z} & \longrightarrow & \cdots \end{array}$$

The bottom map sends $1, \gamma \mapsto 1$, so by commutativity of the diagram, the top arrow is forced to be 2.[3] Just compute kernel and cokernel levelwise!

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 \\ & & {\scriptstyle 1+\gamma}\downarrow\uparrow & & {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 2} & & \\ \mathbb{Z}^\sigma = \mathbb{Z}\{1-\gamma\} & \longrightarrow & \mathbb{Z}[C_2] & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

This confirms $\underline{\pi}_0 \Sigma^\sigma H\underline{\mathbb{Z}} \cong \mathfrak{g}$ and $\underline{\pi}_1 \Sigma^\sigma H\underline{\mathbb{Z}} \cong \underline{\mathbb{Z}}^\sigma$. □

Exercise 2.1.3. Compute $\underline{\pi}_k \Sigma^{n\rho} H\underline{\mathbb{Z}}$.

Proof. First of all, we note $\Sigma^\sigma H\mathfrak{g} \simeq H\mathfrak{g}$ and $\Sigma^\sigma H\underline{\mathbb{Z}}^\sigma \simeq \Sigma H\underline{\mathbb{Z}}$ via exactly the same computational methods as above. This yields

$$\Sigma^\rho H\mathfrak{g} \simeq \Sigma^1 H\mathfrak{g} \quad \text{and} \quad \Sigma^\rho H\underline{\mathbb{Z}}^\sigma \simeq \Sigma^2 H\underline{\mathbb{Z}}.$$

We use the fiber sequence[4]

$$\Sigma^2 H_{C_2}\underline{\mathbb{Z}}^\sigma \longrightarrow \Sigma^\rho H_{C_2}\underline{\mathbb{Z}} \longrightarrow \Sigma^1 H_{C_2}\mathfrak{g}.$$

[2] See also [Zen18, Definition 2.8].

[3] Note that the top map agrees with this transfer map on the right. This is not a coincidence and always happens in these sorts of maps from the induced Mackey functor by exactly this argument.

[4] Here is an error it made. With the above we have a fiber sequence $\Sigma^\sigma H\underline{\mathbb{Z}}^\sigma \to \Sigma^\sigma H\underline{\mathbb{Z}} \to \Sigma^\sigma H\mathfrak{g}$ and now it sounds like we can just apply Ω^σ but the resulting sequence is not a fiber sequence, as is checked on $\underline{\pi}_0$. The mistake is probably that the previous sequence is not given by applying Σ^σ to a fiber sequence but rather arises in a more complicated fashion.

Applying Σ^ρ yields

$$\Sigma^4 H_{C_2}\underline{\mathbb{Z}} \simeq \Sigma^{2+\rho} H_{C_2}\underline{\mathbb{Z}}^\sigma \longrightarrow \Sigma^{2\rho} H_{C_2}\underline{\mathbb{Z}} \longrightarrow \Sigma^{1+\rho} H_{C_2}\mathfrak{g} \simeq \Sigma^2 H_{C_2}\mathfrak{g}$$

This gives $\underline{\pi}_2 = \mathfrak{g}$ and $\underline{\pi}_4 = \underline{\mathbb{Z}}$. One can now go on and obtains

$$\Sigma^{4+(n-2)\rho} H_{C_2}\underline{\mathbb{Z}} \longrightarrow \Sigma^{n\rho} H_{C_2}\underline{\mathbb{Z}} \longrightarrow \Sigma^n H\mathfrak{g},$$

so we can proceed by induction. With a proper analysis of these terms we obtain

$$\underline{\pi}_n = \underline{\pi}_{n+2} = \cdots = \underline{\pi}_{n+2\lfloor \frac{n}{2} \rfloor} = \mathfrak{g}, \; \underline{\pi}_{2n} \cong \begin{cases} \underline{\mathbb{Z}}^\sigma & 2 \nmid n, \\ \underline{\mathbb{Z}} & 2 \mid n. \end{cases}$$

The other terms are 0. □

We chose the $\mathrm{ku}_{\mathbb{R}}$, so we obtained a first quadrant spectral sequence. If were to work with $\mathrm{KU}_{\mathbb{R}}$, then we'd also get the third quadrant, and the slices come from negative ρ-suspensions of Bredon homology. This motivates the following computation.

Example 2.1.4. Let us compute $\Sigma^{-\sigma} H\underline{\mathbb{Z}}$. The cofiber sequence $C_{2+} \to S^0 \to S^\sigma$ dualizes to

$$S^{-\sigma} \longrightarrow S^0 \longrightarrow C_{2+}$$

in $\mathbf{Sp}^{C_2}$ via self-dualizability of the orbits. Tensoring by $H\underline{\mathbb{Z}}$ yields

$$\Sigma^{-\sigma} H\underline{\mathbb{Z}} \longrightarrow H\underline{\mathbb{Z}} \longrightarrow C_{2+} \otimes H\underline{\mathbb{Z}}$$

and thus we obtain an exact sequence

$$\underline{\pi}_0 \Sigma^{-\sigma} H\underline{\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathrm{ind}_e^{C_2} \mathbb{Z} \longrightarrow \underline{\pi}_{-1} \Sigma^{-\sigma} \underline{\mathbb{Z}}$$

This comes out to be

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} & \longrightarrow & 0 \\ & & {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 2} & & {\scriptstyle 1+\gamma}\downarrow\uparrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow[1+\gamma]{} & \mathbb{Z}[C_2] & \longrightarrow & \mathbb{Z}[C_2]/\mathbb{Z} \cong \mathbb{Z}^\sigma \end{array}$$

Thus, $\Sigma^{-\sigma} H\underline{\mathbb{Z}} \simeq \Sigma^{-1} H\underline{\mathbb{Z}}^\sigma$.

You can further desuspend and keep going with the computation.

Exercise 2.1.5. Compute $\Sigma^{-n\rho} H_{C_2}\underline{\mathbb{Z}}$.

Proof. We have already computed $\Sigma^{-\rho} H\underline{\mathbb{Z}} \simeq \Sigma^{-2} H\underline{\mathbb{Z}}^\sigma$ above (Example 2.1.4). The next step is to compute $\Sigma^{-2\rho} H\underline{\mathbb{Z}} \simeq \Sigma^{-3}\Sigma^{-\sigma} H\underline{\mathbb{Z}}^\sigma$. By tensoring with $S^{-\sigma} \to S^0 \to C_{2+}$ we we have the fiber sequence

$$\Sigma^{-\sigma} H\underline{\mathbb{Z}}^\sigma \longrightarrow H\underline{\mathbb{Z}}^\sigma \longrightarrow C_{2+} \otimes H\underline{\mathbb{Z}}^\sigma$$

This leaves us with the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} \\ & & \downarrow\uparrow & & {\scriptstyle 1+\gamma}\downarrow\uparrow & & {\scriptstyle 2}\downarrow\uparrow{\scriptstyle 1} \\ 0 & \longrightarrow & \mathbb{Z}^\sigma & \xrightarrow[1-\gamma]{} & \mathbb{Z}[C_2] & \longrightarrow & \mathbb{Z}[C_2]/\mathbb{Z}\{1-\gamma\} \cong \mathbb{Z} \end{array}$$

where the bottom map is essentially because $\mathrm{H}\underline{\mathbb{Z}}^\sigma \to C_{2+} \otimes \mathrm{H}\underline{\mathbb{Z}}^\sigma = \prod_{C_2} \mathrm{H}\underline{\mathbb{Z}}^\sigma$ hits every factor completely, so on the trivial component it must be 1 and by C_2-equivariance, we must get $-\gamma$. We have $\mathbb{Z}[C_2]/\mathbb{Z}\{1-\gamma\} \cong \mathbb{Z}$ because the C_2-action acts by $\gamma \cdot 1 = \gamma = 1$ in this quotient. Commutativity of the right square gives rise to those maps as depiced where $2 = 1+\gamma$. Thus, $\Sigma^{-\sigma}\mathrm{H}\underline{\mathbb{Z}}^\sigma \simeq \Sigma^{-1}\mathrm{H}\underline{\mathbb{Z}}^*$.

Next, we need to compute $\Sigma^{-3\rho}\mathrm{H}\underline{\mathbb{Z}} \simeq \Sigma^{-5}\Sigma^{-\sigma}\mathrm{H}\underline{\mathbb{Z}}^*$. An analogous fiber sequence gives rise to

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 \\
 & & {\scriptstyle 2}\downarrow\uparrow{\scriptstyle 1} & & {\scriptstyle 1+\gamma}\downarrow\uparrow & & \downarrow\uparrow \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow[1+\gamma]{} & \mathbb{Z}[C_2] & \longrightarrow & \mathbb{Z}[C_2]/\mathbb{Z}\{1+\gamma\} \cong \mathbb{Z}^\sigma
\end{array}$$

and so we identify the right-most term as $Q\mathbb{Z}^\sigma$. Thus, $\Sigma^{-3\rho}\mathrm{H}\underline{\mathbb{Z}} \simeq \Sigma^{-6}\mathrm{H}Q\mathbb{Z}^\sigma$.

Next, we need to compute $\Sigma^{-4\rho}\mathrm{H}\underline{\mathbb{Z}} \simeq \Sigma^{-7}\Sigma^{-\sigma}\mathrm{H}Q\mathbb{Z}^\sigma$. An analogous fiber sequence gives rise to

$$\begin{array}{ccccccc}
\mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} \\
 & & {\scriptstyle 0}\downarrow\uparrow & & {\scriptstyle 1+\gamma}\downarrow\uparrow & & {\scriptstyle 2}\downarrow\uparrow{\scriptstyle 1} \\
0 & \longrightarrow & \mathbb{Z}^\sigma & \xrightarrow[1-\gamma]{} & \mathbb{Z}[C_2] & \longrightarrow & \mathbb{Z}[C_2]/\mathbb{Z}\{1-\gamma\} \cong \mathbb{Z}
\end{array}$$

so we get a fiber sequence

$$\mathrm{H}\mathfrak{g} \longrightarrow \Sigma^{-\sigma}\mathrm{H}Q\mathbb{Z}^\sigma \longrightarrow \Sigma^{-1}\mathrm{H}\underline{\mathbb{Z}}^*$$

The next step would mostly consist of applying $\Sigma^{-\sigma}$ again. For the first term we use $\Sigma^{-\sigma}\mathrm{H}\mathfrak{g} \simeq \mathrm{H}g$ as suggested in the proof of Exercise 2.1.3. The third term we already computed above. So at this point we have all ingredients to induct. There are two cases:

- n odd: $\underline{\pi}_{2n} = Q\mathbb{Z}^\sigma$ and $\underline{\pi}_{2n-2} = \underline{\pi}_{2n-4} = \cdots = \underline{\pi}_{n+3} = \mathfrak{g}$,
- n even: $\underline{\pi}_{2n} \cong \underline{\mathbb{Z}}^*$ and $\underline{\pi}_{2n-1} = \underline{\pi}_{2n-3} = \cdots = \underline{\pi}_{n+3} = \mathfrak{g}$.

We are finally done. □

Example 2.1.6. Now consider $G = C_3$. We have $\rho_{C_3} = 1 \oplus \lambda$ where λ is the 2-dimensional rotation representation. Again, the CW-structure induces a cofiber sequence

$$C_{3+} \longrightarrow S^0 \longrightarrow S^{\succ}$$

where $S^{\succ} = C_3^{\diamond}$ is the **spoke sphere** or also the **eggbeater** (first suggested by Clover May). Note that the notation is a bit misleading since $S^{\succ}$ is not a representation sphere. Attaching another cells yields the cofiber sequence

$$C_{3+} \wedge S^1 \longrightarrow S^{\succ} \longrightarrow S^{\lambda}.$$

Tensor with $\mathrm{H}\underline{\mathbb{Z}}$ and you can compute as before. Essentially the same computation as in Example 2.1.2 yields information about $S^{\succ}$. Then, you do one more computation for S^λ. We learn

$$\underline{\pi}_n(S^{\succ} \otimes \mathrm{H}\underline{\mathbb{Z}}) \cong \begin{cases} \mathfrak{g}_3 & n = 0, \\ \underline{I} & n = 1, \\ 0 & n = 2 \end{cases} \quad \text{and} \quad \underline{\pi}_n(S^{\lambda} \otimes \mathrm{H}\underline{\mathbb{Z}}) \cong \begin{cases} \mathfrak{g}_3 & n = 0, \\ 0 & n = 1, \\ \underline{\mathbb{Z}} & n = 2 \end{cases}$$

where $\mathfrak{g}_3$ is the Mackey functor with $\mathbb{Z}/3$ fixed at the top, also called $B(1,0)$ in HHR notation and $\underline{I} = \ker(\mathbb{Z}[C_3] \to \mathbb{Z})$.

