Slice Filtration and K-Theory

QI ZHU

23 October 2025

Abstract

Much of motivic homotopy theory follows the Grothendieck's dream of a category of motives and relatedly the realization of motivic cohomology. Beilinson and Lichtenstein conjectured the existence of motivic complexes realizing motivic cohomology and a number of desired properties.

To arrive at this story we will follow lectures notes from Bachmann [Bac21] and construct a motivic spectrum KGL representing it (homotopy) K-theory. Then, we give an axiomatic approach to slice filtrations and specialize to Voevodsky's slice filtration. This will allow us to give an ad hoc definition of the motivic cohomology spectrum $H\mathbb{Z}$. With it, we will then discuss a number of historical results about motivic cohomology which were catalystic to the development of motivic homotopy theory.

This is talk 2 given at the Motivic Homotopy Theory Seminar in Bonn, WiSe 2025/26.

Contents

0 Beilinson's Dream

We all have dreams but your dream is not relevant for this talk.[1] Beilison's dream takes the center stage.

Motivated by questions about the zeta function ζ as well as Grothendieck's vision of a category of motives, Beilinson and Lichtenbaum conjectured the existence of motivic cohomology in the 80s [BK25, Introduction]. It is lousely supposed to satisfy a number of desiderata, among others:

(i) It gives rise to an analog of Atiyah-Hirzebruch spectral sequence (2.9).

(ii) It is essentially described by higher Chow groups (3.7).

(iii) There should be a certain range of support (3.9).

(iv) There is a close relation to étale cohomology (3.11).

We will discuss all of those in this talk and therefore see some of the historic motivations for motivic homotopy theory.

1 Motivic K-Theory Spectrum

1.1 Thomason-Trobaugh K-Theory

Recall that for a ring A one can define its algebraic K-theory as $K(A) = \left(\mathbf{Proj}_A^{\mathrm{fg,core}}\right)^{\mathrm{gp}}$. As so often in algebraic geometry, we can extend this to (nice) schemes.

Theorem 1.1. Let S be a regular Noetherian scheme of finite dimension. Then, there exists a motivic space $K \in \mathbf{Spc}(S)$, the so-called **Thomason-Trobaugh K-theory** such that for every $\operatorname{Spec} A \in \mathbf{Sm}_S$ we have $K(\operatorname{Spec} A) \simeq K(A)$.

Proof Idea. One can make the assignment $F : \mathbf{Sm}_S^{\mathrm{op}} \to \mathcal{S}$, $X \mapsto K(\mathcal{O}_X(X))$ functorial. The Thomason-Trobaugh K-theory is $K = L_{\mathrm{mot}} F$.

It remains to show that $F \to L_{\mathrm{mot}} F = K$ is an equivalence on affines. By working with localization formulas – namely [Bac21, Theorem 2.21] and the sheafification formula – it suffices to show that F on is $\mathbb{A}^1$-invariant and Nisnevich-local on affine schemes. This translates to properties of algebraic K-theory, namely $\mathbb{A}^1$-invariance $K(A[t]) \simeq K(A)$ and a Nisnevich descent condition. These are non-trivial properties of the K-theory of regular rings [Bac21, Theorem 2.25]. $\square$

Recall $K(A) \simeq K_0(A) \times \mathrm{BGL}(A)^+$. This can also be extended to schemes.

Construction 1.2. There are presheaves

$$\mathrm{GL}_n : \mathbf{Sm}_S^{\mathrm{op}} \to \mathbf{Grp},\ X \mapsto \mathrm{GL}_n(\mathcal{O}_X(X)) \quad \text{and} \quad \mathrm{GL} = \operatorname*{colim}_n \mathrm{GL}_n\,.$$

Taking classifying spaces sectionwise yields $\mathrm{BGL} \in \mathbf{PSh}(\mathbf{Sm}_S)$.

Fact 1.3 ([Bac21, Theorem 2.28])**.** Let S be a regular Noetherian scheme of finite dimension. Then,

$$K \simeq L_{\mathrm{mot}}(\mathbb{Z} \times \mathrm{BGL}) \in \mathbf{Spc}(S).$$

[1] Sorry.

