Six Functor Formalisms

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Abstract

These are my (live) TeX'd notes for the six functor formalisms seminar in Bonn, WiSe 2025/26. There is a more thorough abstract in the seminar program but it's pretty long, so summarized we are going to cover the following:

We start by setting up six functor formalisms following Heyer-Mann [HM24]. Then, we discuss an example on locally compact Hausdorff spaces via Verdier duality following Volpe [Vol21]. We end with an example from the theory of p-adic Lie groups resulting in the linearization hypothesis after Clausen [Cla25].

My notation and language is not always consistent with the speakers' choices. I also occassionally added some parts which were not included in the actual talks; such parts will always be indicated by a star like Lemma*.

Feel free to send me feedback. :-)

Contents

1 Spans and Six-Functor Formalisms (Thomas Blom)

Thomas changed the title of the talk and uses *span* instead of *correspondence* which has long had a meaning in category theory, namely a functor to [1]. TALK 1 16.10.2025

Kaif: *Also, span is another word that starts with* **Sp**.

1.1 What is a six functor formalism, morally?

Let $\mathscr{C}$ be a 'category of geometric objects', e.g. $\mathscr{C} = \mathbf{LCH}$. A *six functor formalism* on $\mathscr{C}$ consists of:

- an '∞-category of sheaves $D(X)$' for every $X \in \mathscr{C}$,
- $D(X)$ is closed symmetric monoidal, so there is $\otimes, \underline{\mathrm{Map}}(-,-), \mathbb{1}$,
- any $f : Y \to X$ induces a 'pullback' functor $f^* : \mathcal{D}(X) \to \mathcal{D}(Y)$ which has a right adjoint f_*, the *pushforward*,
- for 'any' $f : Y \to X$ there is an *exceptional pushforward* $f_! : D(Y) \to D(X)$ which has a right adjoint $f^!$, called *exceptional pullback*.

These satisfy:

- functoriality,
- pullback is strong symmetric monoidal, in particular $f^*(M \otimes N) \simeq f^*M \otimes f^*N$,
- proper base change: For a pullback square

$$\begin{tikzcd} X \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "f"] \\ Y \arrow[r, "g"'] & W \end{tikzcd}$$

  in $\mathscr{C}$ there is an equivalence $g^! f_* \simeq f'_*(g')^!$ or equivalently $f^* g_! \simeq g'_!(f')^*$ after passing to adjoints,
- projection formula: $M \otimes f_! N \simeq f_!(f^*M \otimes N)$.

Example 1.1. Let $\mathscr{C} = \mathbf{LCH}$. Consider $D(X) = \mathbf{Sh}(X, \mathcal{D}\mathbb{Z})$. A map $f : Y \to X$ then gives

- $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$,
- $(f^*\mathcal{F})(U) = L_{\mathbf{Sh}}\left(\mathrm{colim}_{V \supseteq f(U) \text{ open}} \mathcal{F}(V)\right)$,[1]
- $(f_!\mathcal{F})(U) = \mathrm{colim}_{\substack{K \subseteq f^{-1}(U) \\ K \to U \text{ proper}}} \mathrm{fib}(\mathcal{F}(f^{-1}(U)) \to \mathcal{F}(f^{-1}(U \setminus K)))$[2],
- $f^!$ is kind of mysterious.

Suppose now for simplicity that $\mathscr{C}$ has a terminal object $*$. For $X \in \mathscr{C}$ we will almost always write $p : X \to *$.

[1] You want to evaluate on $f(U)$ but it's not open, so we approximate it by opens. Then, it need not be a sheaf, so we sheafify.

[2] If f is an open embedding, then this is extension by 0.

Definition 1.2. Let D be a six functor formalism on $\mathscr{C}$. For $X \in \mathscr{C}$ and $A \in D(X)$ we define

$$\Gamma(X, A) = p_* A \quad \text{and} \quad \Gamma_c(X, A) = p_! A.$$

Usually, $A = p^* \mathbb{1} = \mathbb{1}$.

Example 1.3. We have

$$H^\bullet(X, \mathbb{Z}) = p_* p^* \mathbb{Z} \quad \text{and} \quad H_c^\bullet(X, \mathbb{Z}) = p_! p^* \mathbb{Z}.$$

You can also do homology and get

$$H_\bullet(X, \mathbb{Z}) = p_! p^! \mathbb{Z} \quad \text{and} \quad H_\bullet^{\mathrm{lf}}(X, \mathbb{Z}) = p_! p^! \mathbb{Z}.$$

