Motivic Homotopy Theory

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Abstract

These are my (live) TeX'd notes for the motivic homotopy theory seminar in Bonn, WiSe 2025/26. The abstract is:

The goal of motivic homotopy theory, as introduced by Morel and Voevodsky, is to bring homotopical techniques into the world of algebraic geometry. The fundamental idea is to replace manifolds by smooth schemes over a base, so that the affine line $\mathbb{A}^1$ plays the role of the interval in usual homotopy theory. We aim to give the participant a feel for this category by first discussing Hoyois' proof of the Hopkins–Morel isomorphism. This passes through several motivic versions of fundamental homotopical constructions such as the identification of the Steenrod algebra for mod p cohomology and the Landweber exact functor theorem, and provides a strong connection between algebraic cobordism and motivic cohomology[1]. In the second half of the seminar[2], we discuss a more recent take on the theory of motivic spectra that in fact does away with the $\mathbb{A}^1$-homotopy invariance entirely. This compromise is motivated by compatibility with algebraic K-theory, and we will see how to set up a motivic analogue of Snaith's theorem that provides a universal property for algebraic K-theory over all base schemes, while passing through a discussion of orientations in this new setting.

My notation and language is not always consistent with the speakers' choices. I also occassionally added some parts which were not included in the actual talks; such parts will always be indicated by a star like Lemma*.

Feel free to send me feedback. :-)

Contents

[1] This result has received renewed attention in recent years: over a base field of positive characteritic p we only understand the motivic Steenrod algebra (and therefore also algebraic cobordism) after inverting p. It is not known how to complete this picture to the characteristic (see [AE25, CF25] for recent innovations in this direction, and cf. the second half of the syllabus for the refined context in which op. cit. plays out), but it should be the foundational computation for setting up the hypothetical prismatic stable homotopy category.

[2] We closely follow these lecture notes and strongly recommend that the speaker use these as a main reference for the structure of the talk due to the fact that some results are improved and re-proved between references.

1 Setting Up (Fabio Neugebauer)

Let S be a scheme, the base scheme. It'll be useful to assume some properties like Noetherianity, finite-dimensionality, etc. throughout the talk, so let us just already do it here. Let $\mathbf{Sm}_S$ denote the category of smooth schemes of finite type over S. The smoothness is particularly relevant for the purity theorem but one can also arrive at the same result by changing not only $\mathbf{Sm}_S$ but also the Grothendieck topology that we will introduce. TALK 1 16.10.2025

1.1 Setup of Motivic Spaces

Let's compare motivic homotopy theory to classical homotopy theory.

$$\mathbf{Mfld} \xmapsto{\text{coherently contract } \mathbb{R}^1} \mathcal{S} \xmapsto{\text{invert } S^1} \mathbf{Sp}$$

$$\mathbf{Sm}_S \xmapsto[\text{coherently contract } \mathbb{A}^1_S]{} \mathbf{Spc}(S) \xmapsto[\text{invert } \mathbb{P}^1]{} \mathcal{SH}(S)$$

Let's make this precise. We wish to define $\mathbf{Spc}(S) = L_{\mathbb{A}^1}\mathbf{Sh}_?(\mathbf{Sm}_S)$ as the ∞-category of motivic spaces.

We need to take sheaves because $\mathbf{Sm}_S$ behaves too badly. What Grothendieck topology do we take? One could try the Zariski topology but this has too few covers. One could try the étale topology which has enough geometry but infinite cohomological dimension. There is something in between:

$$\text{Zar} < \text{Nisnevich} < \text{étale}$$

In the Nisnevich topology the upshot is that descent is essentially some version of excision. We will not formally define the Nisnevich topology but will state a characterization of Nisnevich sheaves:

Theorem 1.1. A presheaf $F : \mathbf{Sm}_S^{\mathrm{op}} \to \mathcal{S}$ is a **Nisnevich sheaf** if and only if:

(i) $F(\emptyset) \simeq *$,

(ii) Let

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V \arrow[d, "p"] \\ U \arrow[r, "i"'] & X \end{tikzcd}$$

be a **Nisnevich square** in $\mathbf{Sm}_S$, i.e. it is cartesian, i is an open immersion, p is étale and $p^{-1}(X - U) \to X - U$ is an isomorphism. Then, the square

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(V) \arrow[d] \\ F(U) \arrow[r] & F(W) \end{tikzcd}$$

is a pullback square.

Corollary 1.2. The Yoneda embedding $\mathbf{Sm}_S \hookrightarrow \mathbf{Sh}^{\mathrm{Nis}}(\mathbf{Sm}_S)$ sends Nisnevich squares to pushouts.

Proof. By the Yoneda Lemma the pullback square in 1.1(ii) becomes

$$\begin{tikzcd}
\mathrm{Map}_{\mathbf{Sh}(\mathbf{Sm}_S)}(よX, F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Map}_{\mathbf{Sh}(\mathbf{Sm}_S)}(よV, F) \arrow[d] \\
\mathrm{Map}_{\mathbf{Sh}(\mathbf{Sm}_S)}(よU, F) \arrow[r] & \mathrm{Map}_{\mathbf{Sh}(\mathbf{Sm}_S)}(よW, F)
\end{tikzcd}$$

for all $F \in \mathbf{Sh}(\mathbf{Sm}_S)$. Thus, $よX \simeq よU \amalg_{よW} よV$ as desired. □

Definition 1.3. A presheaf $F \in \mathbf{PSh}(\mathbf{Sm}_S)$ is called **$\mathbb{A}^1$-invariant** if for all $X \in \mathbf{Sm}_S$ the map

$$\mathrm{pr}^* : F(X) \to F(X \times \mathbb{A}^1)$$

is an equivalence.

Definition 1.4. We denote by $\mathbf{Spc}(S) \subseteq \mathbf{PSh}(\mathbf{Sm}_S)$ is the full subcategory of $\mathbb{A}^1$-invariant Nisnevich sheaves. This is the ∞-category of **motivic spaces**.

Proposition 1.5. The inclusion $\mathbf{Spc}(S) \to \mathbf{PSh}(\mathbf{Sm}_S)$ preserves filtered colimits and admits a finite product-preserving left adjoint

$$L_{\mathrm{mot}} = \operatorname*{colim}_n (L_{\mathrm{Nis}} \to L_{\mathbb{A}^1} L_{\mathrm{Nis}} \to \cdots)$$

with $L_{\mathbb{A}^1} F \simeq \mathrm{colim}_{[n] \in \Delta^{\mathrm{op}}} F(X \times \Delta^n)$.

Proof. These exist for formal reasons but you need the explicit formula to prove the finite product-preservation. Essentially, the key step is to use that sifted colimits commute with finite limits in $\mathcal{S}$. □

Remark 1.6. This functor L_{mot} is not left-exact. In fact, $\mathbf{Spc}(S)$ is not an ∞-topos.

Definition 1.7. A map $H : X \times \mathbb{A}^1 \to Y$ in $\mathbf{Sm}_S$ is called **$\mathbb{A}^1$-homotopy**.

For any $a : *_{\mathbf{Spc}(S)} \simeq S \to \mathbb{A}^1$ we get $H_a : X \to X \times \mathbb{A}^1 \xrightarrow{H} Y$ and for all $a, b \in \mathbb{A}^1(S)$ we have $H_a \simeq H_b$ in $\mathbf{Spc}(S)$. The first map is always an equivalence as a section of $\mathrm{pr} : X \times \mathbb{A}^1 \to X$, in particular it always has the same inverse, so it is always the same map (up to equivalence).

1.2 Motivic Spheres

Definition 1.8. We write $\mathbb{G}_m$ for the pointed S-scheme $(\mathbb{A}^1 - \{0\}, 1)$.

Observation 1.9. The squares

$$\begin{tikzcd}
\mathbb{G}_m \times \mathbb{G}_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{G}_m \times \mathbb{A}^1 \arrow[d] \\
\mathbb{A}^1 \times \mathbb{G}_m \arrow[r] & \mathbb{A}^2 - \{0\}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{G}_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{A}^1 \arrow[d] \\
\mathbb{A}^1 \arrow[r] & \mathbb{P}^1
\end{tikzcd}$$

are Nisnevich squares. So we obtain pushout squares

$$\begin{tikzcd}
\mathbb{G}_m \times \mathbb{G}_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{G}_m \arrow[d] \\
\mathbb{G}_m \arrow[r] & \mathbb{A}^2 - \{0\}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{G}_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & * \arrow[d] \\
* \arrow[r] & \mathbb{P}^1
\end{tikzcd}$$

in $\mathbf{Spc}(S)$ after contracting $\mathbb{A}^1$. We deduce $\mathbb{A}^2 - \{0\} \simeq \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m)$ and $\mathbb{P}^1 \simeq \Sigma\mathbb{G}_m$ in $\mathbf{Spc}(S)_*$. For the first one, you still need to play around a little bit.

Definition 1.10. For integers $d \geq j \geq 0$ the **motivic sphere** are $S^{d,j} = S^{d-j} \wedge \mathbb{G}_m^j \in \mathbf{Spc}(S)_*$.

So $\mathbb{P}^1 \simeq S^{2,1}$ and $\mathbb{A}^2 - \{0\} \simeq S^{3,2}$ by 1.9.

Proposition 1.11. There is an equivalence $S^{2n-1,n} \simeq \mathbb{A}^n - \{0\}$.

In particular,

$$S^{2n,n} \simeq S^1 \wedge S^{2n-1,n} \simeq \mathrm{cofib}(\mathbb{A}^n - \{0\} \to \mathbb{A}^n) = \mathbb{A}^n/(\mathbb{A}^n - \{0\}),$$

i.e. contract the boundary of a disc which should be a sphere.

1.3 Base Change

Let $f : T \to S$ be a map of schemes. We get a functor $\mathbf{Sm}_S \to \mathbf{Sm}_T$, $X \mapsto T \times_S X$ which gives rise to an adjunction

$$\mathbf{PSh}(\mathbf{Sm}_S) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathbf{PSh}(\mathbf{Sm}_T)$$

by left Kan extension. This passes to motivic spaces:

$$\mathbf{Spc}(S) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathbf{Spc}(T)$$

such that

$$\begin{tikzcd} \mathbf{Spc}(S) \arrow[r, "f^*"] & \mathbf{Spc}(T) \\ \mathbf{PSh}(\mathbf{Sm}_S) \arrow[u, "L_{\mathrm{mot}}"] \arrow[r, "f^*"'] & \mathbf{PSh}(\mathbf{Sm}_T) \arrow[u, "L_{\mathrm{mot}}"'] \end{tikzcd}$$

commutes (which is checked on right adjoints).

Remark 1.12. The functor f^* preserves finite products.

Proof. It preserves finite products on the scheme level and so also on presheaves by abstract nonsense [Lur09, Proposition 6.1.5.2]. Since the motivic localizations preserve products, this also descends to motivic spaces. □

Proposition 1.13 (Nisnevich Separation)**.** Let $\{f_i : U_i \to S\}_i$ be a Nisnevich cover. Then, the family $\{f_i^* : \mathbf{Spc}(S) \to \mathbf{Spc}(U_i)\}_i$ is conservative.

1.4 Motivic Thom Spaces

The concept of Thom spaces allows you to study vector bundles despite having contracted $\mathbb{A}^1$.

Definition 1.14.

(i) Let $\mathbf{Vect}_S$ denote the 1-category of vector bundles over S and isomorphisms.

(ii) The **Thom space functor** is $\mathrm{Th} : \mathbf{Vect}_S \to \mathbf{Spc}(S)_*$, $E \mapsto E/(E - \{0\})$.

Or rather: $\mathrm{Th}(E) \simeq E/(E - X)$.

Proposition 1.15. The functor Th is symmetric monoidal.

Proof. We begin by factoring Th into lax symmetric monoidal functors.[3]

$$\begin{tikzcd} \mathbf{Vect}_S \arrow[r] & \mathrm{Ar}(\mathbf{Sm}_S)^{\mathbf{Day}} \arrow[r] & \mathrm{Ar}(\mathbf{Spc}(S))^{\mathbf{Day}} \arrow[r, "\mathrm{cofib}"] & \mathbf{Spc}(S)_* \\ E \arrow[r, mapsto] & (E - \{0\} \to E) \end{tikzcd}$$

Now symmetric monoidality is a property, so it suffices to check for $E, E' \in \mathbf{Vect}_S$ that the map

$$\mathrm{Th}(E) \wedge \mathrm{Th}(E') \to \mathrm{Th}(E \wedge E')$$

is an equivalence. WLOG, E, E' are trivial by Nisnevich separation in which case we get

$$(\mathbb{A}^1/\mathbb{A}^1 - \{0\})^{\wedge n} \simeq (S^{2,1})^{\wedge n} \simeq S^{2n,n} \simeq \mathbb{A}^n/\mathbb{A}^n - \{0\}.$$

□

1.5 Motivic Spectra

Definition 1.16. The category $(\mathcal{SH}(S), \otimes)$ is the initial presentably symmetric monoidal category under $\mathbf{Spc}(S)_*$, i.e. it comes with a functor $\Sigma^\infty : \mathbf{Spc}(S)_* \to \mathcal{SH}(S)$, on which tensoring with $\mathbb{S}^{d,j} = \Sigma^\infty S^{d,j}$ defines an equivalence $\Sigma^{d,j} = \mathbb{S}^{d,j} \otimes - : \mathcal{SH}(S) \to \mathcal{SH}(S)$.