1.2 Algebraic K-Theory Spectrum

Recall from Talk 1 that $\mathcal{SH}(S) \simeq \lim \left(\cdots \xrightarrow{\Omega_{\mathbb{P}^1}} \mathbf{Spc}(S) \xrightarrow{\Omega_{\mathbb{P}^1}} \mathbf{Spc}(S) \right)$. We want to construct a motivic analog of KU, i.e. a motivic spectrum KGL representing algebraic K-theory. To do this we first construct the representing motivic spaces.

Construction 1.4. Let X be an S-scheme. Denote by $\mathbf{Vect}(X)$ the 1-category of vector bundles on X. Then,[2]

$$K(\mathbf{Vect}(X)) = \left(\mathbf{Vect}(X)^{\oplus,\mathrm{core}}\right)^{\mathrm{gp}}$$

is the **direct sum K-theory** of X.

Lemma 1.5. Let S be a regular Noetherian scheme of finite dimension. Then, $K \simeq L_{\mathrm{mot}}K(\mathbf{Vect}(-))$.

Proof. By the Serre-Swan theorem, there is a symmetric monoidal functor $\mathbf{Proj}^{\mathrm{fg}}_{\mathcal{O}_X(X)} \to \mathbf{Vect}(X)$ which is an equivalence on affines. In particular, this induces a map

$$K(\mathcal{O}_-(-)) \to K(\mathbf{Vect}(-))$$

in $\mathbf{PSh}(\mathbf{Sm}_S)$ which is an equivalence on affines. So this is a Zariski equivalence. Thus,

$$K \simeq L_{\mathrm{mot}}K(\mathcal{O}_-(-)) \simeq L_{\mathrm{mot}}L_{\mathrm{Zar}}K(\mathcal{O}_-(-)) \simeq L_{\mathrm{mot}}L_{\mathrm{Zar}}K(\mathbf{Vect}(-)) \simeq L_{\mathrm{mot}}K(\mathbf{Vect}(-))$$

where we use $L_{\mathrm{mot}}L_{\mathrm{Zar}} \simeq L_{\mathrm{mot}}$ which follows from the Nisnevich topology being finer than the Zariski topology. $\square$

Remark 1.6. The definition of K via the direct sum K-theory is more general than the Thomason-Trobaugh K-theory – it doesn't require these regularity conditions on S.

Observation 1.7. The construction of $K(\mathbf{Vect}(X))$ was naturally as a functor to $\mathbf{CGrp}$ which in particular forgets to $\mathcal{S}_*$. In other words, we can obtain natural basepoints via $0 \in K(\mathbf{Vect}(X))$, so we obtain lifts $K(\mathbf{Vect}(-)) \in \mathbf{PSh}(\mathbf{Sm}_S)_*$ and $L_{\mathrm{mot}}K(\mathbf{Vect}(-)) \in \mathbf{Spc}(S)_*$.

Construction 1.8. Consider the tautological line bundle $\gamma = \mathcal{O}_{\mathbb{P}^1}(-1) \in \mathbf{Vect}(\mathbb{P}^1)$. External tensor product of vector bundles yields a natural[3] additive functor $- \otimes \gamma : \mathbf{Vect}(X) \to \mathbf{Vect}(X \times \mathbb{P}^1)$ which induces a map of commutative monoids $\gamma : K(\mathbf{Vect}(X)) \to K(\mathbf{Vect}(X \times \mathbb{P}^1))$. Similarly, there is a map $1 : K(\mathbf{Vect}(X)) \to K(\mathbf{Vect}(X \times \mathbb{P}^1))$ for the trivial line bundle $1 \in \mathbf{Vect}(\mathbb{P}^1)$. Since $K(\mathbf{Vect}(X))$ is grouplike, we can form the difference $\gamma - 1$.

The following is a motivic version of Bott periodicity.

Theorem 1.9 (Motivic Bott Periodicity)**.**

(i) The map $\gamma - 1$ assembles into a map

$$\gamma - 1 : K(\mathbf{Vect}(-)) \to \Omega_{\mathbb{P}^1}K(\mathbf{Vect}(-))$$

in $\mathbf{PSh}(\mathbf{Sm}_S)_*$.

(ii) The induced map

$$\gamma - 1 : L_{\mathrm{mot}}K(\mathbf{Vect}(-)) \to L_{\mathrm{mot}}\Omega_{\mathbb{P}^1}K(\mathbf{Vect}(-)) \to \Omega_{\mathbb{P}^1}L_{\mathrm{mot}}K(\mathbf{Vect}(-))$$

is an equivalence.