Example 1.4 (Künneth)**.** Consider the pullback square

$$\begin{tikzcd} X \times Y \arrow[r, "f_Y"] \arrow[d, "f_X"'] \arrow[dr, "p"] & Y \arrow[d, "p_Y"] \\ X \arrow[r, "p_X"'] & * \end{tikzcd}$$

then

$$\begin{aligned} \Gamma_c(X \times Y) &= p_! \mathbb{1} \\ &\simeq (p_Y)_! (f_A)_! f_X^* \mathbb{1} \\ &\simeq (p_Y)_! p_Y^* (p_X)_! \mathbb{1} \\ &\simeq (p_Y)_! (p_Y^* (p_X)_! \mathbb{1} \otimes \mathbb{1}) \\ &\simeq (p_X)_! \mathbb{1} \otimes (p_Y)_! \mathbb{1} \\ &\simeq \Gamma_c(X) \otimes \Gamma_c(Y) \end{aligned}$$

Example 1.5 (Poincaré duality)**.** Let $\omega_X = p^! \mathbb{1}$. Then, $\Gamma(X, \omega_X) \simeq \underline{\mathrm{Map}}(\Gamma_c(X), \mathbb{1})$.

Proof. We perform a Yoneda argument, so

$$\begin{aligned} \mathrm{Map}(M, \Gamma(X, \omega_X)) &\simeq \mathrm{Map}(M, p_* p^! \mathbb{1}) \\ &\simeq \mathrm{Map}(p_! p^* M, \mathbb{1}) \\ &\simeq \mathrm{Map}(M \otimes p_! \mathbb{1}, \mathbb{1}) \\ &\simeq \mathrm{Map}(M, \underline{\mathrm{Map}}(\Gamma_c(X), \mathbb{1})). \end{aligned}$$

$\square$

If X is orientable, then one can check $\omega_X \simeq \mathbb{1}[n]$, so this suggests Poincaré duality (but certainly this is not yet a proof of the Poincaré duality at this point because you need to prove this equivalence and show the existence of the 6FF and so on).

1.2 What is a six functor formalism, really?

The above 'definition' should hurt from the viewpoint of a homotopy theorist. Let's do it more coherently. Here is a 'pre-definition'.

Definition 1.6. Let $\mathscr{C}$ be an ∞-category with finite limits. Then, a **3FF** is a lax symmetric monoidal functor $D : (\mathbf{Span}(\mathscr{C}), \otimes) \to (\mathbf{Cat}_\infty, \times)$.

We take the cartesian monoidal structure on $\mathscr{C}$ but this does not induce the cartesian monoidal structure on $\mathbf{Span}(\mathscr{C})$.

Notation 1.7. We write

- $f^* = D(X \xleftarrow{f} Y = Y) : D(X) \to D(Y)$,
- $f_! = D(Y = Y \xrightarrow{f} X) : D(Y) \to D(X)$,
- There is a symmetric monoidal structure $(\otimes, \mathbb{1})$ on $D(X)$.[3]

Definition 1.8. A **6FF** is a 3FF such that $f^*, f_!$ admit right adjoints and $\otimes$ is closed.

At this point all exceptional pushforwards exist. The categorical fix is to simply mark those edges for which it should exist.

Definition 1.9. A **geometric setup** is a pair $(\mathscr{C}, \mathscr{E})$ where

(i) $\mathscr{E} \subset \mathscr{C}$ is a wide subcategory,

(ii) pullbacks of maps in $\mathscr{E}$ along any map exist in $\mathscr{C}$ and they lie in $\mathscr{E}$ again,

(iii) $\mathscr{E}$ has pullbacks and $\mathscr{E} \hookrightarrow \mathscr{C}$ preserves these.[4]

(iii') Equivalently: If $f, f \circ g \in \mathscr{E}$, then $g \in \mathscr{E}$.

Definition 1.10. The category **Span**$(\mathscr{C}, \mathscr{E})$ is the full subcategory of **Span**$(\mathscr{C})$ on those spans $\bullet \xleftarrow{f} \bullet \xrightarrow{g} \bullet$ such that $g \in \mathscr{E}$.

Remark 1.11. If we only start with a geometric setup, then we really assume only the existence of pullbacks along maps in $\mathscr{E}$ to be available in $\mathscr{C}$, so it need not have all pullbacks. In that sense, **Span**$(\mathscr{C})$ is ill-defined but this is not so bad, we can freely add in limits – e.g. we can define **Span**$(\mathscr{C}, \mathscr{E})$ as a full subcategory of **Span**$(\mathbf{PSh}(\mathscr{C}))$.

Now, we can redefine 3FF to:

Definition 1.12. Let $\mathscr{C}$ be an ∞-category with finite limits with a geometric setup $(\mathscr{C}, \mathscr{E})$. Then, a **3FF** is a lax symmetric monoidal functor $D : (\mathbf{Span}(\mathscr{C}, \mathscr{E}), \otimes) \to (\mathbf{Cat}_\infty, \times)$.