Existence on $\mathbf{CAlg}(\mathbf{Pr}^L)_{\mathscr{C}/}$ with inverting some compact objects is a formal thing by Robalo [Rob13].

Construction 1.17. Let $f : T \to S$ be a map of schemes. Then, we obtain a symmetric monoidal left adjoint sitting in the square

$$\begin{tikzcd} \mathbf{Spc}(S)_* \arrow[r, "f^*"] \arrow[d, "\Sigma^\infty"'] & \mathbf{Spc}(T) \arrow[d, "\Sigma^\infty"] \\ \mathcal{SH}(S) \arrow[r, dashed, "\exists! f^*"'] & \mathcal{SH}(T) \end{tikzcd}$$

by definition of $\mathcal{SH}(S)$.

Remark 1.18.

(i) Since $\Sigma^{1,0} = \Sigma$ we get that $\mathcal{SH}(S)$ is stable.

(ii) It suffices to invert $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$.

Theorem 1.19. Let

$$\mathbf{Spc}(S)_*[(\mathbb{P}^1)^{-1}] = \operatorname{colim}\left(\mathbf{Spc}(S)_* \xrightarrow{(-)\wedge\mathbb{P}^1} \mathbf{Spc}(S)_* \xrightarrow{(-)\wedge\mathbb{P}^1} \mathbf{Spc}(S)_* \to \cdots\right)$$

in $\mathbf{Pr}^L$.[4] Then, this is an idempotent algebra in $\mathbf{Mod}_{\mathbf{Spc}(S)_*}(\mathbf{Pr}^L)$. Then, the preferred map $\mathbf{Spc}(S)_*[(\mathbb{P}^1)^{-1}] \to \mathcal{SH}(S)$ is an equivalence in $\mathbf{CAlg}(\mathbf{Pr}^L)$.

Proof. This is a categorification of the group completion theorem. Robalo proved that we need to check that $\mathbb{P}^1$ is symmetric, i.e. the cyclic permutation $(1\ 2\ 3) : (\mathbb{P}^1)^{\wedge 3} \to (\mathbb{P}^1)^{\wedge 3}$ is equivalent to the identity [Rob13].

We use $(\mathbb{P}^1)^{\wedge 3} \simeq \mathrm{Th}(\mathbb{A}^3)$. We factor the matrix for $(1\ 2\ 3)$ into elementary matrices $E(a)$ over $\mathbb{Z}$ with $a \in \mathbb{Z}$ which is possible since $\det(1\ 2\ 3) = 1$. Then,

$$\mathbb{A}^1 \times \mathbb{A}^3 \to \mathbb{A}^3,\ (t, x) \mapsto E(ta)(X)$$

is a homotopy to the identity. □

[3] Here, [1] obtains the monoidal structure via max.

[4] I.e. you take the limit of the right adjoints.

1.6 Motivic Thom Spectrum

The symmetric monoidal composite

$$\mathbf{Vect}_S \xrightarrow{\mathrm{Th}} \mathbf{Spc}(S)_* \xrightarrow{\Sigma^\infty} \mathcal{SH}(S).$$

sends all $E \in \mathbf{Vect}_S$ to $\otimes$-invertible objects. On trivial bundles those are spheres where it is true. In general we check this locally by Nisnevich separation. So it factors the group completion

$$J : K(S) = (\mathbf{Vect}_S)^{\mathrm{gp}} \to \mathcal{SH}(S)$$

as a symmetric monoidal functor. Via $K(S) \to \mathbb{Z}$ we can pick out the rank and define $K(S)^0$.

Definition 1.20. The **motivic Thom spectrum** is

$$\mathrm{MGL} = \mathrm{colim}(J : K(S)^0 \to \mathcal{SH}(S)) \in \mathbf{CAlg}(\mathcal{SH}(S)).$$

Remark* 1.21.

(i) This is a bit inaccurate, it only has Thom isomorphisms for bundles over S instead of over all S-schemes. One needs some more parametrizations to make this work properly.

Indeed, let us recall the classical setting [ACB19, Proposition 3.16]. Say some ring spectrum E is MU-oriented, so the composite $\mathrm{BU} \to \mathrm{Pic}\,\mathbb{S} \to \mathrm{Pic}\,E$ is nullhomotopic. Any virtual degree 0 vector bundle is represented by a map $f : X \to \mathrm{BU}$. So the composite

$$X \xrightarrow{f} \mathrm{BU} \xrightarrow{J} \mathrm{Pic}\,\mathbb{S} \xrightarrow{\mathrm{Ind}_{\mathbb{S}}^E} \mathrm{Pic}\,E$$

is nullhomotopic as well. This gives

$$E^\bullet(\mathrm{Th}(f)) \cong E^\bullet(\mathrm{Th}(f) \otimes E) \cong E^\bullet\left(\mathrm{colim}(\mathrm{Ind}_{\mathbb{S}}^E \circ J \circ f)\right) \cong E^\bullet(X),$$

i.e. the Thom isomorphism. The first isomorphism is by adjunction.

Now the same holds in the motivic setting but $K(S)^0$ are only virtual degree 0 vector bundles over S. We want vector bundles over arbitrary S-schemes, so we need some notion parametrizing over S-schemes.

(ii) Here is a more down-to-earth description. Let $f : X \to S$ be a smooth S-scheme. Then, there is a pullback functor $f^* : \mathcal{SH}(S) \to \mathcal{SH}(X)$ essentially by functoriality of $\mathbf{Sm}_{/-}$. It admits a left adjoint $f_\# : \mathcal{SH}(X) \to \mathcal{SH}(S)$. Then, we define

$$\mathrm{MGL}_S = \underset{f \in \mathbf{Sm}_S}{\mathrm{colim}}\, f_\# \,\mathrm{colim}\left(J_X : K(X)^0 \to \mathcal{SH}(X)\right) = \underset{f \in \mathbf{Sm}_S}{\mathrm{colim}}\, f_\# \mathrm{Th}_X(J)$$

as **Voevodsky's algebraic cobordism spectrum**. More information about it is e.g. contained in [BH21, Section 16].

1.7 Homotopy t-Structure

Definition 1.22. Let $\mathcal{X} \in \mathbf{Sh}^{\mathrm{Nis}}(\mathbf{Sm}_S)$. Then, $\underline{\pi}_0(\mathcal{X}) \in \mathbf{Sh}(\mathbf{Sm}_S)$ is defined as the sheafification of $U \mapsto \pi_0(\mathcal{X}(U))$. Similarly, $\underline{\pi}_n(\mathcal{S}, x) \in \mathbf{Sh}(\mathbf{Sm}_S)$ for $(\mathcal{X}, x) \in \mathbf{Sh}(\mathbf{Sm}_S)_*$.

Theorem 1.23. Let k be a perfect field. If $\mathcal{X} \in \mathbf{Spc}(k)_*$, then $\underline{\pi}_i(\mathcal{X}) \in \mathbf{Spc}(S)$ (and all its deloopings) are $\mathbb{A}^1$-invariant for $i \geq 1$.

Definition 1.24. Let $\mathcal{SH}(S)_{\geq 0} \subseteq \mathcal{SH}(S)$ be the category spanned by

$$\{\Sigma^k_{\mathbb{G}_m} \Sigma^\infty_+ X : k \in \mathbb{Z}, X \in \mathbf{Sm}_S\}$$

which you close up under extensions and colimits.

It's formal to obtain that $\mathcal{SH}(S)_{\geq 0}$ is the connective part of a t-structure, the **homotopy t-structure**.

Theorem 1.25 (Morel)**.** Let k be a field and $E \in \mathcal{SH}(k)$.

(i) We have $E \in \mathcal{SH}(S)_{\geq d}$ if and only if $\underline{\pi}_{p,q}(E) = \underline{\pi}_0(\Omega^\infty \Sigma^{-p,-q} E) = 0$ for all $p - q < d$.

(ii) We have $E \in \mathcal{SH}(S)_{\leq d}$ if and only if $\underline{\pi}_{p,q}(E) = 0$ for all $p - q > d$.

There are many interesting t-structures such as the Chow t-structure but this is quite close to the one on **Sp**.

Remark 1.26.

(i) For $f : S \to T$ the functor $f^* : \mathcal{SH}(T) \to \mathcal{SH}(S)$ is t-exact.

(ii) The Betti realization functor $\mathrm{Be}_{\mathbb{R}} : \mathcal{SH}(\mathbb{R}) \to \mathbf{Sp}$, $(X \in \mathbf{Sm}_{\mathbb{R}}) \mapsto X(\mathbb{R})^{\mathrm{an}}$ is t-exact.

1.8 Stable Stems

Fabio is ending with this because I forced him to. For this part let $S = \operatorname{Spec} k$.

Definition 1.27. The **stable stems** are $\pi_i(\mathbb{S})_j = \left[\Sigma^\infty S^i, \Sigma^\infty \mathbb{G}_m^j\right]_{\mathcal{SH}(k)}$.

Example 1.28.

(i) Take $\eta : \mathbb{A}^2 - \{0\} \to \mathbb{P}^1$ which gives $[\eta] \in \pi_0(\mathbb{S})_{-1}$.[5]

(ii) For $a \in k^\times$ we have $a : * \to \mathbb{G}_m$ yielding $[a] \in \pi_0(\mathbb{S})_1$.

Theorem 1.29 (Morel)**.** There is an isomorphism $\pi_0(\mathbb{S})_\bullet \cong \mathbb{Z}\langle [a], [\eta] \rangle / \text{relations} = K^{\mathrm{MW}}(k)_\bullet$.

This is **Milnor-Witt K-theory** and was already defined before this motivic story! We get **Milnor K-theory** via $K^M(k)_\bullet = K^{\mathrm{MW}}(k)_\bullet / [\eta]$ which computes K_0, K_1, K_2 of algebraic K-theory.

2 Slice Filtration & K-Theory (Qi Zhu)

2.1 Beilinson's Dream

TALK 2
23.10.2025

We all have dreams but your dream is not relevant for this talk.[6] Beilison's dream takes the center stage.

Motivated by questions about the zeta function ζ as well as Grothendieck's vision of a category of motives, Beilinson and Lichtenbaum conjectured the existence of motivic cohomology in the 80s [BK25, Introduction]. It is lousely supposed to satisfy a number of desiderata, among others:

(i) It gives rise to an analog of Atiyah-Hirzebruch spectral sequence (2.22).

(ii) It is essentially described by higher Chow groups (2.30).

(iii) There should be a certain range of support (2.32).

(iv) There is a close relation to étale cohomology (2.34).

We will discuss all of those in this talk and therefore see some of the historic motivations for motivic homotopy theory.

[5] To see that this is non-trivial, one can for example use Betti realization which gives the classical Hopf map.
[6] Sorry.

2.2 Motivic K-Theory Spectrum

2.2.1 Thomason-Trobaugh K-Theory

Recall that for a ring A one can define its algebraic K-theory as $K(A) = \left(\mathbf{Proj}_A^{\mathrm{fg,core}}\right)^{\mathrm{gp}}$. As so often in algebraic geometry, we can extend this to (nice) schemes.

Theorem 2.1. Let S be a regular Noetherian scheme of finite dimension. Then, there exists a motivic space $K \in \mathbf{Spc}(S)$, the so-called **Thomason-Trobaugh K-theory** such that for every $\operatorname{Spec} A \in \mathbf{Sm}_S$ we have $K(\operatorname{Spec} A) \simeq K(A)$.

Proof Idea. One can make the assignment $F : \mathbf{Sm}_S^{\mathrm{op}} \to \mathcal{S}$, $X \mapsto K(\mathcal{O}_X(X))$ functorial. The Thomason-Trobaugh K-theory is $K = L_{\mathrm{mot}}F$.

It remains to show that $F \to L_{\mathrm{mot}}F = K$ is an equivalence on affines. By working with localization formulas – namely [Bac21, Theorem 2.21] and the sheafification formula – it suffices to show that F on is $\mathbb{A}^1$-invariant and Nisnevich-local on affine schemes. This translates to properties of algebraic K-theory, namely $\mathbb{A}^1$-invariance $K(A[t]) \simeq K(A)$ and a Nisnevich descent condition. These are non-trivial properties of the K-theory of regular rings [Bac21, Theorem 2.25]. □

Recall $K(A) \simeq K_0(A) \times \mathrm{BGL}(A)^+$. This can also be extended to schemes.

Construction 2.2. There are presheaves

$$\mathrm{GL}_n : \mathbf{Sm}_S^{\mathrm{op}} \to \mathbf{Grp},\ X \mapsto \mathrm{GL}_n(\mathcal{O}_X(X)) \quad \text{and} \quad \mathrm{GL} = \operatorname*{colim}_n \mathrm{GL}_n\,.$$

Taking classifying spaces sectionwise yields $\mathrm{BGL} \in \mathbf{PSh}(\mathbf{Sm}_S)$.