Example 2.1.7. Now $G = C_4$. The difference to C_2 and C_3 is that 4 is not a prime. We obtain a decomposition $\rho_{C_4} \cong 1 \oplus \sigma \oplus \lambda$ where

- σ is the sign representation of $C_4/C_2 \cong C_2$ and essentially pulled back along the quotient map.
- λ is the 2-dimensional rotation representation with rotation by 90°.

An approach to compute $\widetilde{H}_*(S^\lambda; \underline{\mathbb{Z}})$ by attaching cells as before works here as well but an alternative approach is via the cofiber sequence

$$S(\lambda)_+ \longrightarrow S^0 \longrightarrow S^\lambda$$

where $S(\lambda) = S^1_{\text{rot}} = C_4 \cup (C_4 \times e^1)$. Thinking about the attaching map we sort of attach a point with a 90°-rotated one, i.e. $\underline{C}_*(S(\lambda)) = (\text{ind}_e^{C_4} \xrightarrow{1-\gamma} \text{ind}_e^{C_4} \mathbb{Z})$ which works out to be

$$\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\quad 0 \quad} & \mathbb{Z} \\
{\scriptstyle 1+\gamma}\downarrow\uparrow & & {\scriptstyle 1+\gamma}\downarrow\uparrow \\
\mathbb{Z}[C_4/C_2] & \xrightarrow{1-\gamma} & \mathbb{Z}[C_4/C_2] \\
{\scriptstyle 1+\gamma^2}\downarrow\uparrow & & {\scriptstyle 1+\gamma^2}\downarrow\uparrow \\
\mathbb{Z}[C_4] & \xrightarrow[1-\gamma]{} & \mathbb{Z}[C_4]
\end{array}$$

Computing kernel and cokernel yields

$$\begin{array}{ccccccc}
\mathbb{Z} & \longrightarrow & \mathbb{Z} & \xrightarrow{\quad 0 \quad} & \mathbb{Z} & \longrightarrow & \mathbb{Z} \\
{\scriptstyle 1}\downarrow\uparrow{\scriptstyle 2} & & {\scriptstyle 1+\gamma}\downarrow\uparrow & & {\scriptstyle 1+\gamma}\downarrow\uparrow & & {\scriptstyle 2}\downarrow\uparrow{\scriptstyle 1} \\
\mathbb{Z} & \longrightarrow & \mathbb{Z}[C_4/C_2] & \xrightarrow{1-\gamma} & \mathbb{Z}[C_4/C_2] & \longrightarrow & \mathbb{Z} \\
{\scriptstyle 1}\downarrow\uparrow{\scriptstyle 2} & & {\scriptstyle 1+\gamma^2}\downarrow\uparrow & & {\scriptstyle 1+\gamma^2}\downarrow\uparrow & & {\scriptstyle 2}\downarrow\uparrow{\scriptstyle 1} \\
\mathbb{Z} & \longrightarrow & \mathbb{Z}[C_4] & \xrightarrow[1-\gamma]{} & \mathbb{Z}[C_4] & \longrightarrow & \mathbb{Z}
\end{array}$$

The left object is $\underline{\mathbb{Z}}$ and the right one is $\underline{\mathbb{Z}}^*$, the **dual constant Mackey functor**. This computes

$$\underline{H}_n(S(\lambda)) \cong \begin{cases} \underline{\mathbb{Z}} & n = 1, \\ \underline{\mathbb{Z}}^* & n = 0. \end{cases}$$

Thus, we can now run the LES for

$$S(\lambda)_+ \otimes \mathrm{H}\underline{\mathbb{Z}} \longrightarrow \mathrm{H}\underline{\mathbb{Z}} \longrightarrow S^\lambda \otimes \mathrm{H}\underline{\mathbb{Z}}$$

to compute

$$\underline{H}_n(S^\lambda) \cong \begin{cases} \underline{\mathbb{Z}} & n = 2, \\ 0 & n = 1, \\ B(2,0) & n = 0 \end{cases}$$

where $B(2,0)$ is the Mackey functor

$$\begin{array}{c}
\mathbb{Z}/4 \\
\downarrow\uparrow{\scriptstyle 2} \\
\mathbb{Z}/2
\end{array}$$

$$0$$

One could write this as the cofiber sequence from the Postnikov filtration, namely as

$$\Sigma^2 H\underline{\mathbb{Z}} \longrightarrow \Sigma^\lambda H\underline{\mathbb{Z}} \longrightarrow H(B(2,0)).$$

See also [HHR17, Figure 6]. We're still on the way to compute the homology of S^ρ. So now apply Σ^σ to obtain

$$\Sigma^{2+\sigma} H\underline{\mathbb{Z}} \longrightarrow \Sigma^{\lambda+\sigma} H\underline{\mathbb{Z}} \longrightarrow \Sigma^\sigma H(B(2,0)).$$

See also [HHR17, Figure 3]. These are now not so bad and it turns out that the right term has π_0, π_1 while the left term has π_2, π_3, so they don't interact.

Here is one idea to compute $\Sigma^\sigma H_{C_4}\underline{\mathbb{Z}}$. Consider the Mackey functor diagram

$$\begin{array}{c} \underline{M}(C_4) \\ \downarrow\uparrow \\ \underline{M}(C_2) \\ \downarrow\uparrow \\ \underline{M}(e) \end{array}$$

along with the Weyl group actions. One can either chop off the bottom or restrict to the bottom part but lose some of the Weyl group action. In diagrams, this comes from

$$\begin{array}{ccc} \mathbf{Mack}(C_4/C_2) & & \mathbf{Sp}^{C_4/C_2} \\ {\scriptstyle q_*}\uparrow & & \uparrow{\scriptstyle (-)^{C_2}} \\ \mathbf{Mack}(C_4) & & \mathbf{Sp}^{C_4} \\ {\scriptstyle \mathrm{res}^{C_4}_{C_2}}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}^{C_4}_{C_2}} \\ \mathbf{Mack}(C_2) & & \mathbf{Sp}^{C_2} \end{array}$$

which has a spectral incarnation. In the setting $q^* : \mathbf{Sp}^{G/N} \rightleftarrows \mathbf{Sp}^G : (-)^N$ there is the projection formula $(q^*X \otimes Y)^N \simeq X \otimes Y^N$ in $\mathbf{Sp}^{G/N}$. Thus, we compute[5]

$$(S^\sigma \otimes H_{C_4}\underline{\mathbb{Z}})^{C_2} \simeq S^\sigma \otimes H_{C_2}\underline{\mathbb{Z}}$$

which we already computed (Example 2.1.2), namely it has $\underline{\pi}_1 \cong \underline{\mathbb{Z}}^\sigma$ and $\underline{\pi}_0 \cong \mathfrak{g}$. Moreover,

$$\mathrm{Res}^{C_4}_{C_2} \Sigma^\sigma H_{C_4}\underline{\mathbb{Z}} \simeq \Sigma^1 H_{C_2}\underline{\mathbb{Z}}.$$

We deduce

$$\underline{\pi}_0(\Sigma^\sigma H_{C_4}\underline{\mathbb{Z}}) \cong \mathfrak{g} = B(2,1) = \begin{cases} \mathbb{Z}/2 \\ 0 \\ 0 \end{cases}$$

since there are no interactions. Moreover, $\underline{\pi}_1(\Sigma^\sigma H_{C_4}\underline{\mathbb{Z}})$ is

$$\begin{array}{c} 0 \\ \\ \\ \mathbb{Z}^\sigma \\ {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 2} \\ \mathbb{Z}^\sigma \end{array}$$

[5] I think one proves $(H_{C_4}\underline{\mathbb{Z}})^{C_2} \simeq H_{C_2}\underline{\mathbb{Z}}$ by computing $\underline{\pi}_*$.

where we know the middle group with C_2-action from the fixed points, and we know the bottom group (without the C_2-action) plus the maps from the restriction. In particular, the C_2-action on the bottom must make it $\mathbb{Z}^\sigma$, so it the bottom part is C_2-equivariant.

2.2 The (Regular) Slice Filtration

We don't have to change so much from Section 1.3; it's basically all the same.

Definition 2.2.1.

(i) Let $\tau_{\geq n} \subseteq \mathbf{Sp}^G$ denote the full localizing subcategory containing $\mathrm{Ind}_H^G S^{k\rho_H}$ for $H \leq G$ and $k \geq 0$ with $k|H| \geq n$.

(ii) We define $X \leq n-1$ if $[W, X] = 0$ for all $W \in \tau_{\geq n}$.

Example 2.2.2. One can check $\tau_{\geq 0} \simeq \mathbf{Sp}^G_{\geq 0}$.

We get similar properties but add one useful one.

Proposition 2.2.3.

(i) There is an equivalence $P_0^0 X \simeq \mathrm{H}\underline{\pi}_0 X$.

(ii) There is an equivalence $P_1^1 X \simeq \Sigma\mathrm{H}(\underline{\pi}_1 X/\ker \mathrm{res})$.

(iii) There is an equivalence $P_{k+|G|}^{n+|G|}(\Sigma^\rho X) \simeq \Sigma^\rho P_k^n(X)$.

(iv) There is an equivalence $P_k^n \operatorname{Res}_H^G X \simeq \operatorname{Res}_H^G P_k^n X$.

(v) There is an equivalence $\varphi_G^* P_k^n X \simeq P_{k|G|}^{n|G|} \varphi_G^* X$

The last part is a result from Hill's primer and φ_G^* is the geometric inflation functor.

Example 2.2.4. Let $G = C_2$. Then, $\Sigma\mathrm{H}\mathfrak{g} \simeq \varphi_{C_2}^* \Sigma\mathrm{H}\mathbb{F}_2$, as $\mathfrak{g}$ is concentrated in the top degree. So it is a 2-slice.

This result from Hill is very useful in general since it is hard to tell when something is an n-slice. Right now, this result is only stated by geometrically inflating from the trivial group. This can be generalized.

Construction 2.2.5. Let $N \trianglelefteq G$. Then, there exists a family $\mathcal{F}[N]$ such that its universal space satisfies

$$(E\mathcal{F}[N])^H \simeq \begin{cases} \varnothing & H \geq N, \\ * & \text{else.} \end{cases}$$

We obtain a cofiber sequence

$$E\mathcal{F}[N]_+ \longrightarrow S^0 \longrightarrow \widetilde{E\mathcal{F}[N]}$$

which allows us to define two functors

$$\Phi^N(X) = (\widetilde{E\mathcal{F}[N]} \otimes X)^N \quad \text{and} \quad \varphi_N^*(Z) = \widetilde{E\mathcal{F}[N]} \otimes q^* Z$$

where $q : G \to G/N$. This gives an adjunction

$$\mathbf{Sp}^G \underset{\varphi_N^*}{\overset{\Phi^N}{\rightleftarrows}} \mathbf{Sp}^{G/N}$$

Exercise 2.2.6. Let $X \in \mathbf{Sp}^{C_4}$. Then, $\Phi^{C_4} X \simeq \Phi^{C_4/C_2} X^{C_2}$.