[2]We perform everything in $\mathbf{Cat}_\infty$.
[3]Note that here naturality is still 1-categorical and hence can be checked by hand.

Proof.

(i) In 1.8 we constructed a map[4]

$$K(\mathbf{Vect}(-)) \to \Omega_{\mathbb{P}^1_+} K(\mathbf{Vect}(-))$$

in $\mathbf{PSh}(\mathbf{Sm}_S)_*$ which by adjunction corresponds to a map $\mathbb{P}^1_+ \otimes K(\mathbf{Vect}(-)) \to K(\mathbf{Vect}(-))$. We want to produce a map $\mathbb{P}^1 \otimes K(\mathbf{Vect}(-)) \to K(\mathbf{Vect}(-))$. Via the cofiber sequence $*_+ \to \mathbb{P}^1_+ \to \mathbb{P}^1$, we need to show that the composite

$$K(\mathbf{Vect}(-)) \simeq *_+ \otimes K(\mathbf{Vect}(-)) \longrightarrow \mathbb{P}^1_+ \otimes K(\mathbf{Vect}(-)) \longrightarrow K(\mathbf{Vect}(-))$$

is nullhomotopic. But this is induced by the restriction of $\gamma - 1$ to $* = S$ which is 0 since $\gamma|_S = 1$ is the trivial bundle. Adjoining again, we have successfully constructed a map $K(\mathbf{Vect}(-)) \to \Omega_{\mathbb{P}^1} K(\mathbf{Vect}(-))$.

(ii) We only discuss this in the case S is Noetherian, regular and of finite-dimensional, although this is true in general [Bac21, Footnote 12]. Then, by 1.5 this is the statement $K(X) \simeq K(X_+ \wedge \mathbb{P}^1)$ for Thomason-Trobaugh K-theory. This follows from the so-called *projective bundle formula* [Wei13, Theorem V.1.5].[5]

□

Mimicking the topological counterpart, we define:

Definition 1.10. The **(motivic) algebraic K-theory spectrum** is the object

$$\mathrm{KGL} = \mathrm{KGL}_S = ((K, K, \cdots); \gamma - 1 : K \to \Omega_{\mathbb{P}^1} K) \in \mathcal{SH}(S)$$

where we write $K = L_{\mathrm{mot}} K(\mathbf{Vect}(-))$ here for brevity.

Remark 1.11.

(i) So $\Omega^\infty \mathrm{KGL} \simeq K$ with which we have constructed a motivic spectrum representing K. In particular, this recovers algebraic K-theory in case S is regular, Noetherian and of finite dimension (1.5).

(ii) In general, KGL represents Weibel's homotopy K-theory KH, an $\mathbb{A}^1$-invariant approximation to K-theory. We need non $\mathbb{A}^1$-invariant K-theory to fix this defect and it will be a focus towards the end of this seminar.

Remark 1.12. Essentially since everything worked analogous to the topological counterpart one deduces that the complex Betti realization is $\mathrm{Be}_{\mathbb{C}}\, \mathrm{KGL}_{\mathbb{C}} \simeq \mathrm{KU}$ [Ban05, Lemma 4.23]. It turns out that $\mathrm{Be}_{\mathbb{R}}\, \mathrm{KGL} \simeq 0$ as opposed to $\mathrm{Be}_{\mathbb{R}}\, \mathrm{KGL} \simeq \mathrm{KO}$, answering a question from Thomas during the talk [Ban05, Lemma 4.24].

Corollary 1.13.

(i) Let $n \in \mathbb{Z}$, then $\Sigma^{2n,n}\, \mathrm{KGL} \simeq \mathrm{KGL}$.

(ii) Let S be regular, Noetherian and finite-dimensional. For $X \in \mathbf{Sm}_S$ we have

$$[\Sigma^{p,q} \Sigma^\infty_+ X, \mathrm{KGL}_S] \cong \begin{cases} K_{p-2q}(X) & p \geq 2q, \\ 0 & \text{else.} \end{cases}$$

[4] That evaluation on $- \times \mathbb{P}^1$ corresponds to $\Omega_{\mathbb{P}^1_+}$ follows via a Yoneda argument.

[5] I think you use that the cofiber sequence $\mathbb{P}^1 \to X \times \mathbb{P}^1 \to X_+ \wedge \mathbb{P}^1$ induces a fiber sequence on K-theory by the Yoneda Lemma. Then, $K_\bullet(X \times \mathbb{P}^1) = K_\bullet(\mathbb{P}^1_X) \cong K_\bullet(X)[z]/z^2$ by the projective bundle formula. This z part is killed by $K(\mathbb{P}^1)$.