Fact 2.3 ([Bac21, Theorem 2.28])**.** Let S be a regular Noetherian scheme of finite dimension. Then,

$$K \simeq L_{\mathrm{mot}}(\mathbb{Z} \times \mathrm{BGL}) \in \mathbf{Spc}(S).$$

2.2.2 Algebraic K-Theory Spectrum

Recall from Talk 1 that $\mathcal{SH}(S) \simeq \lim\left(\cdots \xrightarrow{\Omega_{\mathbb{P}^1}} \mathbf{Spc}(S) \xrightarrow{\Omega_{\mathbb{P}^1}} \mathbf{Spc}(S)\right)$. We want to construct a motivic analog of KU, i.e. a motivic spectrum KGL representing algebraic K-theory. To do this we first construct the representing motivic spaces.

Construction 2.4. Let X be an S-scheme. Denote by $\mathbf{Vect}(X)$ the 1-category of vector bundles on X. Then,[7]

$$K(\mathbf{Vect}(X)) = \left(\mathbf{Vect}(X)^{\oplus,\mathrm{core}}\right)^{\mathrm{gp}}$$

is the **direct sum K-theory** of X.

Lemma 2.5. Let S be a regular Noetherian scheme of finite dimension. Then, $K \simeq L_{\mathrm{mot}}K(\mathbf{Vect}(-))$.

Proof. By the Serre-Swan theorem, there is a symmetric monoidal functor $\mathbf{Proj}^{\mathrm{fg}}_{\mathcal{O}_X(X)} \to \mathbf{Vect}(X)$ which is an equivalence on affines. In particular, this induces a map

$$K(\mathcal{O}_-(-)) \to K(\mathbf{Vect}(-))$$

in $\mathbf{PSh}(\mathbf{Sm}_S)$ which is an equivalence on affines. So this is a Zariski equivalence. Thus,

$$K \simeq L_{\mathrm{mot}}K(\mathcal{O}_-(-)) \simeq L_{\mathrm{mot}}L_{\mathrm{Zar}}K(\mathcal{O}_-(-)) \simeq L_{\mathrm{mot}}L_{\mathrm{Zar}}K(\mathbf{Vect}(-)) \simeq L_{\mathrm{mot}}K(\mathbf{Vect}(-))$$

where we use $L_{\mathrm{mot}}L_{\mathrm{Zar}} \simeq L_{\mathrm{mot}}$ which follows from the Nisnevich topology being finer than the Zariski topology. □

[7] We perform everything in $\mathbf{Cat}_\infty$.

Remark 2.6. The definition of K via the direct sum K-theory is more general than the Thomason-Trobaugh K-theory – it doesn't require these regularity conditions on S.

Observation 2.7. The construction of $K(\mathbf{Vect}(X))$ was naturally as a functor to **CGrp** which in particular forgets to $\mathcal{S}_*$. In other words, we can obtain natural basepoints via $0 \in K(\mathbf{Vect}(X))$, so we obtain lifts $K(\mathbf{Vect}(-)) \in \mathbf{PSh}(\mathbf{Sm}_S)_*$ and $L_{\mathrm{mot}}K(\mathbf{Vect}(-)) \in \mathbf{Spc}(S)_*$.

Construction 2.8. Consider the tautological line bundle $\gamma = \mathcal{O}_{\mathbb{P}^1}(-1) \in \mathbf{Vect}(\mathbb{P}^1)$. External tensor product of vector bundles yields a natural[8] additive functor $- \otimes \gamma : \mathbf{Vect}(X) \to \mathbf{Vect}(X \times \mathbb{P}^1)$ which induces a map of commutative monoids $\gamma : K(\mathbf{Vect}(X)) \to K(\mathbf{Vect}(X \times \mathbb{P}^1))$. Similarly, there is a map $1 : K(\mathbf{Vect}(X)) \to K(\mathbf{Vect}(X \times \mathbb{P}^1))$ for the trivial line bundle $1 \in \mathbf{Vect}(\mathbb{P}^1)$. Since $K(\mathbf{Vect}(X))$ is grouplike, we can form the difference $\gamma - 1$.

The following is a motivic version of Bott periodicity.

Theorem 2.9 (Motivic Bott Periodicity)**.**

(i) The map $\gamma - 1$ assembles into a map

$$\gamma - 1 : K(\mathbf{Vect}(-)) \to \Omega_{\mathbb{P}^1} K(\mathbf{Vect}(-))$$

in $\mathbf{PSh}(\mathbf{Sm}_S)_*$.

(ii) The induced map

$$\gamma - 1 : L_{\mathrm{mot}}K(\mathbf{Vect}(-)) \to L_{\mathrm{mot}}\Omega_{\mathbb{P}^1}K(\mathbf{Vect}(-)) \to \Omega_{\mathbb{P}^1}L_{\mathrm{mot}}K(\mathbf{Vect}(-))$$

is an equivalence.

Proof.

(i) In 2.8 we constructed a map[9]

$$K(\mathbf{Vect}(-)) \to \Omega_{\mathbb{P}^1_+} K(\mathbf{Vect}(-))$$

in $\mathbf{PSh}(\mathbf{Sm}_S)_*$ which by adjunction corresponds to a map $\mathbb{P}^1_+ \otimes K(\mathbf{Vect}(-)) \to K(\mathbf{Vect}(-))$. We want to produce a map $\mathbb{P}^1 \otimes K(\mathbf{Vect}(-)) \to K(\mathbf{Vect}(-))$. Via the cofiber sequence $*_+ \to \mathbb{P}^1_+ \to \mathbb{P}^1$, we need to show that the composite

$$K(\mathbf{Vect}(-)) \simeq *_+ \otimes K(\mathbf{Vect}(-)) \longrightarrow \mathbb{P}^1_+ \otimes K(\mathbf{Vect}(-)) \longrightarrow K(\mathbf{Vect}(-))$$

is nullhomotopic. But this is induced by the restriction of $\gamma - 1$ to $* = S$ which is 0 since $\gamma|_S = 1$ is the trivial bundle. Adjoining again, we have successfully constructed a map $K(\mathbf{Vect}(-)) \to \Omega_{\mathbb{P}^1} K(\mathbf{Vect}(-))$.

(ii) We only discuss this in the case S is Noetherian, regular and of finite-dimensional, although this is true in general [Bac21, Footnote 12]. Then, by 2.5 this is the statement $K(X) \simeq K(X_+ \wedge \mathbb{P}^1)$ for Thomason-Trobaugh K-theory. This follows from the so-called *projective bundle formula* [Wei13, Theorem V.1.5].[10]

□

[8] Note that here naturality is still 1-categorical and hence can be checked by hand.

[9] That evaluation on $- \times \mathbb{P}^1$ corresponds to $\Omega_{\mathbb{P}^1_+}$ follows via a Yoneda argument.

[10] I think you use that the cofiber sequence $\mathbb{P}^1 \to X \times \mathbb{P}^1 \to X_+ \wedge \mathbb{P}^1$ induces a fiber sequence on K-theory by the Yoneda Lemma. Then, $K_\bullet(X \times \mathbb{P}^1) = K_\bullet(\mathbb{P}^1_X) \cong K_\bullet(X)[z]/z^2$ by the projective bundle formula. This z part is killed by $K(\mathbb{P}^1)$.

Mimicking the topological counterpart, we define:

Definition 2.10. The **(motivic) algebraic K-theory spectrum** is the object

$$\mathrm{KGL} = \mathrm{KGL}_S = ((K, K, \cdots); \gamma - 1 : K \to \Omega_{\mathbb{P}^1} K) \in \mathcal{SH}(S)$$

where we write $K = L_{\mathrm{mot}} K(\mathbf{Vect}(-))$ here for brevity.

Remark 2.11.

(i) So $\Omega^\infty \mathrm{KGL} \simeq K$ with which we have constructed a motivic spectrum representing K. In particular, this recovers algebraic K-theory in case S is regular, Noetherian and of finite dimension (2.5).

(ii) In general, KGL represents Weibel's homotopy K-theory KH, an $\mathbb{A}^1$-invariant approximation to K-theory. We need non $\mathbb{A}^1$-invariant K-theory to fix this defect and it will be a focus towards the end of this seminar.

Remark 2.12. Essentially since everything worked analogous to the topological counterpart one deduces that the complex Betti realization is $\mathrm{Be}_{\mathbb{C}}\,\mathrm{KGL}_{\mathbb{C}} \simeq \mathrm{KU}$ [Ban05, Lemma 4.23]. It turns out that $\mathrm{Be}_{\mathbb{R}}\,\mathrm{KGL} \simeq 0$ as opposed to $\mathrm{Be}_{\mathbb{R}}\,\mathrm{KGL} \simeq \mathrm{KO}$, answering a question from Thomas during the talk [Ban05, Lemma 4.24].

Corollary 2.13.

(i) Let $n \in \mathbb{Z}$, then $\Sigma^{2n,n}\,\mathrm{KGL} \simeq \mathrm{KGL}$.

(ii) Let S be regular, Noetherian and finite-dimensional. For $X \in \mathbf{Sm}_S$ we have

$$[\Sigma^{p,q}\Sigma^\infty_+ X, \mathrm{KGL}_S] \cong \begin{cases} K_{p-2q}(X) & p \geq 2q, \\ 0 & \text{else.} \end{cases}$$

Proof.

(i) We use $\mathcal{SH}(S) \simeq \lim \left(\cdots \xrightarrow{\Omega_{\mathbb{P}^1}} \mathbf{Spc}(S) \xrightarrow{\Omega_{\mathbb{P}^1}} \mathbf{Spc}(S) \right)$. By motivic Bott periodicity (2.9) both $\Sigma^{2n,n}\,\mathrm{KGL}$ and KGL are given by $(\cdots, K, K)$.

(ii) By Bott periodicity from (a), we can shift so far to assume $q = 0$. If $p \geq 0$, then we use adjunctions and the Yoneda Lemma to compute

$$[\Sigma^{p,0}\Sigma^\infty_+ X, \mathrm{KGL}_S] \cong [\Sigma^\infty(S^p \wedge X_+), \mathrm{KGL}_S] \cong [S^p \wedge X_+, K]_* \cong K_p(X).$$

On the other hand,

$$[\Sigma^{-p,0}\Sigma^\infty_+ X, \mathrm{KGL}_S] \cong [\Sigma^{p,p}\Sigma^\infty_+ X, \Sigma^{2p,p}\,\mathrm{KGL}_S] \cong [\mathbb{G}_m^{\wedge p} \wedge X_+, K]_*.$$

Now, there is a cofiber sequence

$$X_+ \longrightarrow (\mathbb{G}_m^{\times p} \times X)_+ \longrightarrow \mathbb{G}_m^{\wedge p} \wedge X_+$$

in $\mathbf{Spc}(S)_*$.[11] Now, let us only consider $p = 1$, for $p \geq 2$ we argue by induction. In that case, we obtain an exact sequence

$$K_1(\mathbb{G}_m \times X) \twoheadrightarrow K_1(X) \longrightarrow [\mathbb{G}_m \wedge X_+, K]_* \longrightarrow K_0(\mathbb{G}_m \times X) \xrightarrow{\sim} K_0(X)$$

where the first arrow is surjective and the last arrow is an isomorphism by *Bass' fundamental theorem*. Thus, the middle term must be 0 by exactness.

□

In particular, for $X = S$ we have $\pi_{p,q}\,\mathrm{KGL}_S \cong \begin{cases} K_{p-2q}(S) & p \geq 2q, \\ 0 & \text{else.} \end{cases}$

[11] If we omit the added basepoints of the first two terms, then this is a cofiber sequence in $\mathbf{Spc}(S)$.

2.3 The Slice Filtration

We know wish to construct a motivic version of the Whitehead filtration.

2.3.1 Axiomatic Approach to Slice Filtrations

Due to the bigrading on motivic spectra, there are multiple imaginable filtrations that generalize the Whitehead filtration. As such, we will begin by giving a general procedure for such constructions. Let me introduce the following ad hoc name. Drew calls this *slice filtration* [Hea19, Definition 2.1].

Definition 2.14 ([GRSOsr12, Section 2.1])**.** Let $\mathscr{C} \in \mathbf{CAlg}(\mathbf{Pr}^L_{\mathrm{st}})$, compactly generated by a set of objects $\mathcal{T}$. Let $\{\mathscr{C}_i\}_{i\in\mathbb{Z}}$ be a family of full subcategories of $\mathscr{C}$. Then, $\{\mathscr{C}_i\}_{i\in\mathbb{Z}}$ is a **slice setup** of $\mathscr{C}$ if the following conditions are satisfied.

(i) For $i \in \mathbb{Z}$ we have $\mathscr{C}_{i+1} \subseteq \mathscr{C}_i$.

(ii) Each $\mathscr{C}_i$ is generated under colimits and extensions by a set of compact objects $\mathcal{K}_i$.

(iii) We have $\mathbb{1} \in \mathscr{C}_0$.

(iv) Each $t \in \mathcal{T}$ is contained in some $\mathscr{C}_i$.

(v) If $c_0 \in \mathscr{C}_0$ and $c_n \in \mathscr{C}_n$, then $c_0 \otimes c_n \in \mathscr{C}_n$.

Observation 2.15. Let $i_q : \mathscr{C}_q \hookrightarrow \mathscr{C}$. Since it is closed under colimits, it admits a right adjoint $r_q : \mathscr{C} \to \mathscr{C}_q$. We put $f_q = i_q \circ r_q : \mathscr{C} \to \mathscr{C}$.