Proof. Recall that $(-)^{C_2}$ is the right adjoint of $p^* = \mathrm{Infl}_{C_4/C_2}^{C_4}$ for $p : C_4 \to C_4/C_2$, so it has a residual C_4/C_2-action. Anyway, let $\mathcal{P}_G$ denote the family of proper subgroups of G. Then,

$$\Phi^{C_4/C_2} X^{C_2} \simeq (\widetilde{E}\mathcal{P}_{C_4/C_2+} \otimes X^{C_2})^{C_2} \simeq \left((p^*\widetilde{E}\mathcal{P}_{C_4/C_2+} \otimes X)^{C_2}\right)^{C_4/C_2} \simeq (p^*\widetilde{E}\mathcal{P}_{C_4/C_2+} \otimes X)^{C_4}.$$

So it suffices to show $p^*\widetilde{E}\mathcal{P}_{C_4/C_2+} \simeq \widetilde{E}\mathcal{P}_{C_4+}$. But indeed,

$$(p^*\widetilde{E}\mathcal{P}_{C_4/C_2+})^H \simeq \widetilde{E}\mathcal{P}_{C_4/C_2+}^{pH} \simeq \begin{cases} S^0 & H = C_4, \\ * & \text{else} \end{cases}$$

which characterizes $\widetilde{E}\mathcal{P}_{C_4+}$. Here, we used that for $p : G \to G/K$ one has $(p^*X)^H \simeq X^{pH}$ as the left side is explicitly given by

$$\mathbf{Orb}_G^{\mathrm{op}} \xrightarrow{p} \mathbf{Orb}_{G/K}^{\mathrm{op}} \xrightarrow{X} \mathcal{S},\ H \mapsto pH \mapsto X^{pH}.$$

□

Proposition 2.2.7 (Hill–Ullman)**.** Let $N \trianglelefteq G$. Then, $\varphi_N^* P_k^n X \simeq P_{k|N|}^{n|N|} \varphi_N^* X$.

Example 2.2.8. Let $G = C_4$.

(i) Let $N = C_2$. We have seen that $\Sigma \mathrm{H}_{C_2}\underline{\mathbb{Z}}$ is a 1-slice (Proposition 1.3.5(iii)). Thus, $\varphi_{C_2}^* \Sigma \mathrm{H}_{C_2}\underline{\mathbb{Z}}$ is a 2-slice which is explicitly $\Sigma^1 \mathrm{H}_{C_4} M$ for the Mackey functor M given by

$$\begin{array}{c} \mathbb{Z} \\ {\scriptstyle 1}\big\downarrow\big\uparrow{\scriptstyle 2} \\ \mathbb{Z} \\ \\ 0 \end{array}$$

(ii) Let $N = C_4$. Again, we use that $\Sigma \mathrm{H}_e \mathbb{Z}$ is a 1-slice. Thus, $\varphi_{C_4}^* \Sigma \mathrm{H}\mathbb{Z}$ is a 4-slice given by $\Sigma \mathrm{H}_{C_4} N$ for the Mackey functor N given by $\mathbb{Z}$ concentrated in the top group.

3 More about Slice Towers

3.1 Examples: $\Sigma^V \mathrm{H}_G \underline{\mathbb{Z}}$

Exercise 3.1.1. Let $0 \le n \le 6$. Then, $\Sigma^n \mathrm{H}_{C_2}\underline{\mathbb{Z}}$ is an n-slice.

Proof. The strategy is to use Proposition 1.3.5(i) – (iii) why delooping a few times and then checking whether we get a 0-slice or a 1-slice which is an explicit algebraic statement about EM-spectra. The main work was done in our computations (Exercise 2.1.5).

- $n = 0$: Certainly, $\mathrm{H}\underline{\mathbb{Z}}$ is a 0-slice by Proposition 1.3.5(i).
- $n = 1$: Also, $\Sigma \mathrm{H}\underline{\mathbb{Z}}$ is a 1-slice by Proposition 1.3.5(iii).
- $n = 2$: This is because $\Sigma^{2-\rho}\mathrm{H}\underline{\mathbb{Z}} \simeq \mathrm{H}\underline{\mathbb{Z}}^\sigma$ is a 0-slice.
- $n = 3$: This is because $\Sigma^{3-\rho} \simeq \Sigma \mathrm{H}\underline{\mathbb{Z}}^\sigma$ is a 1-slice.
- $n = 4$: This is because $\Sigma^{4-2\rho} \simeq \mathrm{H}\underline{\mathbb{Z}}^*$ is a 0-slice.
- $n = 5$: This is because $\Sigma^{5-2\rho} \simeq \Sigma \mathrm{H}\underline{\mathbb{Z}}^*$ is a 1-slice.
- $n = 6$: This is because $\Sigma^{6-3\rho}\mathrm{H}\underline{\mathbb{Z}} \simeq \mathrm{H}Q\mathbb{Z}^\sigma$ is a 0-slice.

Yay. □

It turns out that $0 \leq n \leq 6$ is the precise range where this phenomenon happens. We will take some examples and show that they are not slices. Our strategy is to desuspend enough to ask if it is a 0- or 1-slice and in attempting to see this, we will obtain slice fiber sequences.

Example 3.1.2. Consider $\Sigma^7 H_{C_2}\underline{\mathbb{Z}}$. By Proposition 1.3.5(ii) it is equivalent to ask if it is the 3ρ-suspension of some 1-slice, i.e. if $\Sigma^7 H_{C_2}\underline{\mathbb{Z}} \simeq \Sigma^{3\rho+1} H_{C_2}\underline{M}$ where $\underline{M}$ is some Mackey functor with injective restriction maps (Proposition 1.3.5(iii)).

Equivalently, $\Sigma^{7-3\rho} H_{C_2}\underline{\mathbb{Z}} \simeq \Sigma^1 H_{C_2} Q\mathbb{Z}^\sigma$ using Exercise 2.1.5. But $Q\mathbb{Z}^\sigma$ is the Mackey functor

$$\begin{array}{c} \mathbb{Z}/2 \\ {\scriptstyle 0}\downarrow\uparrow \\ \mathbb{Z}^\sigma \end{array}$$

which is evidently not a 1-slice. The top group is the obstruction to being a 1-slice and killing it off leads to a SES

$$0 \longrightarrow \mathfrak{g} \longrightarrow Q\mathbb{Z}^\sigma \longrightarrow \underline{\mathbb{Z}}^\sigma \longrightarrow 0.$$

It gives rise to a cofiber sequence

$$\Sigma^1 H\mathfrak{g} \longrightarrow \Sigma^1 HQ\mathbb{Z}^\sigma \longrightarrow \Sigma^1 H\underline{\mathbb{Z}}^\sigma$$

which again yields

$$\Sigma^4 H\mathfrak{g} \simeq \Sigma^{1+3\rho} H\mathfrak{g} \longrightarrow \Sigma^7 H\underline{\mathbb{Z}} \longrightarrow \Sigma^{1+3\rho} H\underline{\mathbb{Z}}^\sigma$$

which is the slice fiber sequence of $\Sigma^7 H\underline{\mathbb{Z}}$ by Proposition 1.3.5(iv). Indeed, the left term first of all uses an equivalence from Exercise 2.1.3. The left term is $\Sigma^4 H\mathfrak{g} \simeq \varphi^*_{C_2}\Sigma^4 H\mathbb{F}_2$ which is an 8-slice by Proposition 2.2.7.[6] The right term is a 7-slice because we bumped up the 1-slice $H\underline{\mathbb{Z}}^\sigma$ by $\Sigma^{3\rho}$.

Let's use this to study SliceSS($\Sigma^7 H_{C_2}\underline{\mathbb{Z}}$). Using the computation from Exercise 2.1.3 we arrive at the picture

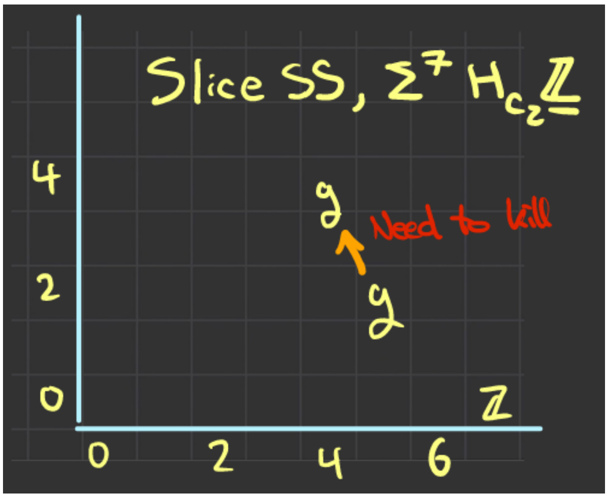


Figure 3: The slice spectral sequence for $\Sigma^7 H_{C_2}\underline{\mathbb{Z}}$.

The left $\mathfrak{g}$ is the 8-slice and the right two terms are the 7-slice. Since we know $\Sigma^7 H\underline{\mathbb{Z}}$, we know that these $\mathfrak{g}$'s must be killed, and the only chance for this is via the orange differential that's drawn in.

[6] See also Example 2.2.4.

Example 3.1.3. Consider $\Sigma^{2\sigma}\mathrm{H}_{C_2}\underline{\mathbb{Z}}$. Is this a 2-slice? Apply $\Sigma^{-\rho}$, so by Proposition 1.3.5(ii) we are equivalently asking if $\Sigma^{\sigma-1}\mathrm{H}_{C_2}\underline{\mathbb{Z}}$ is a 0-slice. We have seen (Example 2.1.2) that it sits in a slice fiber sequence

$$\mathrm{H}_{C_2}\underline{\mathbb{Z}}^{\sigma} \longrightarrow \Sigma^{\sigma-1}\mathrm{H}_{C_2}\underline{\mathbb{Z}} \longrightarrow \Sigma^{-1}\mathrm{H}_{C_2}\mathfrak{g}$$

where the left is a 0-slice and the right is a (-2)-slice. So we already answered that the object in question is not a 0-slice. Applying Σ^{ρ} again gives

$$\Sigma^{2}\mathrm{H}_{C_2}\underline{\mathbb{Z}} \simeq \Sigma^{\rho}\mathrm{H}_{C_2}\underline{\mathbb{Z}}^{\sigma} \longrightarrow \Sigma^{2\sigma}\mathrm{H}_{C_2}\underline{\mathbb{Z}} \longrightarrow \Sigma^{\sigma}\mathrm{H}_{C_2}\mathfrak{g} \simeq \mathrm{H}_{C_2}\mathfrak{g}$$

using the equivalences from Exercise 2.1.3. So SliceSS$(\Sigma^{2\sigma}\mathrm{H}_{C_2}\underline{\mathbb{Z}})$ is the boring already collapsed picture

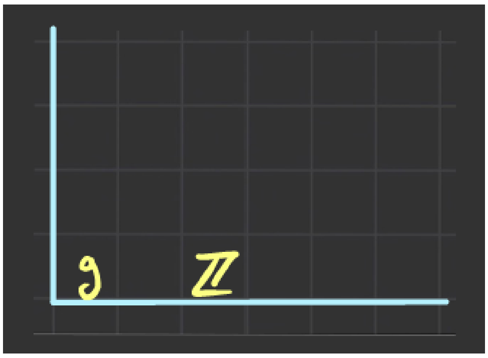


Figure 4: The slice spectral sequence for $\Sigma^{2\sigma}\mathrm{H}_{C_2}\underline{\mathbb{Z}}$.

Observe also that the slices are Eilenberg MacLane objects, so the slice filtration for $\Sigma^{2\sigma}\mathrm{H}_{C_2}\underline{\mathbb{Z}}$ is the Postnikov filtration for $\Sigma^{2\sigma}\mathrm{H}_{C_2}\underline{\mathbb{Z}}$.

These examples show that Σ^{ρ} just shifts things around in the filtration to where they are easy to understand, namely the 0- and 1-slices. This leaves us with Eilenberg-MacLane computations. That's a useful technique in general.

Example 3.1.4. Consider $\Sigma^{\lambda}\mathrm{H}_{C_4}\underline{\mathbb{Z}}$.

- We know that the 2-slice is non-trivial, as detected by $\mathrm{Res}^{C_4}_{e}$ via $|\lambda| = 2$.
- Even better, consider $\mathrm{Res}^{C_4}_{C_2}\Sigma^{\lambda}\mathrm{H}_{C_4}\underline{\mathbb{Z}} \simeq \Sigma^{2\sigma}\mathrm{H}_{C_2}\underline{\mathbb{Z}}$ which has 0-slices and 2-slices.