Proof.

(i) We use $\mathcal{SH}(S) \simeq \lim\left(\cdots \xrightarrow{\Omega_{\mathbb{P}^1}} \mathbf{Spc}(S) \xrightarrow{\Omega_{\mathbb{P}^1}} \mathbf{Spc}(S)\right)$. By motivic Bott periodicity (1.9) both $\Sigma^{2n,n}\,\mathrm{KGL}$ and KGL are given by $(\cdots, K, K)$.

(ii) By Bott periodicity from (a), we can shift so far to assume $q = 0$. If $p \geq 0$, then we use adjunctions and the Yoneda Lemma to compute

$$[\Sigma^{p,0}\Sigma^\infty_+ X, \mathrm{KGL}_S] \cong [\Sigma^\infty(S^p \wedge X_+), \mathrm{KGL}_S] \cong [S^p \wedge X_+, K]_* \cong K_p(X).$$

On the other hand,

$$[\Sigma^{-p,0}\Sigma^\infty_+ X, \mathrm{KGL}_S] \cong [\Sigma^{p,p}\Sigma^\infty_+ X, \Sigma^{2p,p}\,\mathrm{KGL}_S] \cong [\mathbb{G}_m^{\wedge p} \wedge X_+, K]_*.$$

Now, there is a cofiber sequence

$$X_+ \longrightarrow (\mathbb{G}_m^{\times p} \times X)_+ \longrightarrow \mathbb{G}_m^{\wedge p} \wedge X_+$$

in $\mathbf{Spc}(S)_*$.[6] Now, let us only consider $p = 1$, for $p \geq 2$ we argue by induction. In that case, we obtain an exact sequence

$$K_1(\mathbb{G}_m \times X) \twoheadrightarrow K_1(X) \longrightarrow [\mathbb{G}_m \wedge X_+, K]_* \longrightarrow K_0(\mathbb{G}_m \times X) \xrightarrow{\sim} K_0(X)$$

where the first arrow is surjective and the last arrow is an isomorphism by *Bass' fundamental theorem*. Thus, the middle term must be 0 by exactness.

□

In particular, for $X = S$ we have $\pi_{p,q}\,\mathrm{KGL}_S \cong \begin{cases} K_{p-2q}(S) & p \geq 2q, \\ 0 & \text{else.} \end{cases}$

2 The Slice Filtration

We know wish to construct a motivic version of the Whitehead filtration.

2.1 Axiomatic Approach to Slice Filtrations

Due to the bigrading on motivic spectra, there are multiple imaginable filtrations that generalize the Whitehead filtration. As such, we will begin by giving a general procedure for such constructions. Let me introduce the following ad hoc name. Drew calls this *slice filtration* [Hea19, Definition 2.1].

Definition 2.1 ([GRSOsr12, Section 2.1]). Let $\mathscr{C} \in \mathbf{CAlg}(\mathbf{Pr}^L_{\mathrm{st}})$, compactly generated by a set of objects $\mathcal{T}$. Let $\{\mathscr{C}_i\}_{i\in\mathbb{Z}}$ be a family of full subcategories of $\mathscr{C}$. Then, $\{\mathscr{C}_i\}_{i\in\mathbb{Z}}$ is a **slice setup** of $\mathscr{C}$ if the following conditions are satisfied.

(i) For $i \in \mathbb{Z}$ we have $\mathscr{C}_{i+1} \subseteq \mathscr{C}_i$.

(ii) Each $\mathscr{C}_i$ is generated under colimits and extensions by a set of compact objects $\mathcal{K}_i$.

(iii) We have $\mathbb{1} \in \mathscr{C}_0$.

[6] If we omit the added basepoints of the first two terms, then this is a cofiber sequence in $\mathbf{Spc}(S)$.

(iv) Each $t \in \mathcal{T}$ is contained in some $\mathscr{C}_i$.

(v) If $c_0 \in \mathscr{C}_0$ and $c_n \in \mathscr{C}_n$, then $c_0 \otimes c_n \in \mathscr{C}_n$.