Definition 2.16. Let $\mathscr{C} \in \mathbf{CAlg}(\mathbf{Pr}^L_{\mathrm{st}})$ be equipped with a slice setup and $c \in \mathscr{C}$. We wish to construct a map $f_{q+1}c \to f_q c$. By adjunction, it corresponds to a map $f_{q+1}c \to c$ which can be taken as the counit of $i_{q+1} \dashv r_{q+1}$. Then, we obtain a filtration

$$\begin{tikzcd} \cdots \arrow[r] & f_1 c \arrow[r] \arrow[d] & f_0 c \arrow[r] \arrow[d] & f_{-1} c \arrow[r] \arrow[d] & \cdots \arrow[r] & c \\ & s_1 c & s_0 c & s_{-1} c & & \end{tikzcd}$$

is the **slice tower** with **slices** $s_n c = \mathrm{cofib}(f_{n+1}c \to f_n c)$.

In the definition of slice setups (2.14) the conditions (i), (ii) are to get the filtration running, condition (iii) is in some sense a normalization condition, condition (iv) ensures that the induced slice filtrations are exhaustive and condition (v) gives some good compatibilities with multiplicative structures [Hea19, Section 2].

Example 2.17.

(i) For $\mathscr{C} = \mathbf{Sp}$ with $\mathcal{K}_q = \{\mathbb{S}^m : m \geq q\}$ leading to $\mathscr{C}_q \simeq \mathbf{Sp}_{\geq q}$ yields the classical Whitehead tower in $\mathbf{Sp}$.

(ii) Let $\mathcal{K}_q = \{\Sigma^{a,b}\Sigma^\infty_+ X : X \in \mathbf{Sm}_S,\ b \geq q\} \subseteq \mathcal{SH}(S)$. We denote by $\Sigma^{q,q}\mathcal{SH}(S)^{\mathrm{eff}} \subseteq \mathcal{SH}(S)$ be the localizing subcategory generated by $\mathcal{K}_q$. Then, the filtration

$$\cdots \hookrightarrow \Sigma^{q+1,q+1}\mathcal{SH}(S)^{\mathrm{eff}} \hookrightarrow \Sigma^{q,q}\mathcal{SH}(S)^{\mathrm{eff}} \hookrightarrow \cdots$$

defines a slice setup. Associated to it is **(Voevodsky's) slice filtration**.

By construction, $\Sigma^{q,q}\mathcal{SH}(S)^{\mathrm{eff}} \simeq \Sigma^{q,q}(\Sigma^{0,0}\mathcal{SH}(S)^{\mathrm{eff}})$. We also write $\mathcal{SH}(S)^{\mathrm{eff}} = \Sigma^{0,0}\mathcal{SH}(S)^{\mathrm{eff}}$.

Remark 2.18. For the motivic slice filtration you can check $f_q \simeq \Sigma^{q,q} f_0 \Sigma^{-q,-q} : \mathcal{SH}(S) \to \mathcal{SH}(S)$ explicitly, following Bachmann [Bac21, Exercise 3.4].

Remark 2.19. This is not the focus of the talk but certainly the axiomatic construction leads to numerous additional interesting filtration.

(i) Taking $\mathcal{K}_q = \{\Sigma^{q+i,i}\Sigma^\infty_+ X : X \in \mathbf{Sm}_S,\ i \in \mathbb{Z}\} \subseteq \mathcal{SH}(S)$ yields the **homotopy t-structure**.

(ii) Taking $\mathcal{K}_q = \{\Sigma^{2a,a}\Sigma^\infty_+ X : X \in \mathbf{Sm}_S,\ a \geq q\} \subseteq \mathcal{SH}(S)$ we obtain the **very effective slice filtration**. There are also cellular versions of the slice and very effective slice filtrations [Hea19, Section 4].

(iii) In $\mathbf{Sp}^{C_2}$ we define the following slice cells:

- $S^{2q,q\sigma}$ of dimension $2q$,
- $S^{2q-1,q\sigma}$ of dimension $2q-1$,
- $S^q \otimes (C_2)_+$ of dimension q.

Take $P^q\mathbf{Sp}^{C_2} \subseteq \mathbf{Sp}^{C_2}$ be the full subcategory generated under extensions and colimits of slice cells of dimension $\geq q$. This gives rise to the **Hill-Hopkins-Ravenel slice filtration** for $\mathbf{Sp}^{C_2}$. Ignoring cells of the second form gives Ullman's **regular slice filtration**. See [Hea19, Section 5.2]. There is also a version for more general G – on the other hand, C_2 seems fitting in the context of motivic homotopy theory which was also the purpose of [Hea19].

2.3.2 Motivic Examples

Definition 2.20.

(i) The **effective algebraic K-theory spectrum** is the motivic spectrum $\mathrm{kgl} = f_0\,\mathrm{KGL}$.

(ii) The **motivic cohomology spectrum** is the motivic spectrum $H\mathbb{Z} = s_0\,\mathrm{KGL}$.

Remark 2.21. The first definition of motivic cohomology is due to Voevodsky in the mid 90s as certain derived functors of so-called *motivic complexes* of sheaves $\mathbb{Z}(q)$, realizing Beilinson's dream. Afterwards, one may ask for alternative descriptions of this object. It was Voevodsky's first conjecture about his slice filtration [Voe02b, Conjecture 1] that $s_0\,\mathrm{KGL}$ is one such candidate. Here are some other ways of producing $H\mathbb{Z}$ over perfect fields.

(i) Classically singular chains defines a right adjoint $i^* : \mathbf{Sp} \to \mathbf{Ch}(\mathbf{Ab})$ and we can define $H\mathbb{Z} = i_* i^* \mathbb{S}$. Motivically, one can mimic this construction by replacing $\mathbf{Sp}$ with $\mathcal{SH}(k)$ and $\mathbf{Ch}(\mathbf{Ab})$ by the stable ∞-category of motives $\mathbf{DM}(k)$.

(ii) Classically, one can construct $H\mathbb{Z}$ as an infinite loop space via Eilenberg-MacLane spaces which can be viewed as $\mathrm{SP}^\infty(S^n)$ via the Dold-Thom theorem. This also be realized motivically over characteristic 0, i.e. one writes out a sequential $\mathbb{P}^1$-spectrum via Eilenberg MacLane spaces realized through symmetric products.

More naively attempting to take Eilenberg-MacLane objects in the ∞-topos $\mathbf{Sh}_{\mathrm{Nis}}(\mathbf{Sm}_k)$ and then applying $L_{\mathbb{A}^1}$ does not work – this only gives an S^1-spectrum and not an $\mathbb{P}^1$-spectrum.

(iii) Classically, $H\mathbb{Z} \simeq \tau_{\leq 0}\mathbb{S} \simeq \pi_0\mathbb{S}$. Motivically, $H\mathbb{Z} \simeq s_0\mathbb{1}$ is one of Voevodsky's original conjectures about his slice filtration which was shown by Levine. In fact, this combined with another conjecture, also proved by Levine, yields the first conjecture [Voe02c, Lev08].

Remark 2.22. The spectral associated to the slice filtration of KGL is a motivic version of the Atiyah-Hirzebruch spectral sequence [BL99, Voe02c], namely for $X \in \mathbf{Sm}_k$ there is a strongly convergent spectral sequence

$$E_2^{p,q} = H\mathbb{Z}^{p-q,-q}(X) \Rightarrow K_{-p-q}(X).$$

This is the one of the starting points of Hahn-Raksit-Wilson's even filtration and the related motivic filtrations [HRW24].

Lemma 2.23. There are equivalences $f_n \operatorname{KGL} \simeq \Sigma^{2n,n} \operatorname{kgl}$ and $s_n \operatorname{KGL} \simeq \Sigma^{2n,n} H\mathbb{Z}$.

Proof. Via 2.18 we compute

$$\begin{aligned} f_n \operatorname{KGL} &= \Sigma^{n,n} f_0 \Sigma^{-n,-n} \operatorname{KGL} \\ &\simeq \Sigma^{2n,n} \Sigma^{-n,0} f_0 \Sigma^{-n,-n} \operatorname{KGL} \\ &\simeq \Sigma^{2n,n} f_0 \Sigma^{-n,0} \Sigma^{-n,-n} \operatorname{KGL} \\ &\simeq \Sigma^{2n,n} f_0 \operatorname{KGL} \\ &= \Sigma^{2n,n} \operatorname{kgl}. \end{aligned}$$

and

$$\begin{aligned} s_n \operatorname{KGL} &= \operatorname{cofib}(f_{n+1} \operatorname{KGL} \to f_n \operatorname{KGL}) \\ &\simeq \operatorname{cofib}(\Sigma^{2(n+1),n+1} \operatorname{kgl} \to \Sigma^{2n,n} \operatorname{kgl}) \\ &\simeq \Sigma^{2n,n} \operatorname{cofib}(\Sigma^{2,1} \operatorname{kgl} \to \operatorname{kgl}) \\ &\simeq \Sigma^{2n,n} \operatorname{cofib}(f_1 \operatorname{KGL} \to f_0 \operatorname{KGL}) \\ &= \Sigma^{2n,n} H\mathbb{Z}. \end{aligned}$$

□

2.4 Motivic Cohomology & Historic Theorems

2.4.1 Motivic Cohomology Groups

Much of motivic homotopy theory was developed to understand motivic cohomology. Let us introduce some notation before discussing a number of results in the field.

Definition 2.24. Let $E \in \mathcal{SH}(S)$ and $X \in \mathbf{Sm}_S$, then $E^{p,q}(X) = [\Sigma^\infty_+ X, \Sigma^{p,q} E]$ is the **bigraded cohomology theory** represented by E.

Remark 2.25. So $\pi_{p,q} E \cong [\Sigma^{p,q} \Sigma^\infty_+ S, E] \cong E^{-p,-q}(S)$.

Definition 2.26. The bigraded cohomology theory represented by $H\mathbb{Z}$ is called **motivic cohomology** and is denoted by

$$H^{p,q}(X) = H^{p,q}(X;\mathbb{Z}) = H^p(X;\mathbb{Z}(q)) = H\mathbb{Z}^{p,q}(X).$$

The cohomology theory associated to $H\mathbb{Z}/n$ is also called motivic cohomology.

2.4.2 Higher Chow Groups

We first construct an algebro-geometric version of singular homology.

Construction 2.27 (Bloch)**.** Let $X \in \mathbf{Sm}_S$.

(i) We write $Z^d(X) = \mathbb{Z}\{x \in X : \operatorname{codim}(\overline{\{x\}} \subseteq X) = d \iff \dim \mathcal{O}_{X,x} = d\}$.

(ii) If $i : Y \hookrightarrow X$ is a closed immersion, then $c = \sum_n a_n x_n \in Z^d(X)$ is in **good position with respect to i** if the components of $Y \cap \overline{\{x_n\}}$ have codimension $\geq d$ on Y for every n. We write $Z^d(X)_i \subseteq Z^d(X)$ for such cycles. One can construct a pullback map $i^* : Z^d(X)_i \to Z^d(Y)$ [MVW06, Definition 17A.6].

(iii) Let $z^d(X, n) \subseteq Z^d(X \times \Delta^n)$ consists of those cycles in good position with respect to all faces $X \times \Delta^i \subseteq X \times \Delta^n$. Then, we put

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i^* : z^d(X, n) \to z^d(X, n-1).$$

This yields a chain complex and we write $\mathrm{CH}^d(X, n) = H_n(z^d(X, \bullet), \partial)$ for the **higher Chow groups**.

Observation 2.28. Flat maps preserve codimensions of subschemes [Bac21, Remark 4.13], so it induces pullback maps on z^d which descends to pullbacks of higher Chow groups CH^d. In other words, $\mathrm{CH}^d(-, n)$ is contravariantly functorial in flat maps.

Remark 2.29. This generalizes classical Chow groups $\mathrm{CH}^d(X) \cong \mathrm{CH}^d(X, 0)$ [Bac21, Example 4.11].

There are certainly many exciting things to be said about CH^d [Bac21, Section 4.3, 4.4] but we will focus on the connection to motivic cohomology.

Theorem 2.30 ([Voe02a])**.** *Let $X \in \mathbf{Sm}_k$ and $p, q \in \mathbb{Z}$. Then, there are natural isomorphisms*

$$H^{p,q}(X) \cong \mathrm{CH}^q(X, 2q - p).$$

This paper [Voe02a] is a 5-page paper with 100 citations!

Remark 2.31. Besides connecting two seemingly disjoint objects, we can extract many interesting consequences.

(i) We have argued that $\mathrm{CH}^d(-, n)$ is functorial in flat maps (2.28(ii)) but motivic cohomology $H^{p,q}(-)$ is functorial in all maps of schemes, so this functoriality transfers to $\mathrm{CH}^d(-, n)$.

(ii) Since $H^{p,q}(-)$ is represented by a motivic spectrum, we deduce that $\mathrm{CH}^d(-, n)$ is $\mathbb{A}^1$-invariant and satisfies Nisnevich descent.

(iii) By construction, $\mathrm{CH}^d(X, n) = 0$ for $n < 0$. Thus, $H^{p,q}(X) \cong \mathrm{CH}^q(X, 2q - p) = 0$ for $p > 2q$.