We may compute

$$P^0_0\Sigma^{\lambda}\mathrm{H}_{C_4}\underline{\mathbb{Z}} \simeq \mathrm{H}_{C_4}\underline{\pi}_0\Sigma^{\lambda}\mathrm{H}_{C_4}\underline{\mathbb{Z}} \simeq \mathrm{H}_{C_4}B(2,0)$$

by Proposition 2.2.3 and Example 2.1.7. Even more so, in Example 2.1.7 we had found

$$P_1 \simeq \Sigma^{2}\mathrm{H}_{C_4}\underline{\mathbb{Z}} \longrightarrow \Sigma^{\lambda}\mathrm{H}_{C_4}\underline{\mathbb{Z}} \longrightarrow \mathrm{H}_{C_4}B(2,0) \simeq P^0_0.$$

We claim that $\Sigma^{2}\mathrm{H}_{C_4}\underline{\mathbb{Z}}$ is a 2-slice.

- It is ≥ 2: Indeed, it is 2-connective (and $2 \geq 0$), so it is slice 2-connective.
- It is ≤ 2: Recall that we need to show

$$\left[\mathrm{Ind}^{C_4}_{H} S^{k\rho_H}, \Sigma^{2}\mathrm{H}_{C_4}\underline{\mathbb{Z}}\right]^{C_4} \cong 0$$

for $H \leq C_4$ and $k|H| > 2$. By adjunction, this is an H-equivariant question. But now note that $\mathrm{Res}^{C_4}_{C_2} \Sigma^2 \mathrm{H}_{C_4}\underline{\mathbb{Z}} \simeq \Sigma^2 \mathrm{H}_{C_2}\underline{\mathbb{Z}}$ is a 2-slice (Exercise 3.1.1). We are left with the case $H = C_4$. For this, we have to know

$$[\mathbb{S}^{k\rho}, \Sigma^2 \mathrm{H}_{C_4}\underline{\mathbb{Z}}]^{C_4} \cong \pi_{-2}(\Sigma^{-k\rho}\mathrm{H}_{C_4}\underline{\mathbb{Z}}) \cong 0.$$

This can be done and is e.g. in [HHR17, Figure 3].

So the Postnikov filtration for $\Sigma^\lambda \mathrm{H}_{C_4}\underline{\mathbb{Z}}$ is the slice filtration.

One learning of the above example should be that coconnectivity is often a Bredon homology computation.

Exercise 3.1.5. Show that $\Sigma^n \mathrm{H}_{C_4}\underline{\mathbb{Z}}$ is an n-slice for $0 \leq n \leq 4$.

Proof. For $n = 0, 1$ this is clear by Proposition 2.2.3. For $n = 2, 3, 4$ we proceed as above (Example 3.1.4). Everything is the same and we need to read off that $\pi_{-m}(\Sigma^{-k\rho}\mathrm{H}_{C_4}\underline{\mathbb{Z}}) \cong 0$ for $m = 2, 3, 4$. Look at the following picture.

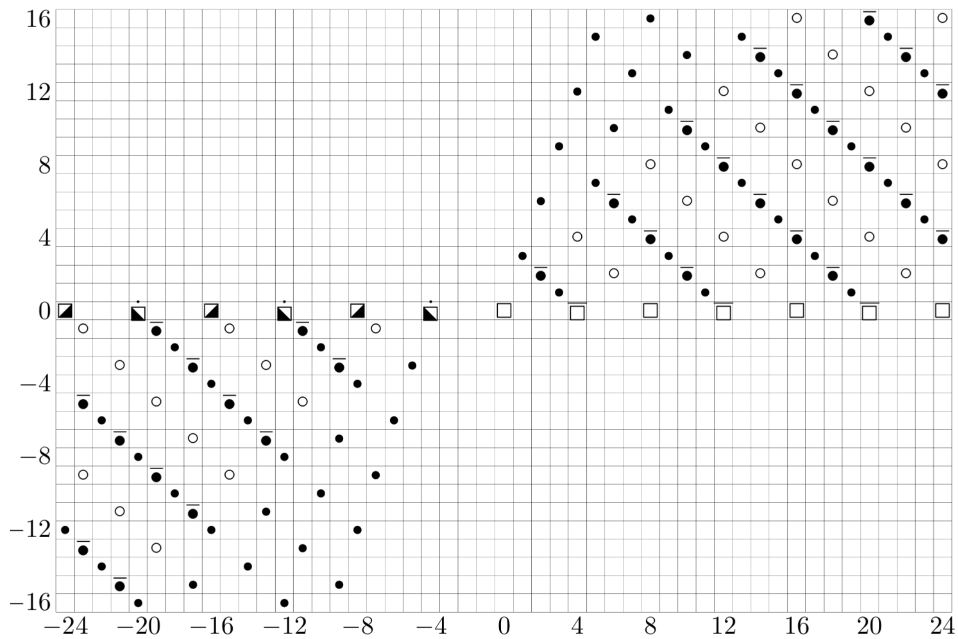


FIGURE 3. The Mackey functor slice spectral sequence for $\bigvee_{n\in\mathbf{Z}} \Sigma^{n\rho_4} H\underline{\mathbf{Z}}$. The symbols are defined in Table 2. The Mackey functor at position $(4n - s, s)$ is $\underline{\pi}_{n(4-\rho_4)-s} H\mathbf{Z} = \underline{H}_{4n-s} S^{n\rho_4}$.

Figure 5: This is [HHR17, Figure 3].

Whatever all these symbols mean, our desired groups are the vertical lines at $-1, -2, -3$. We read off that these vanish. $\square$

Exercise 3.1.6. There is a slice sequence

$$P^8_8 \simeq \Sigma^2 \mathrm{H}_{C_4}\mathfrak{g} \longrightarrow \Sigma^5 \mathrm{H}_{C_4}\underline{\mathbb{Z}} \longrightarrow \Sigma^{3+2\sigma}\mathrm{H}_{C_4}\underline{\mathbb{Z}} \simeq P^5_5.$$

Proof. First, we note that the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\
 & & {\scriptstyle 2}\downarrow\uparrow{\scriptstyle 1} & & {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 2} & & \downarrow\uparrow & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 2} & & {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 2} & & \uparrow & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow[1]{} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
$$

describes a SES of Mackey functors

$$0 \longrightarrow \underline{\mathbb{Z}}(2,1) \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

where $\underline{\mathbb{Z}}(2,1)$ is this Mackey functor on the left. Thus, we obtain a fiber sequence

$$\Sigma^{2+2\sigma} H_{C_4}\mathfrak{g} \longrightarrow \Sigma^{3+2\sigma} H_{C_4}\underline{\mathbb{Z}}(2,1) \longrightarrow \Sigma^{3+2\sigma} HC_4\underline{\mathbb{Z}}$$

By inflating up the C_2-equivalence $\Sigma^{\sigma} H_{C_2}\mathfrak{g} \simeq H_{C_2}\mathfrak{g}$ from Exercise 2.1.3 we also see that the left side is $\Sigma^2 H_{C_4}\mathfrak{g}$. Moreover, $\Sigma^{2-2\sigma} H_{C_4}\underline{\mathbb{Z}} \simeq H_{C_4}\underline{\mathbb{Z}}(2,1)$ by [HHR17, Figure 6].

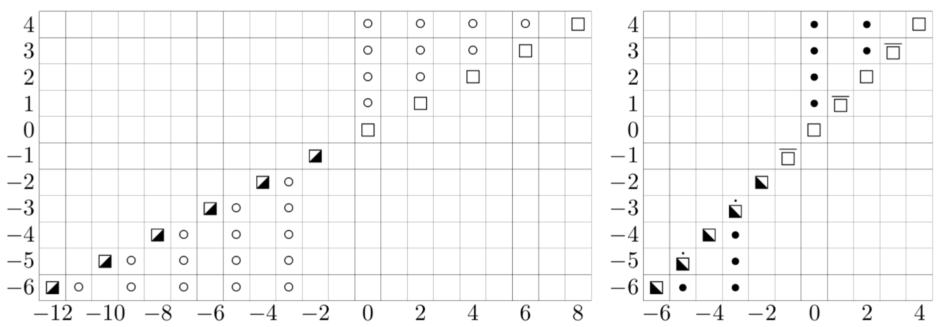


FIGURE 6. Charts for $\underline{H}_i S^{m\lambda}$ and $\underline{H}_i S^{n\sigma}$. The horizontal coordinates are i and the vertical ones are m and n. $S^{m\lambda}$ is on the left and $S^{n\sigma}$ is on the right.

Figure 6: This is [HHR17, Figure 6].

This yields the desired fiber sequence. We still need to show that it is a slice fiber sequence. The left side is quick via Proposition 2.2.3(v), as $\Sigma^2 H_{C_4}\mathfrak{g} \simeq \varphi^*_{C_4}\Sigma^2 H\mathbb{F}_2$. Now onto the right side. We must equivalently show that $\Sigma^{2+\sigma-\lambda} H_{C_4}\underline{\mathbb{Z}}$ is a 1-slice. We do so by explicitly computing its homotopy groups. Consider the cofiber sequence

$$C_4/C_{2+} \longrightarrow S^0 \longrightarrow S^{\sigma}$$

which is an inflated version of the standard cofiber sequence for C_2. So we obtain a cofiber sequence

$$C_4/C_{2+} \otimes \Sigma^{-\lambda} H_{C_4}\underline{\mathbb{Z}} \longrightarrow \Sigma^{-\lambda} H\underline{\mathbb{Z}} \longrightarrow \Sigma^{\sigma-\lambda} H\underline{\mathbb{Z}}$$

We can read off [HHR17, Figure 6] that $\Sigma^{-\lambda} H\underline{\mathbb{Z}} \simeq \Sigma^{-2} H\underline{\mathbb{Z}}^*$. Moreover, the homotopy Mackey functor of the left object is computed by a lift Mackey functor [Zen18, Definition 2.8, p. 44]. Namely, we have

$$\underline{\pi}_*(C_4/C_{2+} \otimes \Sigma^{-\lambda} H_{C_4}\underline{\mathbb{Z}})(C_4/C_4) \cong \underline{\pi}_*(\Sigma^{-\lambda} H_{C_4}\underline{\mathbb{Z}})(C_4/C_2 \times C_4/C_4) \cong \pi_*^{C_2} X$$
$$\underline{\pi}_*(C_4/C_{2+} \otimes \Sigma^{-\lambda} H_{C_4}\underline{\mathbb{Z}})(C_4/C_2) \cong \underline{\pi}_*(\Sigma^{-\lambda} H_{C_4}\underline{\mathbb{Z}})(C_4/C_2 \times C_4/C_2) \cong \pi_*^{C_2} X \oplus \pi_*^{C_2} X$$
$$\underline{\pi}_*(C_4/C_{2+} \otimes \Sigma^{-\lambda} H_{C_4}\underline{\mathbb{Z}})(C_4/e) \cong \underline{\pi}_*(\Sigma^{-\lambda} H_{C_4}\underline{\mathbb{Z}})(C_4/C_2 \times C_4/e) \cong \pi_*^{C_2} X \cong \pi_*^{C_4} X \oplus \pi_*^{C_4} X$$

where we use $C_4/C_2 \times C_4/C_2 \cong C_4 \amalg C_4/C_2$ and $C_4/C_2 \times C_4/e \cong C_4/e \amalg C_4/e$. Using the LES allows us to arrive at

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} & \longrightarrow & 0 \\
\downarrow\uparrow & & {\scriptstyle\Delta}\downarrow\uparrow{\scriptstyle\nabla} & & {\scriptstyle 2}\downarrow\uparrow{\scriptstyle 1} & & \\
\mathbb{Z}\langle(1,-1)\rangle = \mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \overset{\nabla}{\dashrightarrow} & \mathbb{Z} & \longrightarrow & 0 \\
{\scriptstyle 2}\downarrow\uparrow{\scriptstyle 1} & & {\scriptstyle 2}\downarrow\uparrow{\scriptstyle 1} & & {\scriptstyle 2}\downarrow\uparrow{\scriptstyle 1} & & \\
\mathbb{Z}\langle(1,-1)\rangle = \mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow[\nabla]{} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

where the left term is $\underline{\pi}_{-1}(C_4/C_{2+} \otimes \Sigma^{-\lambda} H_{C_4}\underline{\mathbb{Z}})$. This shows that $\Sigma^{2+\sigma-\lambda} H_{C_4}\underline{\mathbb{Z}}$ is a 1-slice. □

Proposition 3.1.7 (Hill–Yarnall)**.** Let $X \in \mathbf{Sp}^G$. Then, $X \geq n$ if and only if $\Phi^H X$ is $\frac{n}{|H|}$-connective for all $H \leq G$.