Observation 2.2. Let $i_q : \mathscr{C}_q \hookrightarrow \mathscr{C}$. Since it is closed under colimits, it admits a right adjoint $r_q : \mathscr{C} \to \mathscr{C}_q$. We put $f_q = i_q \circ r_q : \mathscr{C} \to \mathscr{C}$.

Definition 2.3. Let $\mathscr{C} \in \mathbf{CAlg}(\mathbf{Pr}^L_{\mathrm{st}})$ be equipped with a slice setup and $c \in \mathscr{C}$. We wish to construct a map $f_{q+1}c \to f_q c$. By adjunction, it corresponds to a map $f_{q+1}c \to c$ which can be taken as the counit of $i_{q+1} \dashv r_{q+1}$. Then, we obtain a filtration

$$\begin{tikzcd} \cdots \arrow[r] & f_1 c \arrow[r] \arrow[d] & f_0 c \arrow[r] \arrow[d] & f_{-1} c \arrow[r] \arrow[d] & \cdots \arrow[r] & c \\ & s_1 c & s_0 c & s_{-1} c & & \end{tikzcd}$$

is the **slice tower** with **slices** $s_n c = \mathrm{cofib}(f_{n+1}c \to f_n c)$.

In the definition of slice setups (2.1) the conditions (i), (ii) are to get the filtration running, condition (iii) is in some sense a normalization condition, condition (iv) ensures that the induced slice filtrations are exhaustive and condition (v) gives some good compatibilities with multiplicative structures [Hea19, Section 2].

Example 2.4.

(i) For $\mathscr{C} = \mathbf{Sp}$ with $\mathcal{K}_q = \{\mathbb{S}^m : m \geq q\}$ leading to $\mathscr{C}_q \simeq \mathbf{Sp}_{\geq q}$ yields the classical Whitehead tower in $\mathbf{Sp}$.

(ii) Let $\mathcal{K}_q = \{\Sigma^{a,b}\Sigma^\infty_+ X : X \in \mathbf{Sm}_S,\ b \geq q\} \subseteq \mathcal{SH}(S)$. We denote by $\Sigma^{q,q}\mathcal{SH}(S)^{\mathrm{eff}} \subseteq \mathcal{SH}(S)$ be the localizing subcategory generated by $\mathcal{K}_q$. Then, the filtration

$$\cdots \hookrightarrow \Sigma^{q+1,q+1}\mathcal{SH}(S)^{\mathrm{eff}} \hookrightarrow \Sigma^{q,q}\mathcal{SH}(S)^{\mathrm{eff}} \hookrightarrow \cdots$$

defines a slice setup. Associated to it is **(Voevodsky's) slice filtration**.

By construction, $\Sigma^{q,q}\mathcal{SH}(S)^{\mathrm{eff}} \simeq \Sigma^{q,q}(\Sigma^{0,0}\mathcal{SH}(S)^{\mathrm{eff}})$. We also write $\mathcal{SH}(S)^{\mathrm{eff}} = \Sigma^{0,0}\mathcal{SH}(S)^{\mathrm{eff}}$.

Remark 2.5. For the motivic slice filtration you can check $f_q \simeq \Sigma^{q,q} f_0 \Sigma^{-q,-q} : \mathcal{SH}(S) \to \mathcal{SH}(S)$ explicitly, following Bachmann [Bac21, Exercise 3.4].

Remark 2.6. This is not the focus of the talk but certainly the axiomatic construction leads to numerous additional interesting filtration.

(i) Taking $\mathcal{K}_q = \{\Sigma^{q+i,i}\Sigma^\infty_+ X : X \in \mathbf{Sm}_S,\ i \in \mathbb{Z}\} \subseteq \mathcal{SH}(S)$ yields the **homotopy t-structure**.

(ii) Taking $\mathcal{K}_q = \{\Sigma^{2a,a}\Sigma^\infty_+ X : X \in \mathbf{Sm}_S,\ a \geq q\} \subseteq \mathcal{SH}(S)$ we obtain the **very effective slice filtration**. There are also cellular versions of the slice and very effective slice filtrations [Hea19, Section 4].

(iii) In $\mathbf{Sp}^{C_2}$ we define the following slice cells:

- $S^{2q,q\sigma}$ of dimension $2q$,
- $S^{2q-1,q\sigma}$ of dimension $2q-1$,
- $S^q \otimes (C_2)_+$ of dimension q.