This result also allows us to compute weight 0 and weight 1 motivic cohomology after further higher Chow group computations [Bac21, Exercise 4.2, Theorem 4.14]. One obtains

$$H^{\bullet,0}(X) \cong \mathbb{Z}^{\pi_0 X}[0] \quad \text{and} \quad H^{p,1}(X) \cong \begin{cases} \mathrm{Pic}(X) & p = 2, \\ \mathcal{O}_X(X)^\times & p = 1, \\ 0 & \text{else.} \end{cases}$$

Conjecture 2.32 (Beilinson-Soulé Vanishing Conjecture)**.** *Is $H^{p,q}(X) \cong 0$ for $p < 0$?*

2.4.3 Bloch-Kato Conjecture

Let ℓ be an integer invertible in k. The Kummer exact sequence yields a connecting homomorphism $\partial : k^\times \to H^1_{\text{ét}}(k, \mu_\ell)$. Very briefly, via multiplicativity of étale cohomology, the definition of Milnor K-theory (with Artin reciprocity) and ℓ-torsion of $H^\bullet_{\text{ét}}(k, \mu_\ell^{\otimes \bullet})$, this induces a map

$$\begin{tikzcd}
k^\times \otimes \cdots \otimes k^\times \arrow[r, "\partial^n"] \arrow[d] & H^n_{\text{ét}}(k, \mu_\ell^{\otimes n}) \\
K^M_n(k) \arrow[ur, dashed, "\partial^n"] \arrow[d] & \\
K^M_n(k)/\ell \arrow[uur, dotted, bend right, "\partial_n"'] &
\end{tikzcd}$$

the so-called **Galois symbol** or **norm residue map**.

Theorem 2.33 (Norm Residue Theorem/(Motivic) Bloch-Kato Conjecture)**.** Let ℓ be an integer invertible in k. The map $\partial_n : K^M_n(k)/\ell \to H^n_{\text{ét}}(k, \mu_\ell^{\otimes n})$ is an isomorphism.

For $\ell = 2$ this was first conjectured by Milnor and as such is the *Milnor conjecture*. For $n = 2$ this is the *Merkurjev-Suslin theorem*, as this case was first proven by them and the first major advance in the resolution of this theorem. Voevodsky first proved the Milnor conjecture which earned him a fields medal and he later went on to prove the entire theorem with ideas from Rost.

Theorem 2.34 (Beilinson-Lichtenbaum Conjecture, Rost-Voevodsky)**.** Let $X \in \mathbf{Sm}_k$ and $\ell \in \mathbb{Z}$ be invertible in k. Then, $H^{p,q}(X, \mathbb{Z}/\ell) \cong H^p_{\text{ét}}(X, \mu_\ell^{\otimes q})$ for $p \leq q$.

This implies the norm residue theorem. Indeed, we first mention:

Theorem 2.35 (Nesterenko-Suslin '90, Totaro '92)**.** We have $\mathrm{CH}^d(k, n) \cong \begin{cases} 0 & n < d, \\ K^M_d(k) & n = d. \end{cases}$

So combining this result (2.35) with Levine's comparison of $H^{p,q}$ and CH^d (see 2.30) we see that the Milnor K-group term from the norm residue theorem (2.33) is identified with the motivic cohomology term from the Beilinson-Lichtenbaum conjecture (2.34).

Combining the Bloch-Kato conjecture with the motivic Atiyah-Hirzebruch spectral sequence also gave the resolution of the Quillen-Lichtenbaum conjecture which related étale cohomology to algebraic K-theory.

The resolution of this conjecture was a huge leap in the development of motivic homotopy theory. It required motivic versions of Spanier-Whitehead duality and the Steenrod algebra. Especially the latter is a focus point of this seminar.

3 $7 \leq n$ Functors for $\mathcal{SH}$ (Lucas Piessevaux)

We will discuss a number of functors on $\mathcal{SH}$. In fact, there are at least seven relevant ones: $f_\#, f_*, f^*, f_!, f^!, \otimes, \underline{\mathrm{Map}}$. TALK 3 30.10.2025

Recall from the previous two talks for a (qcqs) scheme S that $\mathbf{Spc}(S) = L_{\text{mot}}\mathbf{PSh}(\mathbf{Sm}_S)$ and $\mathcal{SH}(S) = \mathbf{Spc}(S)_*[(\mathbb{P}^1)^{\otimes -1}]$. This contains examples like KH and $\mathbb{Z}(n)^{\text{mot}}$ which is mysterious but in nice enough cases contains the examples $H^i_{\text{ét}}(-; \mu_\ell^{\otimes j})$ for $i \leq j$ and $R\Gamma_{\text{Zar}}(-, W_r\Omega^i_{\log})$.

Our goals today are:

- $\mathcal{SH}^* : \mathbf{Sch}^{\text{op}} \to \mathbf{CAlg}(\mathbf{Pr}^L_{\text{st}})$ with $\mathscr{E} = $ (lft),

- motivic properties: $\mathbb{A}^1$-invariance, gluing, Thom twists.

In particular, this is not supposed to be an exercise in the theory of Heyer-Mann [HM24] but rather we want to apply the formalism to prove motivic properties!

3.1 The Functors 1-4 and 7: $(\#, *, \otimes)$

We begin with the easier functors, namely the $*$'s, $\otimes$, $\underline{\mathrm{Map}}$ and # which are functors 1-4 and number 7.

3.1.1 Closed Monoidality and Push-Pull $(*, \otimes)$

- $\otimes$: Equipping $\mathbf{Spc}(S)$ with a cartesian symmetric monoidal structure, we obtain a symmetric monoidal structure on $\mathbf{Spc}(S)_*$ given by the smash product. This gives rise to $\mathcal{SH}(S) \in \mathbf{CAlg}(\mathbf{Pr}^L_{\mathrm{st}})$.[12]
- $*$: For $f : T \to S$ we get $T \times_S - : \mathbf{Sm}_S \to \mathbf{Sm}_T$ which preserves $\mathbb{A}^1_X \to X$ and Nisnevich squares. So it induces an adjunction

$$\mathbf{Spc}(S) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathbf{Spc}(T)$$

  where f_* is induced by restriction and f^* is obtained via left Kan extension (and localization). On representables we obtain $f^*X_+ \simeq (T \times_S X)_+$, whence f^* is symmetric monoidal and $f^*\mathbb{P}^1_S \simeq \mathbb{P}^1_T$.

Proposition 3.1. Given $f : T \to S$ there exists an adjunction

$$\mathcal{SH}(S) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{SH}(T)$$

such that:

(i) $f^*\Sigma^{\infty-n}_{\mathbb{P}^1} X_+ \simeq \Sigma^{\infty-n}_{\mathbb{P}^1}(T \times_S X)_+$,

(ii) f^* is strong symmetric monoidal and strongly cocontinuous.[13]

Proof. The composite

$$\begin{tikzcd} \mathbf{Spc}(S)_* \arrow[r] & \mathbf{Spc}(T)_* \arrow[d] \\ & \mathcal{SH}(T) \end{tikzcd}$$

sends $\mathbb{P}^1_S \mapsto \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^1_T$, so the universal property of stabilization induces the adjunction. It moreover gives rise to the formula in (i). For (ii) the strong symmetric monoidality is inherited from the unstable setting and strong cocontinuity follows from $\Sigma^{\infty-n}_{\mathbb{P}^1}(T \times_S X)_+ \in \mathcal{SH}(T)^\omega$, meaning that it sends a family of compact generators to compact objects. $\square$

Remark 3.2. Note that the formula 3.1(i) uniquely determines f^* since $\mathcal{SH}(S)$ is generated by these $\Sigma^{\infty-n}_{\mathbb{P}^1} X_+$.

[12] In particular, $\Sigma^\infty_+ X \otimes \Sigma^\infty_+ Y \simeq \Sigma^\infty_+(X \times Y)$.
[13] I.e. preserves compact objects.

3.1.2 Forgetful Functor

Onto the 7th functor.

If $f : T \to S$ is a smooth map, then we have a functor

$$\mathbf{Sm}_T \to \mathbf{Sm}_S,\ (X \to T) \mapsto (X \to T \to S).$$

By left Kan extension we obtain an adjunction

$$\mathbf{PSh}(\mathbf{Sm}_T) \underset{f^*}{\overset{f_\#}{\rightleftarrows}} \mathbf{PSh}(\mathbf{Sm}_S).$$

Remark 3.3.

(i) One can check on representables that for smooth f, this f^* agrees with the f^* defined in the push-pull section.

(ii) Given $X \in \mathbf{Sm}_T$ we have $f_\#(X \to T) = (X \to T \to S) \in \mathbf{PSh}(\mathbf{Sm}_S)$ on representables by left Kan extension. In that regard, we are just forgetting.

Now, the smooth projection formula (SPF).

Proposition 3.4 (SPF)**.** Given a smooth $f : T \to S$ there exists an adjunction

$$\mathbf{Spc}(T)_* \underset{f^*}{\overset{f_\#}{\rightleftarrows}} \mathbf{Spc}(S)_*$$

of $\mathbf{Spc}(S)_*$-modules where we view $\mathbf{Spc}(T)_* \in \mathbf{Mod}_{\mathbf{Spc}(S)_*}$ via f^*.

Proof. The $\mathbf{Spc}(S)_*$-linearity is an equivalence $f_\#(X \otimes f^*Y) \simeq f_\# X \otimes Y$, i.e. the smooth projection formula. Note that all functors involved commute with colimits and Σ^{-n}, so we may check the formula on representables. There, it's asking for $X \times_T (T \times_S Y) \cong X \times_S Y$ in $\mathbf{Sch}_S$ which is pullback pasting. □

Corollary 3.5. Given a smooth $f : T \to S$ there exists an adjunction

$$\mathcal{SH}(T) \underset{f^*}{\overset{f_\#}{\rightleftarrows}} \mathcal{SH}(S)$$

with $f_\# \Sigma^{\infty-n}_{\mathbb{P}^1} X_+ \simeq \Sigma^{\infty-n}_{\mathbb{P}^1} X_+$.

Proof. Basechanging the $\mathbf{Spc}(S)_*$-algebra $\mathcal{SH}(S)$ we obtain an $\mathcal{SH}(S)$-linear adjunction

$$\mathbf{Spc}(T)_* \otimes_{\mathbf{Spc}(S)_*} \mathcal{SH}(S) \underset{f^*}{\overset{f_\#}{\rightleftarrows}} \mathcal{SH}(S)$$

but the left side is seen to be

$$\mathbf{Spc}(T)_* \otimes_{\mathbf{Spc}(S)_*} \mathbf{Spc}(S)_*[(\mathbb{P}^1_S)^{\otimes -1}] \simeq \mathbf{Spc}(T)_*[(f^*\mathbb{P}^1_S)^{\otimes -1}] \simeq \mathcal{SH}(T).$$

The formula is as before. □

Remark* 3.6. The equivalence $f_\# \Sigma^{\infty-n}_{\mathbb{P}^1} X_+ \simeq \Sigma^{\infty-n}_{\mathbb{P}^1} X_+$ uniquely determines $f_\#$.

The next result (smooth base change) involves Beck-Chevalley maps, one of which is a mate of the other. To check that both are equivalences, it suffices to check that one of them is. This will be true for many results in this talk and we will only write down one of the transformations later.

Proposition 3.7 (SBC)**.** Given a pullback square

$$\begin{tikzcd} T' \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & S' \arrow[d, "p"] \\ T \arrow[r, "f"'] & S \end{tikzcd}$$

with smooth p, q the BC transformations

$$q_\# g^* \Rightarrow f^* p_\# \quad \text{and} \quad f^* p_* \Rightarrow q_* g^*$$

are equivalences.

Proof. It suffices to check $q_\# g^* \Rightarrow f^* p_\#$, then its mate automatically becomes an equivalence by abstract nonsense. All functors here commute with colimits and desuspensions, so we are going to evaluate on X_+ with $X \in \mathbf{Sm}_{S'}$. We need to show $T' \times_{S'} X \simeq T \times_S X$ as T-schemes. This follows from the pasting of the pullback squares

$$\begin{tikzcd} T' \times_{S'} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ T' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S' \arrow[d] \\ T \arrow[r] & S \end{tikzcd}$$

so we are done. □

Emma: *This is just parametrized cocompleteness of presheaf topoi.*

Corollary 3.8. Let $j : U \hookrightarrow X$ be an open immersion. Then,

$$\begin{tikzcd} U \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, hook, "j"] \\ U \arrow[r, hook, "j"'] & X \end{tikzcd}$$

is a pullback, so the smooth base change formula tells us $\mathrm{id} \simeq j^* j_\#$ and $j^* j_* \simeq \mathrm{id}$, so $j_\#$ and j_* are fully faithful.

3.2 Motivic Properties

3.2.1 Homotopy Invariance

Warning 3.9. The assignment $S \mapsto \mathcal{SH}(S)$ does not invert $\mathbb{A}^1$-equivalences. What does is the functor

$$\mathbf{Sm}_S \to \mathcal{SH}(S), \ (f : X \to S) \mapsto \Sigma^\infty_{\mathbb{P}^1} X_+.$$

Sven remarks that this the classical analog is $\mathbf{Sh}(*) \not\simeq \mathbf{Sh}(\mathbb{R})$.

Lemma 3.10. Let $p : E \to S$ be an affine bundle, then p^* is fully faithful.