Here, $\frac{n}{|H|}$-*connective* means $\left\lceil \frac{n}{|H|} \right\rceil$-connective. There is also a version for genuine fixed points but that one unfortunately does not work for $n < 0$.

Example 3.1.8. Let $G = C_2$.

(i) We get $X \geq 1$ if and only if $\mathrm{Res}^{C_2}_e X$ is 1-connective and $\Phi^{C_2} X$ is $\frac{1}{2}$-connective, i.e. 1-connective.

(ii) We get $X \geq 2$ if and only if $\mathrm{Res}^{C_2}_e X$ is 2-connective and $\Phi^{C_2} X$ is 1-connective.

Example 3.1.9. Let $G = C_4$. Then, $X \geq -3$ if and only if

$$\mathrm{Res}^{C_4}_e X \geq 3,\ \Phi^{C_2} X \geq -\frac{3}{2},\ \Phi^{C_4} X \geq -\frac{3}{4}.$$

Let's try specific examples for X.

Example 3.1.10. Consider $\Sigma^1 H_{C_2}\mathfrak{g}$. Then, $\mathrm{Res}^{C_2}_e \Sigma^1 H_{C_2}\mathfrak{g} \simeq *$ is n-connective for all n and

$$\Phi^{C_2}\Sigma^1 H_{C_2}\mathfrak{g} \simeq \Sigma^1 \Phi^{C_2} H_{C_2}\mathfrak{g} \simeq \Sigma^1 H\mathbb{F}_2$$

using symmetric monoidality of Φ^G and it being inverse to taking geometric spectra.[7] This is 1-connective. Altogether, $\Sigma^1 H_{C_2}\mathfrak{g} \geq 2$ by Example 3.1.8(ii).

Example 3.1.11. Consider $\Sigma^{-\bar{\rho}} H_{C_4}\underline{\mathbb{Z}} \simeq \Sigma^{-\sigma-\lambda} H_{C_4}\underline{\mathbb{Z}}$. We compute

$$\begin{aligned}
\mathrm{Res}^{C_4}_e \Sigma^{-\sigma-\lambda} H_{C_4}\underline{\mathbb{Z}} &\simeq \Sigma^{-3} H\underline{\mathbb{Z}} \geq -3, \\
\Phi^{C_2}\Sigma^{-\sigma-\lambda} H_{C_4}\underline{\mathbb{Z}} &\simeq \Sigma^{-1}\Phi^{C_2} H_{C_2}\underline{\mathbb{Z}} \geq -1, \\
\Phi^{C_4}\Sigma^{-\sigma-\lambda} H_{C_4}\underline{\mathbb{Z}} &\simeq \Phi^{C_4} H_{C_4}\underline{\mathbb{Z}} \geq 0
\end{aligned}$$

where we use symmetric monoidality of geometric fixed points and the Σ^{-1} in the second line comes from $-\sigma$ as it is really pulled back along $C_4/C_2 \to C_2$. That the geometric fixed points of connective spectra is connective for example follows from the converse of Hill-Yarnall (Proposition 3.1.7). We deduce $\Sigma^{-\sigma-\lambda} H_{C_4}\underline{\mathbb{Z}} \geq -3$ by Example 3.1.9.

[7] Alternatively, note that $a : H_{C_2}\mathfrak{g} \to \Sigma^\sigma H_{C_2}\mathfrak{g}$ is an equivalence, as seen in Exercise 2.1.3. Thus,

$$\Phi^{C_2} H_{C_2}\mathfrak{g} \cong \operatorname*{colim}_n \Sigma^{n\sigma}(H_{C_2}\mathfrak{g})^{C_2} \simeq (H_{C_2}\mathfrak{g})^{C_2} \simeq H\mathbb{F}_2.$$

3.2 RO(G)-Grading

Instead of grading over $\mathbb{Z}$, we may grade over the **real representation ring** of G, namely $\mathrm{RO}(G) = \mathbb{Z}\{\text{iso classes of irreps of } G \text{ over } \mathbb{R}\}$. It is often better to choose representatives.

Example 3.2.1.

(i) $\mathrm{RO}(C_2) = \mathbb{Z}\{1, \sigma\}$,

(ii) $\mathrm{RO}(C_3) = \mathbb{Z}\{1, \lambda\}$,

(iii) $\mathrm{RO}(C_4) = \mathbb{Z}\{1, \sigma, \lambda\}$,

(iv) $\mathrm{RO}(C_2 \times C_2) = \mathbb{Z}\{1, p_1^*\sigma, m^*\sigma, p_2^*\sigma\}$.

We can now invoke RO(G)-grading. Let $V, W \in \mathbf{Rep}(G)$, then

$$\pi_{V-W}(X) = [\mathbb{S}^{V-W}, X]^G \quad \text{and} \quad \pi^H_{V-W}(X) = [\mathrm{Res}^G_H \mathbb{S}^{V-W}, \mathrm{Res}^G_H X]^H.$$

We obtain an RO(G)-graded Mackey functor $\underline{\pi}_\star X$ where this fancy star notation was pioneered by Hu–Kriz.

Example 3.2.2. We compute $\underline{\pi}_{-\sigma}\mathrm{H}_{C_2}\underline{\mathbb{Z}} \cong \underline{\pi}_0\Sigma^\sigma \mathrm{H}_{C_2}\underline{\mathbb{Z}} \cong \mathfrak{g}$ by Example 2.1.2.

Exercise 3.2.3. A map of C_2-spectra $f : X \to Y$ is an $\underline{\pi}_*$-isomorphism if and only if it is an $\pi^{C_2}_\star$-isomorphism.

Proof. Use the cofiber sequence $C_{2+} \to S^0 \to S^\sigma$ and its dual. If you're a $\underline{\pi}_*$-isomorphism, then we can use the 5-Lemma and induction to get all other RO(G)-graded homotopy groups via the LES from the cofiber sequence. Conversely, we have all the $\pi^{C_2}_\star$ and need π^e_* which again we get from the 5-Lemma. □

If R is a ring in $\mathbf{Sp}^G$, then we obtain a map $\pi_V(R) \otimes \pi_W(R) \to \pi_{V\oplus W}(R)$. If R is a commutative ring in $\mathbf{Sp}^G$, then $\pi_\star R$ is graded-commutative but this is much more complicated than in the ordinary setting.

Example 3.2.4. For $G = C_2$ we obtain a map

$$\pi_{i+k\sigma}(R) \otimes \pi_{j+\ell\sigma}(R) \to \pi_{i+j+(k+\ell)\sigma}(R)$$

with graded-commutativity $\alpha\beta = (-1)^{ij}\varepsilon^{k\ell}\beta\alpha$. Here, $\varepsilon = \tau_W : \mathbb{S}^\sigma \otimes \mathbb{S}^\sigma \simeq \mathbb{S}^\sigma \otimes \mathbb{S}^\sigma$ which corresponds to a map $\mathbb{S} \to \mathbb{S}$, i.e. an element in $A(C_2)$, after choosing certain preferred equivalences. This is even an unit in $A(C_2)$.

A special case is

$$\mathbb{S} \longrightarrow \mathrm{ku}_\mathbb{R} \longrightarrow \mathrm{H}_{C_2}\underline{\mathbb{Z}}$$

which on $\underline{\pi}_0$ gives $\underline{A} \to \underline{\mathbb{Z}} \cong \underline{\mathbb{Z}}$, $\varepsilon \mapsto -1$. So in these rings ε is a little simpler.

Let us enumerate some important elements.

Construction 3.2.5. Let $V \in \mathbf{Rep}(G)$

(i) We write $a_V : S^0 \to S^V$, so $a_V \in \pi_{-V}(\mathbb{S})$.

(ii) If V is orientable, there exists a class $u_V \in \pi_{|V|-V}\mathrm{H}_G\underline{\mathbb{Z}} \cong \widetilde{H}^0(S^{V-|V|}; \underline{\mathbb{Z}})$. It satisfies $\mathrm{Res}^G_e u_V = \pm 1$.

Remark 3.2.6. The cofiber sequences $C_{2+} \to S^0 \xrightarrow{a} S^\sigma$ and $S^{-\sigma} \xrightarrow{\Sigma^{-\sigma} a} S^0 \to C_{2+}$ yield

$$\mathrm{im}(\mathrm{tr}_e^{C_2}) = \ker a \quad \text{and} \quad \ker(\mathrm{res}_e^{C_2}) = \mathrm{im}\, a.$$

Example 3.2.7. Let $G = C_2$. Then, 2σ is orientable, so $u_{2\sigma} \in \underline{\pi}_{2-2\sigma}\mathrm{H}_{C_2}\underline{\mathbb{Z}}$ exists.

Example 3.2.8. Here is a picture of $\underline{\pi}_\star \mathrm{H}_{C_2}\underline{\mathbb{Z}}$.

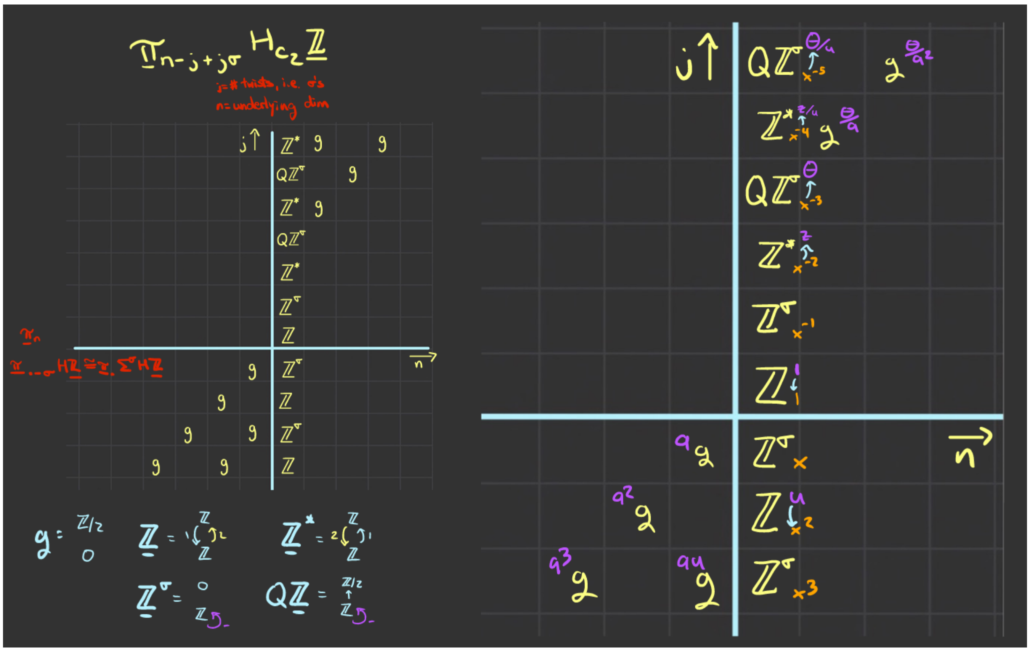


Figure 7: The $\mathrm{RO}(C_2)$-graded homotopy of $\mathrm{H}_{C_2}\underline{\mathbb{Z}}$.

This is a culmination of our computations in Exercise 2.1.3 and Exercise 2.1.5. We can see a few phenomena in this picture, e.g. related to Remark 3.2.6. For example:

- We have $\theta \in \ker(\mathrm{res}_e^{C_2}) = \mathrm{im}\, a$, so it must be a-divisible which is why these the term $\frac{\theta}{a}$ makes sense.
- In the coordinate $(0,0)$ we have $\mathrm{tr}_e^{C_2} 1 = 2$ in the constant Mackey functor and since $\mathrm{im}(\mathrm{tr}_e^{C_2}) = \ker a$, we deduce $2a = 0$. This is compatible with how we have a $\mathbb{Z}/2$ in $\mathfrak{g}$.