Take $P^q\mathbf{Sp}^{C_2} \subseteq \mathbf{Sp}^{C_2}$ be the full subcategory generated under extensions and colimits of slice cells of dimension $\geq q$. This gives rise to the **Hill-Hopkins-Ravenel slice filtration** for $\mathbf{Sp}^{C_2}$. Ignoring cells of the second form gives Ullman's **regular slice filtration**. See [Hea19, Section 5.2]. There is also a version for more general G – on the other hand, C_2 seems fitting in the context of motivic homotopy theory which was also the purpose of [Hea19].

2.2 Motivic Examples

Definition 2.7.

(i) The **effective algebraic K-theory spectrum** is the motivic spectrum $\mathrm{kgl} = f_0 \,\mathrm{KGL}$.

(ii) The **motivic cohomology spectrum** is the motivic spectrum $H\mathbb{Z} = s_0 \,\mathrm{KGL}$.

Remark 2.8. The first definition of motivic cohomology is due to Voevodsky in the mid 90s as certain derived functors of so-called *motivic complexes* of sheaves $\mathbb{Z}(q)$, realizing Beilinson's dream. Afterwards, one may ask for alternative descriptions of this object. It was Voevodsky's first conjecture about his slice filtration [Voe02b, Conjecture 1] that $s_0 \,\mathrm{KGL}$ is one such candidate. Here are some other ways of producing $H\mathbb{Z}$ over perfect fields.

(i) Classically singular chains defines a right adjoint $i^* : \mathbf{Sp} \to \mathbf{Ch}(\mathbf{Ab})$ and we can define $H\mathbb{Z} = i_* i^* \mathbb{S}$. Motivically, one can mimic this construction by replacing $\mathbf{Sp}$ with $\mathcal{SH}(k)$ and $\mathbf{Ch}(\mathbf{Ab})$ by the stable ∞-category of motives $\mathbf{DM}(k)$.

(ii) Classically, one can construct $H\mathbb{Z}$ as an infinite loop space via Eilenberg-MacLane spaces which can be viewed as $\mathrm{SP}^\infty(S^n)$ via the Dold-Thom theorem. This also be realized motivically over characteristic 0, i.e. one writes out a sequential $\mathbb{P}^1$-spectrum via Eilenberg MacLane spaces realized through symmetric products.

More naively attempting to take Eilenberg-MacLane objects in the ∞-topos $\mathbf{Sh}_{\mathrm{Nis}}(\mathbf{Sm}_k)$ and then applying $L_{\mathbb{A}^1}$ does not work – this only gives an S^1-spectrum and not an $\mathbb{P}^1$-spectrum.

(iii) Classically, $H\mathbb{Z} \simeq \tau_{\leq 0}\mathbb{S} \simeq \pi_0 \mathbb{S}$. Motivically, $H\mathbb{Z} \simeq s_0 \mathbb{1}$ is one of Voevodsky's original conjectures about his slice filtration which was shown by Levine. In fact, this combined with another conjecture, also proved by Levine, yields the first conjecture [Voe02c, Lev08].

Remark 2.9. The spectral associated to the slice filtration of KGL is a motivic version of the Atiyah-Hirzebruch spectral sequence [BL99, Voe02c], namely for $X \in \mathbf{Sm}_k$ there is a strongly convergent spectral sequence

$$E_2^{p,q} = H\mathbb{Z}^{p-q,-q}(X) \Rightarrow K_{-p-q}(X).$$

This is the one of the starting points of Hahn-Raksit-Wilson's even filtration and the related motivic filtrations [HRW24].

Lemma 2.10. There are equivalences $f_n \,\mathrm{KGL} \simeq \Sigma^{2n,n} \,\mathrm{kgl}$ and $s_n \,\mathrm{KGL} \simeq \Sigma^{2n,n} H\mathbb{Z}$.

Proof. Via 2.5 we compute

$$\begin{aligned}
f_n \,\mathrm{KGL} &= \Sigma^{n,n} f_0 \Sigma^{-n,-n} \,\mathrm{KGL} \\
&\simeq \Sigma^{2n,n} \Sigma^{-n,0} f_0 \Sigma^{-n,-n} \,\mathrm{KGL} \\
&\simeq \Sigma^{2n,n} f_0 \Sigma^{-n,0} \Sigma^{-n,-n} \,\mathrm{KGL} \\
&\simeq \Sigma^{2n,n} f_0 \,\mathrm{KGL} \\
&= \Sigma^{2n,n} \,\mathrm{kgl}\,.
\end{aligned}$$