Proof. Since p is smooth, the adjunction $p_\# \dashv p^*$ exists. Consider $\varepsilon : p_\# p^* \Rightarrow \mathrm{id}$. All functors commute with colimits, so this is determined by the value on representables. Evaluate on $\Sigma^\infty_{\mathbb{P}^1} X_+$ with $X \in \mathbf{Sm}_S$. Then, we need to check $\Sigma^\infty_{\mathbb{P}^1}(E \times_S X)_+ \xrightarrow{\simeq} \Sigma^\infty_{\mathbb{P}^1} X_+$ in $\mathcal{SH}(S)$. We may first trivialize E and assume $E = \mathbb{A}^n_S$ by Nisnevich descent and then this is $\mathbb{A}^1$-invariance. □

3.2.2 Thom Twists

Recall that for a finite locally free sheaf $\mathcal{E} \to S$ we take the cofiber sequence

$$(\mathbb{V}(E) \setminus S)_+ \longrightarrow \mathbb{V}(E)_+ \longrightarrow \mathrm{Th}_S(E)$$

as the defining sequence for $\mathrm{Th}_S(E)$.

Lemma 3.11. This refines to an assignment

$$\begin{tikzcd} \mathbf{Vect}_S^{\mathrm{core}} \arrow[r, "\mathrm{Th}_S(-)"] \arrow[d] & \mathrm{Pic}(\mathcal{SH}(S)) \\ K(S) \arrow[ur] & \end{tikzcd}$$

where $K(S)$ is Q-construction K-theory.

Proof. By Nisnevich separation (1.13) of $\mathcal{SH}$ we can always descend to the affine case $S = \operatorname{Spec} R$, so $P \subseteq R^{\oplus n}$. Consider the SES

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

and write $\pi : T \to S$ as the moduli of splittings of this SES. This is an affine bundle because locally its fibers are of the form $\underline{\mathrm{hom}}_S(E'', E')$. So π^* is fully faithful. $\square$

Definition 3.12. We write $K(S) \to \mathrm{Pic}(\mathcal{SH}(S)) \simeq \mathrm{Aut}_{\mathcal{SH}(S)}(\mathcal{SH}(S))$, $\mathcal{E} \mapsto \langle \mathcal{E} \rangle$.

Example 3.13. We have $\mathcal{O}_{\mathbb{P}^1} \mapsto [2](1) = \Sigma^{2,1}$.

3.2.3 Purity

This is the ur-theorem of $\mathcal{SH}$ and the reason Morel-Voevodsky setup $\mathcal{SH}$ the way they do. Their insight is that their definition allows us to perform things like deformation to the normal cone.

Definition 3.14.

(i) A **smooth closed pair** (X, Z) is a closed immersion $Z \hookrightarrow X$ in $\mathbf{Sm}_S$.

(ii) A **map of smooth closed pairs** $(X', Z') \to (X, Z)$ is a map $X' \to X$ that induces a pullback square

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, hook] \\ X' \arrow[r] & X \end{tikzcd}$$

on closed subschemes.

(iii) A map $f : (X', Z') \to (X, Z)$ is **weakly excisive** if

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & X' \arrow[r] & X'/(X' \setminus Z') \arrow[d] \\ Z \arrow[r] & X \arrow[r] & X/(X \setminus Z) \end{tikzcd}$$

is cocartesian in $\mathbf{Spc}(S)$.

Example 3.15. Nisnevich squares and zero sections of affine bundles are examples.

The point is that there is a slightly larger class of squares than the Nisnevich squares which are sent to cocartesian squares including things like blow-up squares.

Construction 3.16. Let (X, Z) be a smooth closed pair.

(i) The blowup

$$\begin{tikzcd} E = p^{-1}(Z) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Bl}_Z(X) \arrow[d, "p"] \\ Z \arrow[r, hook, "i"'] & X \end{tikzcd}$$

is a map of smooth closed pairs.

(ii) Let $D_Z X = \mathrm{Bl}_{Z\times 0}(X \times \mathbb{A}^1) \setminus \mathrm{Bl}_{Z\times 0}(X \times 0)$ be the **deformation to the normal cone**. This has a closed immersion $Z \times \mathbb{A}^1 \hookrightarrow D_Z X$ which forms a smooth closed pair.

Blowups are universal smooth closed pairs such that the exceptional divisor is an effective Cartier divisor.

Here is the idea: The scheme $D_Z X$ is a family over $\mathbb{A}^1$ with generic fiber X and special fiber the normal cone $N_Z X$ of Z in X, as suggested by its nomenclature. Pictorially:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & D_Z X \arrow[d] & N_Z X \arrow[l, hook'] \arrow[d] \\ 1 \arrow[r] & \mathbb{A}^1 & 0 \arrow[l] \end{tikzcd}$$

This induces a cospan $(X, Z) \to (D_Z X, Z \times \mathbb{A}^1) \leftarrow (N_Z X, Z)$.

Theorem 3.17. These maps are weakly excisive.

Proof Sketch. Assume $(X, Z) = (\mathbb{V}(\mathcal{E}), Z)$. Then, $(\mathrm{Bl}_Z X, p^{-1}(Z)) = (\mathbb{V}(\mathcal{O}_{\mathbb{P}_Z(\mathcal{E})}(1)), \mathbb{P}_Z(\mathcal{E}))$. Then,

$$\begin{tikzcd} \mathbb{P}_Z(\mathcal{E}) \arrow[r] \arrow[d] & \mathbb{V}(\mathcal{O}_{\mathbb{P}_Z(\mathcal{E})}(1)) \arrow[r] \arrow[d] \arrow[dl, phantom, "\ulcorner", very near start] & \mathbb{V}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))/\mathbb{V}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \setminus \mathbb{P}(\mathcal{E}) \arrow[d] \arrow[dl, phantom, "\ulcorner", very near start] \\ Z \arrow[r] & \mathbb{V}(\mathcal{E}) \arrow[r] & \mathbb{V}(\mathcal{E})/(\mathbb{V}(\mathcal{E}) \setminus Z) \end{tikzcd}$$

Now, use pasting. The right square is cocartesian in $\mathbf{PSh}(\mathbf{Sm}_S)$ and the left square is cocartesian in $L_{\mathbb{A}^1}\mathbf{PSh}(\mathbf{Sm}_S)$. □

The above is hard, so we are sketchy. Applying weak excisiveness, we have constructed a zig-zag of equivalences giving rise to:

Corollary 3.18 (Purity)**.** Let (X, Z) be a smooth closed pair. Then, there is an equivalence

$$\frac{X}{X \setminus Z} \simeq \frac{N_Z X}{N_Z X \setminus Z} = \mathrm{Th}_Z(N_Z X)$$

in $\mathcal{SH}(S)$.

3.2.4 Localization/Gluing

We want to study what happens in the setting

$$U \xhookrightarrow{\text{open}} X \xhookleftarrow{\text{closed}} Z$$

which gives rise to a stable recollement

$$\mathcal{SH}(U) \rightleftarrows \mathcal{SH}(X) \rightleftarrows \mathcal{SH}(Z).$$

Recall that the term $K(\mathbf{Perf}(X)_Z)$ shows up when trying to compute the K-theory of open-closed decompositions. On the other hand, $\mathrm{KH} \not\simeq K$ and in particular $\mathbb{A}^1$-invariance yields $\mathrm{KH}(\mathbf{Perf}(X)_Z) \simeq \mathrm{KH}(\mathbf{Perf}(Z))$.

Remark 3.19. The small étale site ét_S is a topological/nil invariant. Moreover, $(\mathbf{Sm}_S^{\mathrm{Nis}}, L_{\mathbb{A}^1})$ is also topologically invariant.

Theorem 3.20. Given an open-closed decomposition $U \xhookrightarrow{j} X \xhookleftarrow{i} Z$ we get a stable recollement

$$\begin{tikzcd} \mathcal{SH}(U) \arrow[r, shift left, "j_\#"'] & \mathcal{SH}(X) \arrow[l, shift left, "j^*"'] \arrow[r, shift left, "i^*"] & \mathcal{SH}(Z). \arrow[l, shift left, "i_*"] \end{tikzcd}$$

Lemma 3.21. Let $E \in \mathbf{Spc}(S)$. Then, the square

$$\begin{tikzcd} j_\# j^* E \arrow[r] \arrow[d] & E \arrow[d] \\ U \arrow[r] \arrow[ur, phantom, "\ulcorner"] & i_* i^* E \end{tikzcd}$$

is cocartesian.

Construction 3.22. Given $X \in \mathbf{Sm}_S$ and $t : Z \to X_Z$. We set

$$\Phi_S(X,t) : \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set},\ \Phi_S(X,t)(Y) = \begin{cases} \mathrm{Hom}_S(Y,X) \times_{\mathrm{Hom}_S(Y_Z,X_Z)} \{Y_Z \to Z \xrightarrow{t} X_Z\} & Y_Z \neq \varnothing, \\ * & Y_Z = \varnothing \end{cases}$$

as the moduli space of maps into X that factor through Z on the special fiber.

Remark 3.23. There is an equivalence $\Phi_S(X,t) \simeq \left(X \amalg_{X_U} U\right) \times_{i_* X_Z} S$ in $\mathbf{PSh}(\mathbf{Sm}_S)$.

Lemma 3.24. Up to L_{Nis} the presheaf $\Phi_S(X,t)$ is invariant under étale neighbourhoods of $t(Z)$.

Proof. Consider

$$\begin{tikzcd} Z \arrow[r, "t'"] \arrow[d, equal] & X'_Z \arrow[r] \arrow[d] & X' \arrow[d, "p"] \\ Z \arrow[r, "t"'] & X_Z \arrow[r] & X \end{tikzcd}$$

with étale p, then $\Phi_S(p)$ is a Nisnevich equivalence. These are presheaves of sets. Let's show that it's an effective epimorphism. Let $Y \to X$ be a class in $\Gamma(Y; \Phi_S(X,t))$ and set

$$\begin{tikzcd} & Y' = X' \times_X Y \arrow[dl, "q \text{ étale}"'] \arrow[dr, "g"] & \\ Y & & X' \end{tikzcd}$$

then

$$\begin{tikzcd} q^{-1}(Y_U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, "q"] \\ Y_U \arrow[r, hook] & Y \end{tikzcd}$$

is a Nisnevich square. Note that $f|_{Y'}$ is lifted by g and that $f|_{Y_U}$ lifts trivially. Check the same for the diagonal. □

Lemma 3.25. Up to $L_{\mathbb{A}^1}$ the presheaf $\Phi_S(X, t)$ is invariant under affine bundles.

Proof. Given a vector bundle $E \to S$ consider $t : S \to \mathbb{V}(E)$ and t_Z. We'd like to show $\Phi_S(\mathbb{V}(E), t_Z) \simeq S$. Consider

$$\mathbb{A}^1 \times \Phi_S(\mathbb{V}(E), t_Z) \to \Phi_S(\mathbb{V}(E), t_Z),\ (a, f) \mapsto af.$$

This is a homotopy between id and

$$\Phi_S(\mathbb{V}(E), t_Z) \longrightarrow S \xrightarrow{t} \Phi_S(\mathbb{V}(E), t_Z).$$

□

Proof of 3.20. Everything commutes with colimits,[14] so we can reduce to representables. The claim is equivalent to $X \amalg_{X_U} U \to i_* X_Z$ to be a motivic equivalence for all $X \in \mathbf{Sm}_S$. By universality of colimits this is equivalent to $(X \amalg_{X_U} U) \times_{i_* X_Z} S \to S$ is a motivic equivalence, i.e. $\Phi_S(X, t) \simeq S$ for every t.

- Use Nisnevich descent to reduce to S affine.
- Replace $t(Z)$ by an étale neighbourhood $Z \hookrightarrow V \hookrightarrow X$ such that $p : V \to S$ is étale at Z with restriction $t_V : Z \to V_Z$.
- Pick it small enough for it to admit $h : V \hookrightarrow \mathbb{V}(E)$ such that h is étale at $t(Z)$.
- Use $\mathbb{A}^1$-invariance.

□

Corollary 3.26. The functor $i_* : \mathcal{SH}(Z) \to \mathcal{SH}(X)$ is fully faithful.

Proof. We have $i^* j_\# \simeq 0$ which can be checked on representables using that U and Z do not intersect. So the same holds for their right adjoint $j^* i_* \simeq 0$. Then, the gluing sequence (3.21) for $i_* E$ gives a cofiber sequence

$$0 \longrightarrow i_* E \xrightarrow{\eta i_*} i_* i^* i_* E,$$

so ηi_* is an equivalence. By the triangle identities also $i_* \varepsilon$ is an equivalence. Hence, we're left to show conservativity of i_*. □

[14] We didn't show this for i_* but let's blackbox this.

3.2.5 Proper Base Change

The idea is

$$\mathrm{PBC} = \mathrm{CBC} + \mathrm{SBC} + \text{ambidexterity}.$$

Let $p : X \to Y$ which we factor as $X \xhookrightarrow{i} P \xrightarrow{q} Y$ where i is a closed immersion and q is smooth and proper. We know base change (3.7) for $q_\#$ and ambidexterity is a relation between $q_\#$ and q_*. So the strategy will be to discuss ambidexterity and closed base change.

Theorem 3.27 (Ambidexterity)**.** Let $f : X \to Y$ be smooth and proper. Consider the diagram

$$\begin{tikzcd} X \arrow[dr, "\Delta_f"] & & \\ & X \times_Y X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

then

$$\mathrm{Nm}_f : f_\# \simeq f_\#(\pi_2)_*(\Delta_f)_* \Rightarrow f_*(\pi_1)_\#(\Delta_f)_* \simeq f_*\langle \Omega_f \rangle$$

is an equivalence.