The slice spectral sequence

$$E_2^{s,t} = \underline{\pi}_{t-s} P_t^t X \Rightarrow \underline{\pi}_{t-s} X,\ d_r : \underline{\pi}_n P_t^t X \to \underline{\pi}_{n-1} P_{t+r-1}^{t+r-1} X$$

also has an $\mathrm{RO}(G)$-graded form.

Observation 3.2.9. There is a spectral sequence

$$E_2^{s,V} = \underline{\pi}_{V-s} P_V^V X \Rightarrow \underline{\pi}_{V-s} X,\ d_r : \underline{\pi}_{V-s} P_V^V X \to \underline{\pi}_{V-s-1} P_{V+r-1}^{V+r-1} X.$$

If R is a commutative ring in $\mathbf{Sp}^G$, then we obtain

$$E_2^{s,V} \otimes E_2^{s',V'} = \underline{\pi}_{V-s} P_V^V R \otimes \underline{\pi}_{V'-s'} P_{V'}^{V'} R \to \underline{\pi}_{V+V'-s-s'} P_{V+V'}^{V+V'} R = E_2^{s+s',V+V'}.$$

Now, let's reminisce back to the slice SS for $\mathrm{ku}_{\mathbb{R}}$ as discussed in Section 1.4. It turns out that there is a class $r_1 \in \pi_\rho \Sigma^\rho \mathrm{H}_{C_2}\underline{\mathbb{Z}}$ such that

$$\underline{\pi}_\star \Sigma^\rho \mathrm{H}_{C_2}\underline{\mathbb{Z}} \cong \underline{\pi}_\star \mathrm{H}_{C_2}\underline{\mathbb{Z}}\{r_1\}.$$

We can now enumerate some elements on the E_3-page of SliceSS($ku_{\mathbb{R}}$):

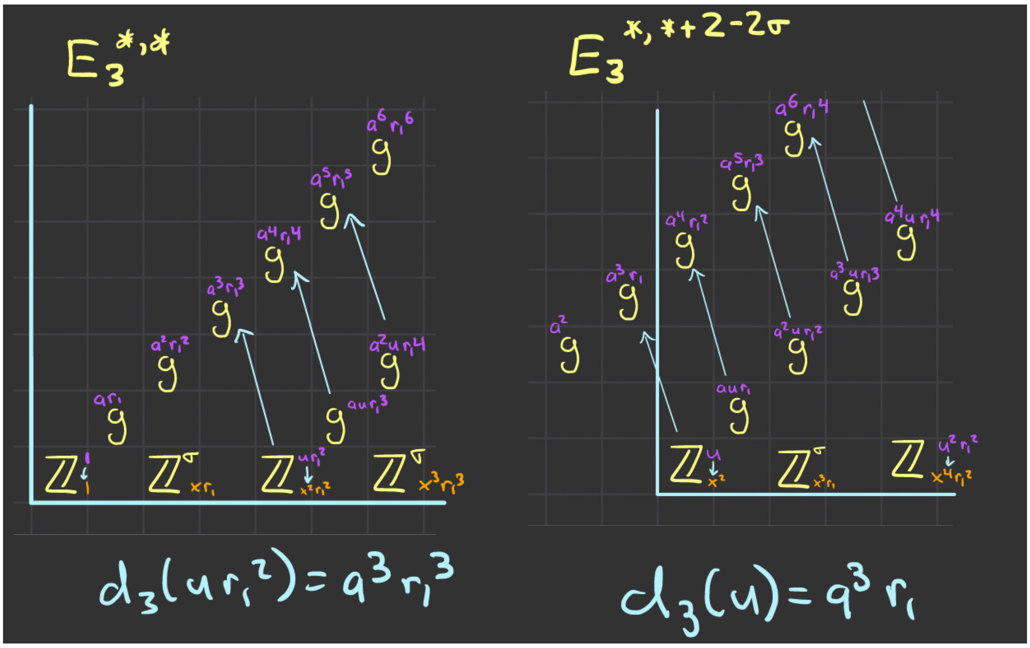
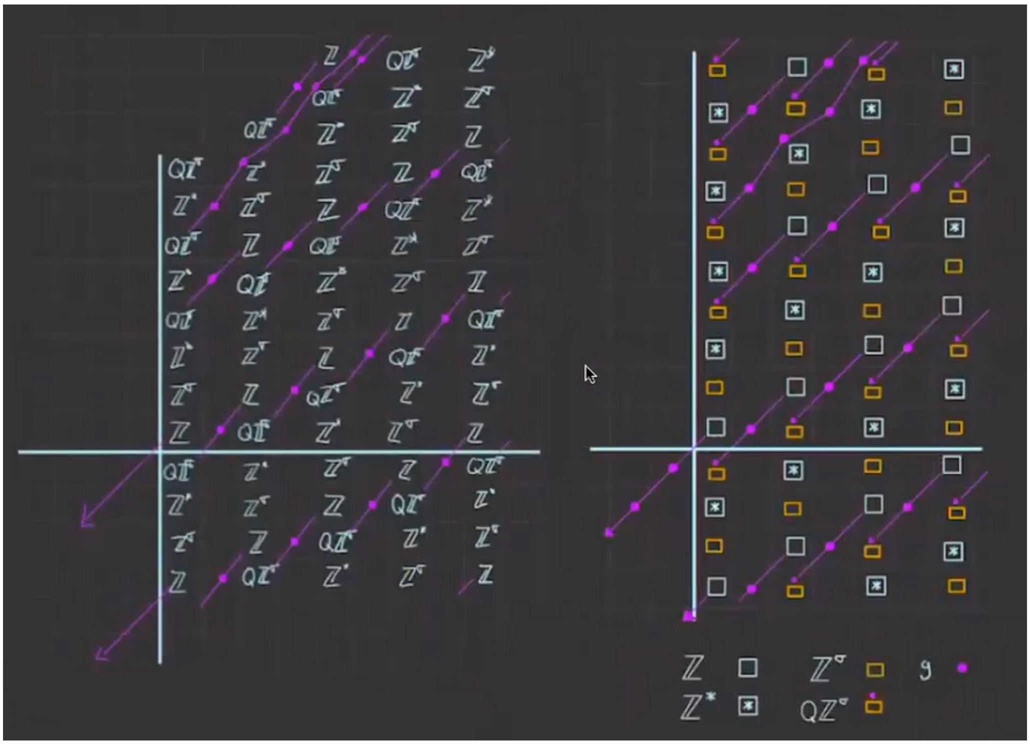


Figure 8: A part of the spectral sequence.

We can also slightly twist the spectral sequence in RO(C_2)-grading on the right and see that the differential already comes from u.

Here is a depiction of $\underline{\pi}_\star ku_{\mathbb{R}}$ together with the a-towers.



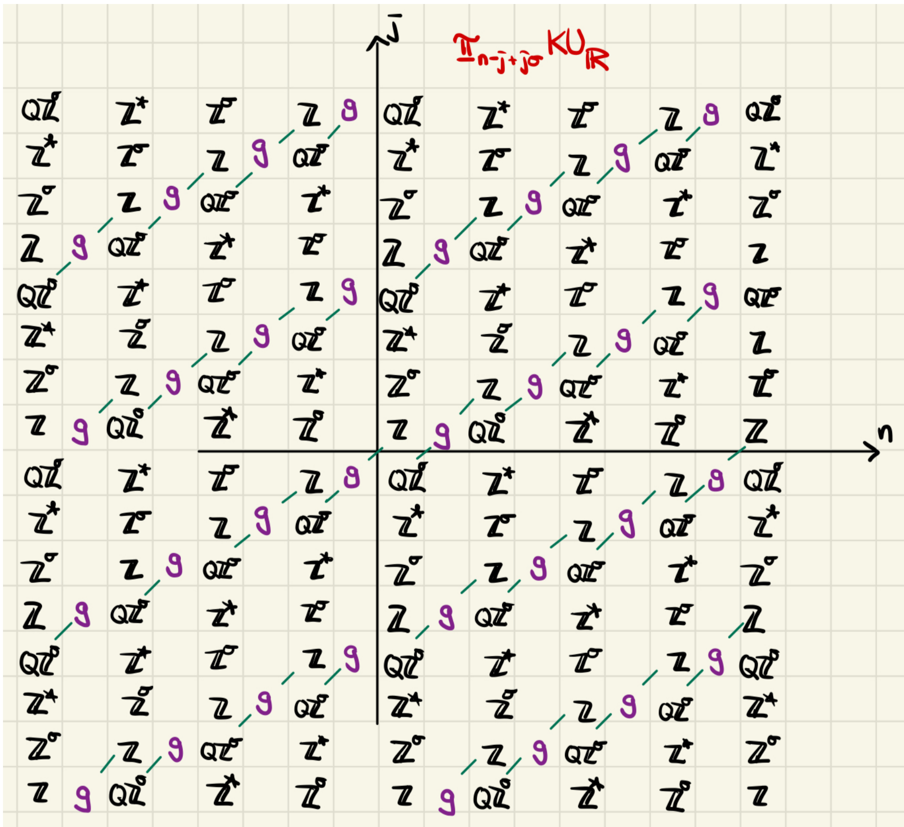
The row right above the x-axis are $\underline{\pi}_* ku_{\mathbb{R}}$ which we have computed with the slice spectral sequence (Section 1.4). Via the ρ-periodicity of $KU_{\mathbb{R}}$, we can propagate this onto the entire picture.

This drawing for example allows us to read off the geometric fixed points. We see these a towers which often eventually get cut off. The geometric fixed points consist of precisely the non-a-torsion elements.

Corollary 3.2.10. We have $\Phi^{C_2}(\mathrm{ku}_{\mathbb{R}}) \simeq \mathrm{H}\mathbb{F}_2[u^2]$.

Exercise 3.2.11. Draw the $RO(C_2)$-graded homotopy groups $\underline{\pi}_\star \, \mathrm{KU}_{\mathbb{R}}$.

Proof. We draw the following picture:



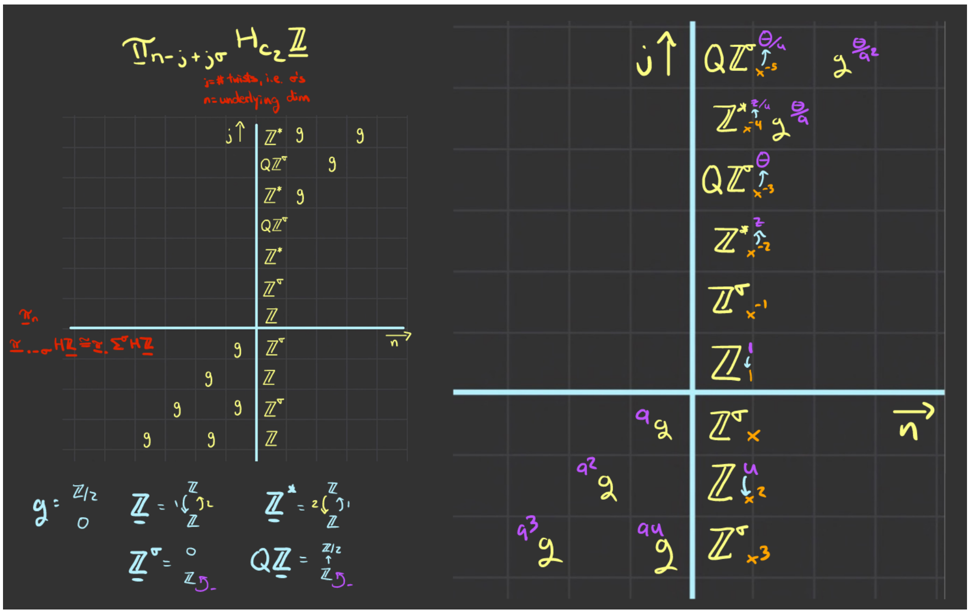
The zero'th horizontal line was computed via the slice spectral sequence in Section 1.4. Real Bott periodicity then gives $\underline{\pi}_\sigma \, \mathrm{KU}_{\mathbb{R}} \cong \underline{\pi}_{-1} \, \mathrm{KU}_{\mathbb{R}}$, so this allows us to propagate the picture in the j-direction. We have drawn in the pre-Euler classes which are all eventually cut off. This indicates $\Phi^{C_2} \, \mathrm{KU}_{\mathbb{R}} \simeq 0$. □

We will now start a new page for the new section to fit in some pictures on one page.