and

$$\begin{aligned}
s_n \,\mathrm{KGL} &= \mathrm{cofib}(f_{n+1} \,\mathrm{KGL} \to f_n \,\mathrm{KGL}) \\
&\simeq \mathrm{cofib}(\Sigma^{2(n+1),n+1} \,\mathrm{kgl} \to \Sigma^{2n,n} \,\mathrm{kgl}) \\
&\simeq \Sigma^{2n,n} \,\mathrm{cofib}(\Sigma^{2,1} \,\mathrm{kgl} \to \mathrm{kgl}) \\
&\simeq \Sigma^{2n,n} \,\mathrm{cofib}(f_1 \,\mathrm{KGL} \to f_0 \,\mathrm{KGL}) \\
&= \Sigma^{2n,n} H\mathbb{Z}.
\end{aligned}$$

□

3 Motivic Cohomology & Historic Theorems

3.1 Motivic Cohomology Groups

Much of motivic homotopy theory was developed to understand motivic cohomology. Let us introduce some notation before discussing a number of results in the field.

Definition 3.1. Let $E \in \mathcal{SH}(S)$ and $X \in \mathbf{Sm}_S$, then $E^{p,q}(X) = [\Sigma^\infty_+ X, \Sigma^{p,q} E]$ is the **bigraded cohomology theory** represented by E.

Remark 3.2. So $\pi_{p,q} E \cong [\Sigma^{p,q}\Sigma^\infty_+ S, E] \cong E^{-p,-q}(S)$.

Definition 3.3. The bigraded cohomology theory represented by $H\mathbb{Z}$ is called **motivic cohomology** and is denoted by

$$H^{p,q}(X) = H^{p,q}(X;\mathbb{Z}) = H^p(X;\mathbb{Z}(q)) = H\mathbb{Z}^{p,q}(X).$$

The cohomology theory associated to $H\mathbb{Z}/n$ is also called motivic cohomology.

3.2 Higher Chow Groups

We first construct an algebro-geometric version of singular homology.

Construction 3.4 (Bloch)**.** Let $X \in \mathbf{Sm}_S$.

(i) We write $Z^d(X) = \mathbb{Z}\{x \in X : \operatorname{codim}(\overline{\{x\}} \subseteq X) = d \iff \dim \mathcal{O}_{X,x} = d\}$.

(ii) If $i : Y \hookrightarrow X$ is a closed immersion, then $c = \sum_n a_n x_n \in Z^d(X)$ is in **good position with respect to i** if the components of $Y \cap \overline{\{x_n\}}$ have codimension $\geq d$ on Y for every n. We write $Z^d(X)_i \subseteq Z^d(X)$ for such cycles. One can construct a pullback map $i^* : Z^d(X)_i \to Z^d(Y)$ [MVW06, Definition 17A.6].

(iii) Let $z^d(X,n) \subseteq Z^d(X \times \Delta^n)$ consists of those cycles in good position with respect to all faces $X \times \Delta^i \subseteq X \times \Delta^n$. Then, we put

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i^* : z^d(X,n) \to z^d(X,n-1).$$

This yields a chain complex and we write $\mathrm{CH}^d(X,n) = H_n(z^d(X,\bullet),\partial)$ for the **higher Chow groups**.

Observation 3.5. Flat maps preserve codimensions of subschemes [Bac21, Remark 4.13], so it induces pullback maps on z^d which descends to pullbacks of higher Chow groups CH^d. In other words, $\mathrm{CH}^d(-,n)$ is contravariantly functorial in flat maps.

Remark 3.6. This generalizes classical Chow groups $\mathrm{CH}^d(X) \cong \mathrm{CH}^d(X,0)$ [Bac21, Example 4.11].

There are certainly many exciting things to be said about CH^d [Bac21, Section 4.3, 4.4] but we will focus on the connection to motivic cohomology.

Theorem 3.7 ([Voe02a])**.** Let $X \in \mathbf{Sm}_k$ and $p,q \in \mathbb{Z}$. Then, there are natural isomorphisms

$$H^{p,q}(X) \cong \mathrm{CH}^q(X, 2q-p).$$

This paper [Voe02a] is a 5-page paper with 100 citations!

Remark 3.8. Besides connecting two seemingly disjoint objects, we can extract many interesting consequences.