An equivariant analog is the Wirthmüller isomorphism or a sort of Atiyah duality.

Theorem 3.28 (CBC)**.** Consider a pullback square

$$\begin{tikzcd} Y_Z \arrow[r, hook, "k"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\ Z \arrow[r, hook, "i"'] & S \end{tikzcd}$$

where i, k are closed immersions, then $f^* i_* \overset{\simeq}{\Rightarrow} k_* g^*$.

Proof. The functor i_* is fully faithful (3.26), i^* is epic, so we may precompose with i^* and prove this equivalence then. This part is some quick abstract nonsense. □

Theorem 3.29 (CPF)**.** Let i be a closed immersion, then i_* is i^*-linear, i.e. $i_* X \otimes Y \xrightarrow{\simeq} i_*(X \otimes i^* Y)$.

Proof. Let j denote the open complement. Then, i^*, j^* are jointly conservative and we check the abstrat nonsense there. □

Theorem 3.30 (SCBC)**.** Consider

$$\begin{tikzcd} Z' \arrow[r, hook, "k"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Z \arrow[r, hook, "i"'] & X \end{tikzcd}$$

with closed immersions i, k and smooth f, g, then $f_\# k_* \overset{\simeq}{\Rightarrow} i_* g_\#$.

Proof. Use that k_* is fully faithful (3.26). □

3.3 Assembling the Six Functors

What do we know? We have a geometric setup $(\mathbf{Sch}^{\mathrm{qcqs}}, \mathrm{lft})$ with a suitable decomposition $I =$ (open immersions) and $P =$ (proper maps) by Nagata compactification. The functor

$$\mathcal{SH}^* : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{CAlg}(\mathbf{Pr}^L)$$

is sheafy, so locally of finite type maps is enough Nisnevich locally.

- I: For $j \in I$ we set $j_! = j_\#$ as left adjoint to j^*. Need BC = SBC and PF = SPF.
- P: For $p \in P$ we set $p_* = p_*$ as right adjoint to p^*. Need BC = CBC + (SBC + ambidexterity) and PF = CPF + (SPF + ambidexterity).
- $I \cap P$: Need BC = SPBC $\Longleftarrow$ SCBC + (SSBC + ambidexterity)

Corollary 3.31. The functor $\mathcal{SH}$ extends to a Nisnevich sheafy 6FF on $(\mathbf{Sch}^{\mathrm{qcqs}}, \mathrm{lft})$.

4 Transfers, Motivic Cohomology & EM-Spectra (Thomas Blom)

Let $R \in \mathbf{CRing}$. Consider TALK 4 13.11.2025

$$\begin{tikzcd}
\mathbf{Sm}_S \arrow[rr, "\Gamma"] \arrow[d] & & \mathbf{Cor}(S,R) \arrow[d] \\
\mathbf{Spc}(S) \arrow[rr, shift left, "R_{\mathrm{tr}}"] \arrow[d, shift right, "\Sigma^\infty"'] & & \mathbf{Spc}_{\mathrm{tr}}(S,R) \arrow[ll, shift left, "U_{\mathrm{tr}}"] \\
\mathcal{SH}(S) \arrow[rr, shift left, "R_{\mathrm{tr}}"] \arrow[u, shift right, "\Omega^\infty"'] \arrow[dr, shift right, "HR\otimes-"'] & & \mathcal{SH}_{\mathrm{tr}}(S,R) \arrow[ll, shift left, "U_{\mathrm{tr}}"] \arrow[u, "\Omega^\infty"'] \arrow[dl, shift left, "\Psi"] \\
& \mathbf{Mod}_{HR} \arrow[ul, shift right] \arrow[ur, shift left, "\Phi"] &
\end{tikzcd}$$

Today: Explain the entire part that is not the left column.

(1) $\mathbf{Cor}(S,R)$,

(2) $\mathbf{Spc}_{\mathrm{tr}}(S,R)$,

(3) $\mathcal{SH}_{\mathrm{tr}}(S,R)$,

(4) $HR \circlearrowleft HA$,

(5) Motivic cohomology.

4.1 Correspondences

Idea: Motivic cohomology admits more functoriality than just contravariant functoriality in maps of schemes. This is encoded by $\mathbf{Cor}(S,R)$. There are also wrong-way maps like in classical algebraic topology, called transfers.

The category $\mathbf{Cor}(S,R)$ has the same objects as $\mathbf{Sm}_S$. Its morphisms are 'cycles' in $X \times_S Y$ over X. One example is

$$\mathbb{A}^1 \longleftarrow \mathrm{Spec}(k[x,y]/(x^2 - f(y))) \longrightarrow \mathbb{A}^1.$$

Maps are $\mathrm{Map}_{\mathbf{Cor}(S,R)}(X,Y) = C_0(X \times_S Y/X) \otimes R$ where roughly $C_0(X \times_S Y/X)$ is the free abelian group on all closed integral subschemes $Z \hookrightarrow X \times_Y Y$ such that $Z \to X \times_S Y \to X$ is finite and onto on an irreducible component.[15]

These can be composed using 'pullback/intersection'. Any $f : X \to Y$ in $\mathbf{Sm}_S$ admits a graph Γ_f. This defines a functor $\Gamma : \mathbf{Sm}_S \to \mathbf{Cor}(S,R)$.

Voevodsky defines mixed motives as certain presheaves on $\mathbf{Cor}(S,R)$.

4.2 Spaces with Transfers

The category $\mathbf{Cor}(S,R)$ is additive (with $\amalg$) and symmetric monoidal (with $\times$). Imposing Nisnevich sheaves and $\mathbb{A}^1$-invariance gives

$$\mathbf{Spc}_{\mathrm{tr}}(S,R) \subseteq \mathbf{PSh}^{\Sigma}(\mathbf{Cor}(S,R))$$

where the right side is something like simplicial presheaves (and in Hoyois' paper it actually is). Here, the Nisnevich topology on $\mathbf{Sm}_S$ induces one on $\mathbf{Cor}(S,R)$.

This is modelled by '$\mathbb{A}^1$-invariant non-negative chain complexes of Nisnevich sheaves' with transfers in $\mathbf{Mod}_R$.[16]

Remark 4.1. Alternatively, the functor $\Gamma : \mathbf{Sm}_S \to \mathbf{Cor}(S,R)$ induces by Yoneda extension a functor $\mathbf{PSh}(\mathbf{Sm}_S) \to \mathbf{PSh}^{\Sigma}(\mathbf{Cor}(S,R))$ which is an algebra through Day convolution. Can define $\mathbf{Spc}_{\mathrm{tr}}(S,R) = \mathbf{PSh}^{\Sigma}(\mathbf{Cor}(S,R)) \otimes_{\mathbf{PSh}(\mathbf{Sm}_S)} \mathbf{Spc}_*(S)$.

4.3 Spectra with Transfers

Definition 4.2. Let $\mathcal{SH}_{\mathrm{tr}}(S,R) = \mathbf{Spc}_{\mathrm{tr}}(S,R)[(R_{\mathrm{tr}}\mathbb{P}^1)^{-1}] \simeq \mathbf{Spc}_{\mathrm{tr}}(S,R) \otimes_{\mathbf{Spc}_*(S)} \mathcal{SH}(S)$.

4.4 Eilenberg-MacLane Spectra

Definition 4.3. Let $HR = U_{\mathrm{tr}}(\mathbb{1}_{\mathcal{SH}_{\mathrm{tr}}}(S,R))$.

By general module nonsense we get an adjunction

$$\mathbf{Mod}_{HR} \underset{\Psi}{\overset{}{\rightleftarrows}} \mathbf{Mod}_{\mathbb{1}}(\mathcal{SH}_{\mathrm{tr}}(S,R)) = \mathcal{SH}_{\mathrm{tr}}(S,R)$$

Theorem 4.4 (Ostvaer–Röndigs)**.** This restricts to an equivalence between the full subcategories of cellular objects.

Observe:

$$\begin{tikzcd} \mathcal{D}_{\geq 0}(R) \arrow[r, shift left, "\mathrm{const}"] \arrow[d] & \mathbf{Spc}_{\mathrm{tr}}(S,R) \arrow[l, shift left] \arrow[d] \\ \mathcal{D}(R) \arrow[r, shift left, "\mathrm{const}"] & \mathcal{SH}_{\mathrm{tr}}(S,R) \arrow[l, shift left] \end{tikzcd}$$

so for $A \in \mathcal{D}(R)$ we get that $HA = U_{\mathrm{tr}}\,\mathrm{const}(A)$ is canonically an HR-module.

Definition 4.5. We write $K(A(q),p) = U_{\mathrm{tr}}(R_{\mathrm{tr}}S^{p,q} \otimes_R \mathrm{const}\, A)$.

[15] This is slightly wrong, $Z \to X$ needs to dominate an irreducible component.
[16] This secretly uses hypercompleteness of $\mathbf{Sh}_{\mathrm{Nis}}(\mathbf{Sm}_S)$.

4.5 Motivic Cohomology

Theorem 4.6. Let A be an R-module and $X \in \mathbf{Sm}_S$ be essentially smooth over a field k.[17] Then,

$$H^{p,q}(X, A) \cong [\Sigma^\infty_+ X, \Sigma^{p,q} HA].$$

Proof Idea. 'Essentially by definition' one obtains $H^{p,q}(X, A) \cong [X_+, K(A(q), p)]$ and deduce the stable version from this. Hoyois uses the definition

$$H^{p,q}(X; \mathbb{Z}) = H^{p-q}_{\mathrm{Zar}}\left(X, \mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m^{\otimes q})[-q]\right)$$

or perhaps some slightly different indexing. □

5 Motivic Steenrod Algebra (David Wiedemann)

Let k be a perfect field and $\ell \neq \operatorname{char} k$ with $c = c(k)$ the characteristic exponent. TALK 5 20.11.2025

Let $\mathcal{M}^{\bullet,\bullet}$ be the bi-graded algebra of stable motivic cohomology operations with $\mathbb{Z}/\ell$-coefficients. These contain 'reduced power operations' $P^i \in \mathcal{M}^{2i(\ell-1),i(\ell-1)}$ and Bockstein operation $\beta \in \mathcal{M}^{1,0}$ as well as $H^{\bullet,\bullet}(k, \mathbb{Z}/\ell)$. Let $\mathcal{A}^{\bullet,\bullet} \subseteq \mathcal{M}^{\bullet,\bullet}$ be the algebra spanned by these.

Theorem 5.1. Let S/k be a Noetherian scheme of finite Krull dimension which is essentially smooth over k.

(i) The map $\mathcal{A}^{\bullet,\bullet} \to \mathcal{M}^{\bullet,\bullet}$ is an isomorphism with explicit basis as an $H^{\bullet,\bullet}(S, \mathbb{Z}/\ell)$-module given by $\{\beta^{\varepsilon_r} P^{i_r} \cdots P^{i_1} \beta^{\varepsilon_0} : r \geq 0, i_j \geq 0, \varepsilon_j \in \{0,1\}, i_{j+1} \geq \ell i_j + \varepsilon_j\}$.

(ii) The map $(H\mathbb{Z}/p)^{\bullet,\bullet}(H\mathbb{Z}/p) \to \mathcal{M}^{\bullet,\bullet}$ is an isomorphism.

(iii) There is an equivalence of $H\mathbb{Z}/\ell$-modules $H\mathbb{Z}/\ell \otimes H\mathbb{Z}/\ell \simeq \bigoplus_\alpha \Sigma^{p_\alpha,q_\alpha} H\mathbb{Z}/\ell$.

Definition 5.2. Let S be a scheme and $\mathbf{Sch}_S$ be the category of separated finite-type schemes. We call a full subcategory $\mathscr{C} \subseteq \mathbf{Sch}_S$ **admissible** if:

(i) $S, \mathbb{A}^1_S \in \mathscr{C}$,

(ii) If $X \in \mathscr{C}$ and $U \to X$ is a finite étale map, then $U \in \mathscr{C}$,

(iii) $\mathscr{C}$ is closed under finite (co-)products.

Examples are smooth schemes or normal schemes.

Theorem 5.3 (Fundamental square)**.** Let R be a $\mathbb{Z}_{(l)}$-algebra and $i : \mathscr{C} \to \mathscr{D}$ is an inclusion admissible subcategories with $\mathscr{C} \subseteq \mathbf{Sm}_k$. Then, the square

$$\begin{tikzcd}
\mathcal{H}^*_{\mathrm{Nis},\mathbb{A}^1}(\mathscr{D}) \arrow[r, "i^*"] \arrow[d, "R^{\mathrm{tr}}"'] & \mathcal{H}^*_{\mathrm{Nis},\mathbb{A}^1}(\mathscr{C}) \arrow[d, "R^{\mathrm{tr}}"] \\
\mathcal{H}^{\mathrm{tr}}_{\mathrm{Nis},\mathbb{A}^1}(\mathscr{D}, R) \arrow[r, "i^*"'] & \mathcal{H}^{\mathrm{tr}}_{\mathrm{Nis},\mathbb{A}^1}(\mathscr{C}, R)
\end{tikzcd}$$

commutes.

We will prove this later.

[17] This probably means that it's a filtered colimit of smooth things.