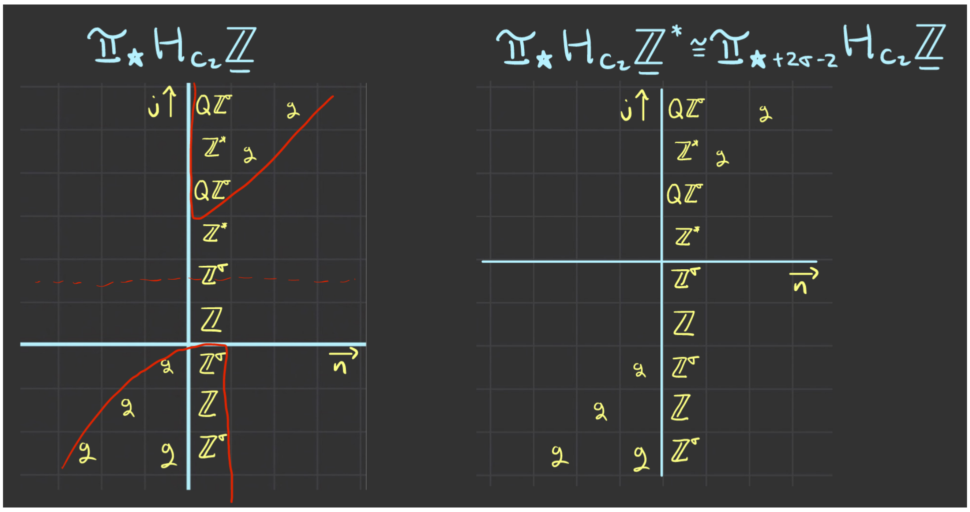
4 Duality & A Glimpse of HHR

4.1 Duality

Let's go back to the picture of $\underline{\pi}_\star \mathrm{H}_{C_2}\underline{\mathbb{Z}}$ from Example 3.2.8.



For example, we may identify $\underline{\pi}_{2\sigma-2}\mathrm{H}_{C_2}\underline{\mathbb{Z}} \cong \underline{\mathbb{Z}}^*$ in there. Even more so: The horizontal line it lies in only has this one Mackey functor, so $\Sigma^{2-2\sigma}\mathrm{H}_{C_2}\underline{\mathbb{Z}} \simeq \mathrm{H}_{C_2}\underline{\mathbb{Z}}^*$. In particular, we see the following picture:



So the right is just a shift of the left. Starring at these pictures a bit longer reveils a sort of symmetry. There these two beginnings of a cone and it seems like there is a horizontal line which is at the center of the symmetry. These are drawn in on the left.

They are explained by duality. Let us recall this non-equivariantly first.

Construction 4.1.1. Since $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Q}$ are injective, we get cohomology theories

$$\mathrm{Hom}_{\mathbf{Ab}}(\pi_{-*}(-), \mathbb{Q}/\mathbb{Z}) \quad \text{and} \quad \mathrm{Hom}_{\mathbf{Ab}}(\pi_{-*}(-), \mathbb{Q}).$$

By Brown representability, we obtain a map of spectra $\mathrm{H}\mathbb{Q} = I_{\mathbb{Q}} \to I_{\mathbb{Q}/\mathbb{Z}}$.[8] Moreover, we define $I_{\mathbb{Z}} = \mathrm{fib}(I_{\mathbb{Q}} \to I_{\mathbb{Q}/\mathbb{Z}})$. Thus, we obtain a fiber sequence

$$\begin{array}{ccccc} \mathrm{map}_{\mathbf{Sp}}(X, I_{\mathbb{Z}}) & \longrightarrow & \mathrm{map}_{\mathbf{Sp}}(X, I_{\mathbb{Q}}) & \longrightarrow & \mathrm{map}_{\mathbf{Sp}}(X, I_{\mathbb{Q}/\mathbb{Z}}) \\ \| & & \| & & \| \\ I_{\mathbb{Z}}X & \longrightarrow & I_{\mathbb{Q}}X & \longrightarrow & I_{\mathbb{Q}/\mathbb{Z}}X \end{array}$$

The left is the **Anderson dual**, the right is the **Brown–Comenetz dual**.

Proposition 4.1.2. Let $M \in \mathbf{Ab}$.

(i) If M is torsion, then $I_{\mathbb{Q}/\mathbb{Z}}\mathrm{H}M \simeq \mathrm{H}M$.

(ii) If M is torsion, then $I_{\mathbb{Z}}\mathrm{H}M \simeq \Sigma^{-1}I_{\mathbb{Q}/\mathbb{Z}}\mathrm{H}M \simeq \Sigma^{-1}\mathrm{H}M$.

(iii) If M is torsion-free, then $I_{\mathbb{Z}}\mathrm{H}M \simeq \mathrm{H}M$.

Proof. Part (ii) uses the fiber sequence from Construction 4.1.1 and (i). □

In the generic case of $M_{\mathrm{tor}} \hookrightarrow M \twoheadrightarrow M_{\mathrm{free}}$ we obtain a fiber sequence

$$\mathrm{H}M_{\mathrm{free}} \simeq I_{\mathbb{Z}}\mathrm{H}M_{\mathrm{free}} \longrightarrow I_{\mathbb{Z}}\mathrm{H}M \longrightarrow I_{\mathbb{Z}}\mathrm{H}M_{\mathrm{tor}} \simeq \Sigma^{-1}\mathrm{H}M_{\mathrm{tor}}.$$

This also works equivariantly. It was first studied in [Ric16] by sort of doing in levelwise in the Mackey functor.

Observation 4.1.3. There are equivalences

$$I_{\mathbb{Z}}\mathrm{H}\underline{\mathbb{Z}} \simeq \mathrm{H}\underline{\mathbb{Z}}^{*} \quad \text{and} \quad I_{\mathbb{Z}}\mathrm{H}\underline{\mathbb{Z}}^{\sigma} \simeq \mathrm{H}\underline{\mathbb{Z}}^{\sigma} \quad \text{and} \quad I_{\mathbb{Q}/\mathbb{Z}}\mathrm{H}\mathfrak{g} \simeq \mathrm{H}\mathfrak{g}.$$

Note how restriction and transfer are flipped. The fiber sequence $\mathrm{H}\mathfrak{g} \to \mathrm{H}Q\underline{\mathbb{Z}}^{\sigma} \to \mathrm{H}\underline{\mathbb{Z}}^{\sigma}$ now leads to

$$\Sigma^{-1}\mathrm{H}\mathfrak{g} \simeq I_{\mathbb{Z}}\mathrm{H}\mathfrak{g} \longleftarrow I_{\mathbb{Z}}\mathrm{H}Q\underline{\mathbb{Z}}^{\sigma} \longleftarrow I_{\mathbb{Z}}\mathrm{H}\underline{\mathbb{Z}}^{\sigma} \simeq \mathrm{H}\underline{\mathbb{Z}}^{\sigma}.$$

This allows us to compute the homotopy Mackey functor of the middle term which suggests a spectrification in the following.

Exercise 4.1.4. There is an equivalence $I_{\mathbb{Z}}\mathrm{H}Q\underline{\mathbb{Z}}^{\sigma} \simeq \Sigma^{\sigma-1}\mathrm{H}\underline{\mathbb{Z}}$.

Non-Proof. Naively, we can try to use the cofiber sequence $\Sigma^{-1}\mathrm{H}\mathfrak{g} \leftarrow I_{\mathbb{Z}}HQ\underline{\mathbb{Z}}^{\sigma} \leftarrow \mathrm{H}\underline{\mathbb{Z}}^{\sigma}$ from Observation 4.1.3. Applying $\Sigma^{1-\sigma}$ and using $\Sigma^{\sigma}\mathrm{H}\mathfrak{g} \simeq \mathrm{H}\mathfrak{g}$ from Exercise 2.1.3 yields

$$\mathrm{H}\mathfrak{g} \longleftarrow \Sigma^{1-\sigma}I_{\mathbb{Z}}\mathrm{H}Q\underline{\mathbb{Z}}^{\sigma} \longleftarrow \Sigma^{1-\sigma}\mathrm{H}\underline{\mathbb{Z}}^{\sigma} \simeq \mathrm{H}\underline{\mathbb{Z}}^{*}$$

where the equivalence on the right is computed in Example 2.1.4. We need to show that the middle term is $\mathrm{H}\underline{\mathbb{Z}}$. I thought that one could compute this via the LES

$$\begin{array}{ccccc} \mathbb{Z}/2 & \longleftarrow & ? & \longleftarrow & \mathbb{Z} \\ & & \downarrow\uparrow & & {\scriptstyle 2}\downarrow\uparrow{\scriptstyle 1} \\ 0 & \longleftarrow & ?? & \longleftarrow & \mathbb{Z} \end{array}$$

and certainly $?? = \mathbb{Z}$. But the top row sounds like an extension problem which is not solvable here: both possible options are possible. □

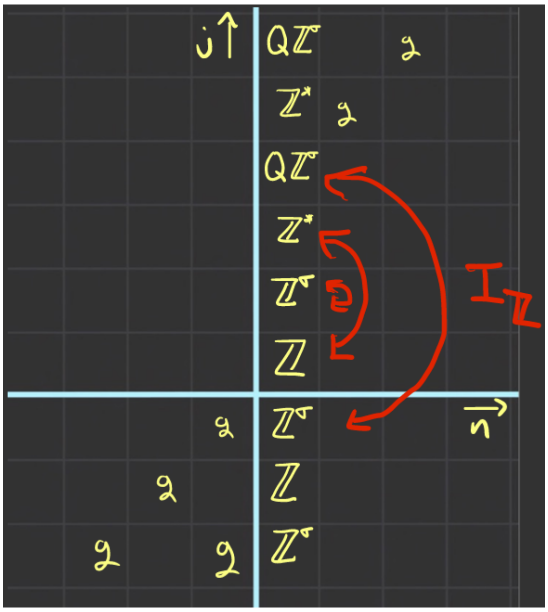
[8] We need to see $\mathrm{Hom}_{\mathbf{Ab}}(\pi_{-n}X, \mathbb{Q}) \cong H^{n}(X; \mathbb{Q})$ to get $\mathrm{H}\mathbb{Q} \simeq I_{\mathbb{Q}}$. But both are cohomology theories and they agree on $\mathbb{S}$.

Proof. We use $HQ\underline{\mathbb{Z}}^\sigma \simeq \Sigma^{3-3\sigma}H\underline{\mathbb{Z}}$ and $H\underline{\mathbb{Z}}^* \simeq \Sigma^{2-2\sigma}H\underline{\mathbb{Z}}$ from Exercise 2.1.5. Thus,

$$I_{\mathbb{Z}}HQ\underline{\mathbb{Z}}^\sigma \simeq I_{\mathbb{Z}}\Sigma^{3-3\sigma}H\underline{\mathbb{Z}} \simeq \Sigma^{3\sigma-3}H\underline{\mathbb{Z}}^* \simeq \Sigma^{\sigma-1}$$

using Observation 4.1.3. □

Our discussion, particularly Observation 4.1.3, shows that $I_{\mathbb{Z}}$ is responsible for the symmetry phenomena on the pictures:



One use is that now one only has to do one half of the Bredon homology computation and can deduce the other half from duality.

These dualities are also related to the slice filtration. First, the Brown–Comenetz dual.

Proposition 4.1.5 (Ullman)**.** There is an equivalence $P^n_k(I_{\mathbb{Q}/\mathbb{Z}}X) \simeq I_{\mathbb{Q}/\mathbb{Z}}P^{-k}_{-n}X$.

The Anderson dual is also useful, in the free case.