(i) We have argued that $\mathrm{CH}^d(-,n)$ is functorial in flat maps (3.5(ii)) but motivic cohomology $H^{p,q}(-)$ is functorial in all maps of schemes, so this functoriality transfers to $\mathrm{CH}^d(-,n)$.

(ii) Since $H^{p,q}(-)$ is represented by a motivic spectrum, we deduce that $\mathrm{CH}^d(-,n)$ is $\mathbb{A}^1$-invariant and satisfies Nisnevich descent.

(iii) By construction, $\mathrm{CH}^d(X,n) = 0$ for $n < 0$. Thus, $H^{p,q}(X) \cong \mathrm{CH}^q(X, 2q-p) = 0$ for $p > 2q$.

This result also allows us to compute weight 0 and weight 1 motivic cohomology after further higher Chow group computations [Bac21, Exercise 4.2, Theorem 4.14]. One obtains

$$H^{\bullet,0}(X) \cong \mathbb{Z}^{\pi_0 X}[0] \quad \text{and} \quad H^{p,1}(X) \cong \begin{cases} \mathrm{Pic}(X) & p = 2, \\ \mathcal{O}_X(X)^\times & p = 1, \\ 0 & \text{else.} \end{cases}$$

Conjecture 3.9 (Beilinson-Soulé Vanishing Conjecture)**.** Is $H^{p,q}(X) \cong 0$ for $p < 0$?

3.3 Bloch-Kato Conjecture

Let ℓ be an integer invertible in k. The Kummer exact sequence yields a connecting homomorphism $\partial : k^\times \to H^1_{\text{ét}}(k, \mu_\ell)$. Very briefly, via multiplicativity of étale cohomology, the definition of Milnor K-theory (with Artin reciprocity) and ℓ-torsion of $H^\bullet_{\text{ét}}(k, \mu_\ell^{\otimes \bullet})$, this induces a map

$$\begin{tikzcd}
k^\times \otimes \cdots \otimes k^\times \arrow[r, "\partial^n"] \arrow[d] & H^n_{\text{ét}}(k, \mu_\ell^{\otimes n}) \\
K^M_n(k) \arrow[ur, dashed, "\partial^n"] \arrow[d] & \\
K^M_n(k)/\ell \arrow[uur, dotted, "\partial_n"'] &
\end{tikzcd}$$

the so-called **Galois symbol** or **norm residue map**.

Theorem 3.10 (Norm Residue Theorem/(Motivic) Bloch-Kato Conjecture)**.** Let ℓ be an integer invertible in k. The map $\partial_n : K^M_n(k)/\ell \to H^n_{\text{ét}}(k, \mu_\ell^{\otimes n})$ is an isomorphism.

For $\ell = 2$ this was first conjectured by Milnor and as such is the *Milnor conjecture*. For $n = 2$ this is the *Merkurjev-Suslin theorem*, as this case was first proven by them and the first major advance in the resolution of this theorem. Voevodsky first proved the Milnor conjecture which earned him a fields medal and he later went on to prove the entire theorem with ideas from Rost.

Theorem 3.11 (Beilinson-Lichtenbaum Conjecture, Rost-Voevodsky)**.** Let $X \in \mathbf{Sm}_k$ and $\ell \in \mathbb{Z}$ be invertible in k. Then, $H^{p,q}(X, \mathbb{Z}/\ell) \cong H^p_{\text{ét}}(X, \mu_\ell^{\otimes q})$ for $p \leq q$.

This implies the norm residue theorem. Indeed, we first mention:

Theorem 3.12 (Nesterenko-Suslin '90, Totaro '92)**.** We have $\mathrm{CH}^d(k,n) \cong \begin{cases} 0 & n < d, \\ K^M_d(k) & n = d. \end{cases}$

So combining this result (3.12) with Levine's comparison of $H^{p,q}$ and CH^d (see 3.7) we see that the Milnor K-group term from the norm residue theorem (3.10) is identified with the motivic cohomology term from the Beilinson-Lichtenbaum conjecture (3.11).

Combining the Bloch-Kato conjecture with the motivic Atiyah-Hirzebruch spectral sequence also gave the resolution of the Quillen-Lichtenbaum conjecture which related étale cohomology to algebraic K-theory.

The resolution of this conjecture was a huge leap in the development of motivic homotopy theory. It required motivic versions of Spanier-Whitehead duality and the Steenrod algebra. Especially the latter is a focus point of this seminar.

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