Theorem 5.4. Let S be essentially smooth over k and A be a finitely generated $\mathbb{Z}[1/c]$-module and F be a field of characteristic $\neq c$. Let $p \geq 2$ and $q \geq 0$. Then, $F^{\mathrm{tr}}K(A(q),p)_{\mathbf{Sm}_S}$ is a direct sum of $F^{\mathrm{tr}}S^{a,b}$ with $a \geq 2b$ and $b \geq q$. If $L \in \mathcal{H}^{\mathrm{tr}}_{\mathbb{A}^1,\mathrm{Nis}}(\mathbf{Sm}_S, F)$ is a direct sum of $F^{\mathrm{tr}}S^{a,b}$ for $a \geq 2b, b \geq \dim S$, it is called **split proper Tate of weight** $\geq \dim S$.

Proof Sketch. For admissible $\mathscr{C}$ let $K_{\mathscr{C}} = K(A(q),p)_{\mathscr{C}}$. If $\mathscr{C}$ is the subcategory of normal schemes over S, then Voevodsky shows the result. Let $i : \mathbf{Sm}_S \hookrightarrow \mathscr{C}$, then

$$i^*F^{\mathrm{tr}}K_{\mathscr{C}} \simeq F^{\mathrm{tr}}i^*K_{\mathscr{D}} \simeq F^{\mathrm{tr}}K_{\mathbf{Sm}_S}$$

using the fundamental square. But $i_! \dashv i^*$ and $c_!$ is fully faithful. □

Fix $H = H\mathbb{Z}/\ell$ and $K_n = K(\mathbb{Z}/\ell(n), 2n)$.

Corollary 5.5. The map $H^\bullet H \to \mathcal{M}^{\bullet,\bullet}$ is an isomorphism.

Proof. There is a $\lim^1$-sequence

$$0 \longrightarrow \lim^1_n \widetilde{H}^{p-1+2n,q+n}(K_n, \mathbb{Z}/\ell) \longrightarrow H^{p,q}H \longrightarrow \lim_n \widetilde{H}^{p+2n,q+n}(K_n, \mathbb{Z}/\ell) \longrightarrow 0$$

By the theorem above $(\mathbb{Z}/\ell)^{\mathrm{tr}}K_n \simeq \Sigma^{2n,n}H_n$ for a split proper Tate H_n of weight ≥ 0. One computes that

$$\widetilde{H}^{p-1+2n,q-1+n}(K_n, \mathbb{Z}/\ell) \cong [\Sigma^\infty H_n, \Sigma^{p-1,q}(\mathbb{Z}/\ell)^{\mathrm{tr}}\mathbb{1}].$$

It is a general fact that

$$\bigoplus_n H_n \longrightarrow \bigoplus_n H_n \longrightarrow \operatorname{colim}_n H_n$$

is split by Voevodsky. So the $\lim^1$-term vanishes. □

Corollary 5.6. There is a splitting $H \otimes H \simeq \bigwedge_\alpha \Sigma^{p_\alpha,q_\alpha}H$ as H-modules.

Proof. The object $\operatorname{colim}_n H_n$ is split proper Tate by work of Voevodsky of weight ≥ 0 and $\operatorname{colim}_n H_n \simeq \bigoplus_\alpha (\mathbb{Z}/\ell)^{\mathrm{tr}}S^{p_\alpha,q_\alpha}$. Thus,

$$(\mathbb{Z}/\ell)^{\mathrm{tr}}(H) \simeq \Sigma^\infty \operatorname*{colim}_n H_n \simeq (\mathbb{Z}/\ell)^{\mathrm{tr}} \bigoplus_\alpha \Sigma^\infty S^{p_\alpha,q_\alpha}.$$

So $(\mathbb{Z}/\ell)^{\mathrm{tr}}(H)$ is cellular which is $\Phi(H \otimes H)$ for $\Phi : \mathbf{Mod}_H \to \mathcal{SH}^{\mathrm{tr}}(\mathscr{C}, \mathbb{Z}/\ell)$. Therefore, we get $H \otimes H \simeq H \otimes \bigoplus_\alpha \Sigma^\infty S^{p_\alpha,q_\alpha}$. □

5.1 Comparison with Étale Steenrod Algebra

Let k be algebraically closed. Let $a_{\text{ét}} : \mathcal{H}^*(\mathbf{Sch}_k) \to \mathcal{H}^*_{\text{ét}}(\mathbf{Sch}_k)$ and $u_{\text{ét}} : D(\mathbf{Sm}_k) \to D(\mathbf{Sm}_k)$. There is an isomorphism

$$\mathbb{Z}(1)[1] \simeq よ_{\mathbb{G}_m} \in D_{\mathrm{Nis}}(\mathbf{Sm}_k).$$

By Kummer, if $m \in k^\times$, there is an isomorphism $a_{\text{ét}}\mathbb{Z}/m(1) \simeq \mu_m$ which implies

$$a_{\text{ét}}(\mathbb{Z}/m(q)[p]) \simeq \mu_m^{\otimes q}[p].$$

So $\widetilde{H}^{p,q}(X, \mathbb{Z}/m) \cong \widetilde{H}^p_{\mathrm{Nis}}(X, \mathbb{Z}/m(q)) \to \widetilde{H}^p_{\text{ét}}(X, \mu_m^{\otimes q})$ for all pointed presheaves.

Theorem 5.7. This is an isomorphism for $p \leq q$.

Corollary 5.8. Let $X \in \mathcal{H}^*(\mathbf{Sm}_k)$. Étale sheafification induces an isomorphism

$$\widetilde{H}^{\bullet,\bullet}(X, \mathbb{Z}/\ell)[\tau^{-1}] \cong H^\bullet_{\text{ét}}(X, \mu_\ell^{\otimes \bullet})$$

where $\tau \in \mu_\ell(k)$ is a primitive root of unity with $\tau \in H^{0,1}(\operatorname{Spec} k, \mathbb{Z}/\ell)$.

Let $\mathcal{H}_{\text{ét}}^{\bullet,\bullet} = \lim_n H^{\bullet+2n}(K_n^{\text{ét}}, \mu_\ell^{\otimes \bullet+n})$ with connecting maps coming from $\mathbb{P}^1 \otimes K_n^{\text{ét}} \to K_{n+1}^{\text{ét}}$. Since k is algebraically closed, μ_ℓ is constant, so $K_n^{\text{ét}}$ is constant. Since $\operatorname{Spec} k$ has no non-trivial covers, $R\Gamma : \mathcal{H}_{\text{ét}}(\mathbf{Sm}_k) \to \mathcal{S}$ is given by evaluation at $\operatorname{Spec} k$. Thus, $\mathrm{id} \to R\Gamma \circ c$ is an isomorphism. Thus,

$$\widetilde{H}^\bullet(K_n^{\text{ét}}, A) \cong H_{\text{ét}}^\bullet\left(K(\mu_\ell(k)^{\otimes 2n}, n), A\right) \cong \widetilde{H}_{\mathbf{Top}}^\bullet\left(K(\mu_\ell(k)^{\otimes 2n}, n), A\right).$$

Take $A = \mathbb{Z}/\ell$. This gives an isomorphism of algebras $\chi : \mathcal{A}^\bullet \to \mathcal{M}_{\text{ét}}^{\bullet,0}$ where $\mathcal{A}^\bullet$ is the topological Steenrod algebra.

This allows us to define $P_{\text{ét}}^i = \tau^{i(\ell-1)}\chi(P^i)$. Note that $\mathcal{H}_{\text{ét}}^{\bullet,\bullet}$ is generated by $P_{\text{ét}}^i, \beta_{\text{ét}}^i$ and $\tau^{\pm 1}$. Use that $\tau : \mathcal{M}_{\text{ét}}^{\bullet,n} \to \mathcal{H}_{\text{ét}}^{\bullet,n+1}$. This defines a map $\psi : \mathcal{A}^{\bullet,\bullet} \to \mathcal{H}_{\text{ét}}^{\bullet,\bullet}$.

Lemma 5.9. The diagram

$$\begin{tikzcd} \mathcal{A}^{\bullet,\bullet} \arrow[r] \arrow[dr, "\psi"'] & \mathcal{H}^{\bullet,\bullet} \arrow[d] \\ & \mathcal{H}_{\text{ét}}^{\bullet,\bullet} \end{tikzcd}$$

commutes.

Proof of 5.1.

(i) The theorem holds over any perfect field by Voevodsky, but we deal with algebraically closed k. It suffices to prove:

(a) The map $\mathcal{A}^{\bullet,\bullet}/\tau \to \mathcal{H}^{\bullet,\bullet}/\tau$ is injective.

(b) The map $\mathcal{A}^{\bullet,\bullet}[\tau^{-1}] \to \mathcal{H}^{\bullet,\bullet}[\tau^{-1}]$ is surjective.

We omit how to see that this is sufficient.

(a) This was shown by Voevodsky. For every $P = \sum_I a_I P^I$ a space X and $w \in H^{\bullet,\bullet}(X, \mathbb{Z}/\ell)$ such that $P(w) \neq 0$. The example $X = (B\mu_\ell)^N$ for $N \gg 0$ works.

(b) Consider the commutative diagram

$$\begin{tikzcd} \mathcal{A}^{\bullet,\bullet}[\tau^{-1}] \arrow[r] \arrow[dr] & \mathcal{H}^{\bullet,\bullet}[\tau^{-1}] \arrow[r] \arrow[d, dashed, "\varphi"] & \widetilde{H}^{\bullet+2n,\bullet+n}(K_n, \mathbb{Z}/\ell)[\tau^{-1}] \arrow[d] \\ & \mathcal{H}_{\text{ét}}^{\bullet,\bullet} \arrow[r] & \widetilde{H}_{\text{ét}}^{\bullet+2n}(K_n^{\text{ét}}, \mathbb{Z}/\ell) \end{tikzcd}$$

It suffices to see that φ is injective. Let $x = (x_0, x_1, \cdots) \in \mathcal{H}^{\bullet,\bullet}[\tau^{-1}]$ and $\operatorname{supp} \varphi(x) = 0$. Thus, $x_n \mapsto 0$ in $\widetilde{H}^{\bullet+2n,n}(K_n, \mathbb{Z}/\ell)[\tau^{-1}]$. But $(\mathbb{Z}/\ell)^{\text{tr}} K_n$ is split proper Tate, so there is no τ-torsion, whence $x_n = 0$.

(iii) We compute

$$\mathcal{H}^{\bullet,\bullet} \cong [H, \Sigma^{\bullet,\bullet} H] \cong [H \otimes H, \Sigma^{\bullet,\bullet} H]_H \cong \prod_\alpha H^{\bullet - p_\alpha, \bullet, q_\alpha}.$$

Omit the result for general base schemes.

□

5.2 Back to Fundamental Square

Definition 5.10.

(i) An **fpsl cover** is a faithfully flat cover $f : U \to X$ such that $f_*\mathcal{O}_U$ is free of rank prime to ℓ.

(ii) An **ldh-cover** is a cover which cdh-locally is fpsl.

Theorem 5.11. Let $X \in \mathbf{Sch}_k$. There exists an ldh-cover by smooth & quasi-projective schemes.

Proposition 5.12. The map $i^* : \mathcal{H}^{\mathrm{tr}}(\mathbf{Sch}_k, R) \to \mathcal{H}^{\mathrm{tr}}(\mathbf{Sm}_k, R)$ preserves $R^{\mathrm{tr}}W_{\mathrm{ldh}}$-local equivalences.

Finally, the hard part of the theorem:

Corollary 5.13. Let R be a $\mathbb{Z}_{(\ell)}$-algebra. Then,

$$\begin{tikzcd} \mathcal{H}^{\mathrm{tr}}(\mathbf{Sm}_k, R) \arrow[rr] \arrow[dr] & & \mathcal{H}^{\mathrm{tr}}_{\mathrm{Nis},\mathbb{A}^1}(\mathbf{Sm}_k, R) \\ & \mathcal{H}^{\mathrm{tr}}_{\mathrm{ldh}}(\mathbf{Sm}_k, R) \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Proof of 5.3. By Voevodsky we can assume that $\mathscr{C} = \mathbf{Sm}_k$ and $\mathscr{D} = \mathbf{Sch}_k$. It suffices to show that $R^{\mathrm{tr}}i^*F \to i^*R^{\mathrm{tr}}F$ is an isomorphism for $F \in \mathcal{H}^*_{\mathrm{Nis}}(\mathscr{C})$. Consider

$$\begin{tikzcd} R^{\mathrm{tr}}i^*i_!i^* \arrow[r] \arrow[d] & R^{\mathrm{tr}}i^* \arrow[d] \\ i'^*R^{\mathrm{tr}}i_!i^* \arrow[r] & i^*R^{\mathrm{tr}} \end{tikzcd}$$

The top and left maps are equivalences. It's left to show that the bottom map is an equivalence. When restricted to $F \in \mathbf{Sm}_k$ the map $i_!i^*F \to F$ is an equivalence. So $i_!i^*F \to F$ is a ldh-local equivalence. So $i^*R^{\mathrm{tr}}i_!i^*F \to i^*R^{\mathrm{tr}}F$ is an $R^{\mathrm{tr}}W_{\mathrm{ldh}}$-local equivalence, but these become equivalences in $\mathcal{H}^{\mathrm{tr}}_{\mathrm{Nis},\mathbb{A}^1}(\mathscr{D}, R)$. $\square$

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