Example 4.1.6. Consider $\Sigma^4H\mathfrak{g} \simeq \Sigma^{1+3\rho}H\mathfrak{g} \to \Sigma^7H\underline{\mathbb{Z}} \to \Sigma^{1+3\rho}H\underline{\mathbb{Z}}^\sigma$ as computed in Example 3.1.2. Thus, we get

$$\begin{array}{ccccc} I_{\mathbb{Z}}\Sigma^{1+3\rho}H\underline{\mathbb{Z}}^\sigma & \longrightarrow & I_{\mathbb{Z}}\Sigma^7H\underline{\mathbb{Z}} & \longrightarrow & I_{\mathbb{Z}}\Sigma^4H\mathfrak{g} \\ \| & & \| & & \| \\ \Sigma^{-1-3\rho}H\underline{\mathbb{Z}}^\sigma & & \Sigma^{-7}H\underline{\mathbb{Z}}^* & & \Sigma^{-4-1}I_{\mathbb{Q}/\mathbb{Z}}H\mathfrak{g} \\ & & \| & & \| \\ & & \Sigma^{-5-2\sigma}H\underline{\mathbb{Z}} & & \Sigma^{-5}H\mathfrak{g} \end{array}$$

using Observation 4.1.3. On the left we have a (-7)-slice, on the right a (-10)-slice. Applying $\Sigma^{2\rho}$ leads us to

$$\Sigma^{-1-\rho}H\underline{\mathbb{Z}}^\sigma \longrightarrow \Sigma^{-3}H\underline{\mathbb{Z}} \longrightarrow \Sigma^{-3}H\mathfrak{g}$$

where the left is a (-3)-slice and the right is a (-6)-slice. Thus, we used the Anderson dual and the slice filtration of $\Sigma^7H\underline{\mathbb{Z}}$ to learn about the slice filtration of $\Sigma^{-3}H\underline{\mathbb{Z}}$.

4.2 Real Complex Bordism

There are analogues of $\mathrm{KU}_{\mathbb{R}}$.

(i) In the 60's Landweber defined $\mathrm{MU}_{\mathbb{R}} \in \mathbf{Sp}^{C_2}$.

(ii) In the 70's Araki worked out the formal group theory and defined $\mathrm{BP}_{\mathbb{R}} \in \mathbf{Sp}^{C_2}$.

(iii) In the 90's Hu–Kriz wrote an influential paper and computed $\pi_\star \mathrm{BP}_{\mathbb{R}}$. They write that this was first computed in unpublished work of Araki.

These objects were highly popularized in [HHR09].

Proposition 4.2.1 (Hu–Kriz)**.**

(i) There is an equivalence $\Phi^{C_2} \mathrm{BP}_{\mathbb{R}} \simeq \mathrm{H}\mathbb{F}_2$.

(ii) The Mackey functor $\underline{\pi}_{n\rho} \mathrm{BP}_{\mathbb{R}}$ is constant.

Since $\pi_* \mathrm{BP} \cong \mathbb{Z}_{(2)}[v_1, v_2, \cdots]$ with $|v_i| = (2^i - 1)2$ we get

$$\pi_{*\rho} \mathrm{BP}_{\mathbb{R}} \cong \mathbb{Z}_{(2)}[\overline{v}_1, \overline{v}_2, \cdots] \quad \text{with} \quad |\overline{v}_i| = (2^i - 1)\rho.$$

Since $\Phi^{C_2} \mathrm{BP}_{\mathbb{R}} \simeq \mathrm{H}\mathbb{F}_2$, we deduce that all $\overline{v}_i$'s are a-torsion.

Theorem 4.2.2 (Slice Theorem, HHR)**.** The non-trivial slices of $\mathrm{BP}_{\mathbb{R}}$ are

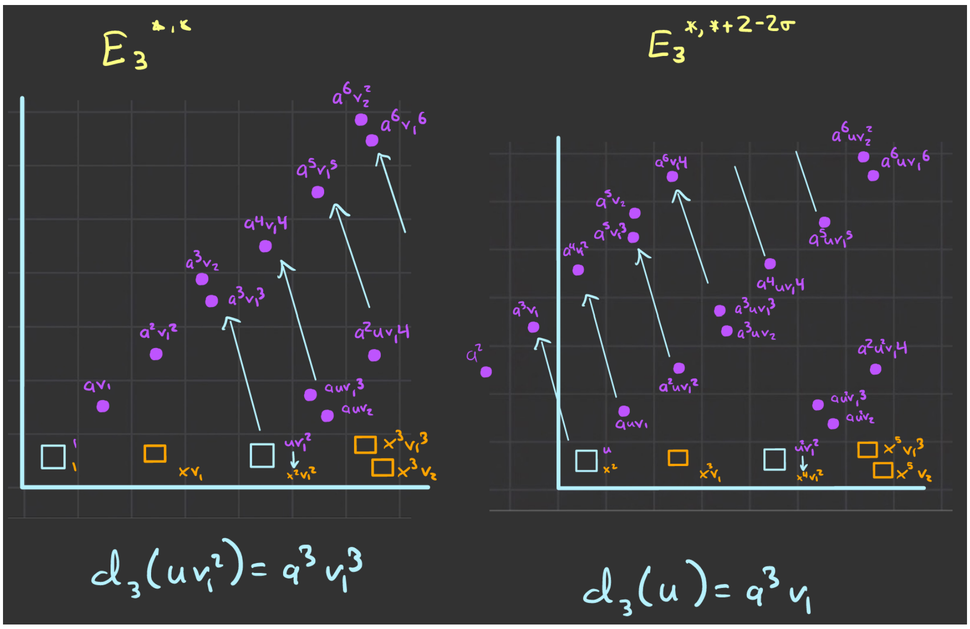
$$P^{2n}_{2n} \mathrm{BP}_{\mathbb{R}} \simeq \bigoplus_{\text{monomials in } (v_1, v_2, \cdots)} \Sigma^{n\rho} \underline{\mathbb{Z}}_{(2)}.$$

Example 4.2.3. We get

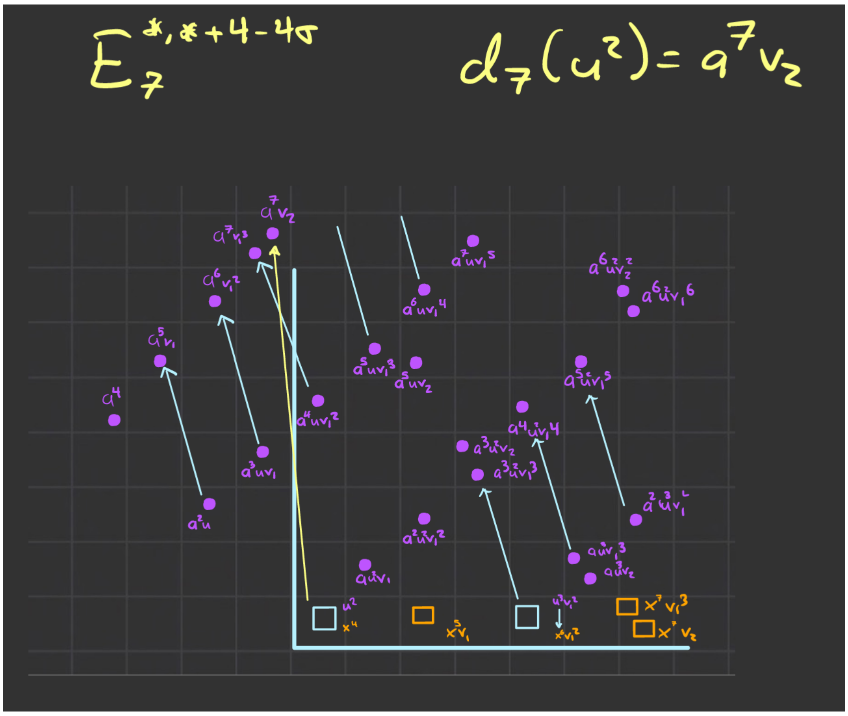
$$\begin{aligned} P^2_2 \mathrm{BP}_{\mathbb{R}} &\simeq \Sigma^{\rho} \mathrm{H}\underline{\mathbb{Z}}_{(2)}\{\overline{v}_1\} \simeq P^2_2 \mathrm{ku}_{\mathbb{R}} \\ P^4_4 \mathrm{BP}_{\mathbb{R}} &\simeq \Sigma^{2\rho} \mathrm{H}\underline{\mathbb{Z}}_{(2)}\{\overline{v}_1^2\} \simeq P^4_4 \mathrm{ku}_{\mathbb{R}}, \\ P^6_6 \mathrm{BP}_{\mathbb{R}} &\simeq \Sigma^{3\rho} \mathrm{H}\underline{\mathbb{Z}}_{(2)}\{\overline{v}_1^3, \overline{v}_2\}. \end{aligned}$$

Example 4.2.4.

(i) Here are some pictures of the E_3-page of the $RO(C_2)$-graded $\mathrm{SliceSS}(\mathrm{BP}_{\mathbb{R}})$.



(ii) Now the E_7-page.



Theorem 4.2.5 (Slice Differential Theorem, HHR)**.** Let $n \in \mathbb{N}$. Then, $d_{2^{n+2}-1}(u^{2^n}) = a^{2^{n+2}-1}\overline{v}_{n+1}$.

Example 4.2.6. So $d_3(u) = a^3\overline{v}_1$, $d_7(u^2) = a^7\overline{v}_2$, $d_{15}(u^4) = a^{15}\overline{v}_3$, $d_{31}(u^8) = a^{31}\overline{v}_4$.

Here is a (non-exhaustive) collection of elements that survive:

- a^k for $k \geq 0$.
- $\overline{v}_n$ for $n \geq 1$.
- $2u^k$ for $k \geq 0$.
- $u^2\overline{v}_1, u^4\overline{v}_2, \cdots$.

About the last list: Theoretically, $u^2\overline{v}_1$ hits $a^7\overline{v}_1\overline{v}_2$ via d_7 and $u^4\overline{v}_2$ by $a^{15}\overline{v}_2\overline{v}_3$ via d_{15}. But these elements already die earlier in the spectral sequence. For instance, the first element is killed by d_3.

4.3 Norms

The induction $\text{Ind}_H^G : \mathcal{S}_*^H \to \mathcal{S}_*^G$, $X \mapsto \text{Ind}_H^G X = G_+ \wedge_H X = \bigvee_{G/H} X$ is about addition. There is a multiplicative version, called the **norm** which is

$$N_H^G : \mathcal{S}_*^H \to \mathcal{S}_*^G,\ X \mapsto N_H^G X = \bigwedge_{G/H} X.$$

There is a spectral version of this, first defined by Greenlees–May, but made popular by Hill–Hopkins–Ravenel [HHR09]. For $C_2 \leq G$ we have a functor $N_{C_2}^G : \mathbf{Sp}^{C_2} \to \mathbf{Sp}^{C_2}$.

Definition 4.3.1. Let $G = C_{2^n}$. Then, $MU_{\mathbb{R}}^{((G))} = N_{C_2}^G MU_{\mathbb{R}}$ and $BP_{\mathbb{R}}^{((G))} = N_{C_2}^G BP_{\mathbb{R}}$.

By the slice theorem (Theorem 4.2.2) we can obtain $P^{2k}_{2k}\,\mathrm{MU}_{\mathbb{R}}^{((G))} \simeq \mathrm{Ind}_{C_2}^{G}\, S^{k\rho} \otimes \mathrm{H}_G\underline{\mathbb{Z}}$.

A word on [HHR09]. They define a spectrum

$$\mathrm{MU}_{\mathbb{R}}^{((C_8))}[D^{-1}] \quad \text{with} \quad D \in \pi_{19\rho}\,\mathrm{MU}_{\mathbb{R}}^{((C_8))}.$$

The formalism of slice spectral sequences was developed to understand this spectrum. One can show that this has a small gap in homotopy which one then propagates periodically by inverting this element D. Then, the strategy is to show that the higher Kervaire classes vanish here and would be detected if they were non-zero. Unfortunately, the periodicity is just large enough that the last Kervaire class in dimension 126 is not resolved with this strategy.

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