Proposition 1.13. If $\mathscr{C}$ has finite products, then $\times$ defines a symmetric monoidal structure on **Span**$(\mathscr{C}, \mathscr{E})$.

If $\mathscr{C}$ does not have finite products, then one obtains an ∞-operad **Span**$(\mathscr{C}, \mathscr{E})$. Heyer-Mann spend a lot of effort on this [HM24, Proposition 2.3.3] but according to Thomas one could also just enlarge **Span**$(\mathscr{C}, \mathscr{E})$ to a category with finite products and then take the suitable suboperad.

1.3 Sanity Check

Let's unpack our abstract definition and recover all those desired properties of 6FFs.

(i) The composite

$$(\mathscr{C}^{\mathrm{op}})^{\amalg} \longrightarrow \mathbf{Span}(\mathscr{C}, \mathscr{E}) \longrightarrow \mathbf{Cat}_\infty^\times$$

[3] In $\mathscr{C}^\times$ every objects admits a preferred cocommutative coalgebra structure via the diagonal and by the backwards functoriality via $(-)^*$ it becomes a commutative algebra in **Span**$(\mathscr{C})$. So the lax symmetric monoidality of D sends this to a commutative algebra in **Cat**.

[4] In part (ii) the pullback need not be a pullback in $\mathscr{E}$ because the unique maps coming from the universal property are in $\mathscr{C}$ and need not be in $\mathscr{E}$.

is lax symmetric monoidal.[5] By [Lur17, Theorem 2.4.3.18] this is equivalent to a functor $\mathscr{C}^{\mathrm{op}} \to \mathbf{CMon}(\mathbf{Cat}_\infty)$. This gives $\otimes$ and strong symmetric monoidality on f^*.

(ii) There is an equivalence

$$D(X \xleftarrow{f} Y = Y) \circ D(Z \xleftarrow{g} X = X) \simeq D(Z \xleftarrow{gf} Y = Y)$$

which gives functoriality on f^*. Similarly $f_!$.

(iii) There is an equivalence

$$D(X \xleftarrow{f} Y \xrightarrow{g} Z) \simeq D(Y = Y \xrightarrow{g} Z) \circ D(X \xleftarrow{f} Y = Y) \simeq g_! f^*.$$

With

$$\begin{tikzcd}
 & & X \arrow[dl, "f'"'] \arrow[dr, "g'"] & & \\
 & Y \arrow[dl, equal] \arrow[dr, "g"'] & & W \arrow[dl, "f"] \arrow[dr, equal] & \\
Y & & Z & & W
\end{tikzcd}$$

we obtain $g'_!(f')^* \simeq f^* g_!$, i.e. proper base change.

(iv) The projection formula is a bit trickier.

$$\begin{tikzcd}
 & & Y \arrow[dl, "(f,\mathrm{id}_Y)"'] \arrow[dr, "f"] & & \\
 & X \times Y \arrow[dl, equal] \arrow[dr, "\mathrm{id}_X \times f"'] & & X \arrow[dl, "\Delta_X"] \arrow[dr, equal] & \\
X \times Y & & X \times X & & X
\end{tikzcd}$$

Thus,

$$\Delta_X^* \circ (\mathrm{id} \times f)_! \simeq f_!(f, \mathrm{id})^* \simeq f_! \circ \Delta_Y^* \circ (f \times \mathrm{id})^*.$$

Using symmetric monoidality of pullbacks we deduce $(-) \otimes f_!(-) \simeq f_!(f^*(-) \otimes (-))$.

1.4 Constructing a 6FF

Often, one can construct some pullback/pushforwards, etc. by hand – see e.g. our example for **LCH** – and so it is desirable that one could glue these together into a 6FF.

Definition 1.14. Let $(\mathscr{C}, \mathscr{E})$ be a geometric setup. A **suitable decomposition** of $\mathscr{E}$ is a pair $I, P \subset \mathscr{E}$ of wide subcategories such that:

(i) $P \circ I = \mathscr{E}$,

(ii) $(\mathscr{C}, I), (\mathscr{C}, P)$ are geometric setups,

(iii) Any $f \in I \cap P$ is n-truncated for some $n \geq -2$.

Condition (iii) is rather technical but at least in the example **LCH** is not relevant since this is an honest $(1,1)$-category.

[5] The first functor is strong symmetric monoidal.

Example 1.15. Let $\mathscr{C} = \mathbf{LCH}$. Any $f : X \to Y$ is a composite

$$X \xrightarrow{i} \overline{X} \xrightarrow{p} Y$$

of an open embedding and a proper map. Indeed, take $\overline{X}$ as the closure of $\Gamma(f) \subseteq X^+ \times Y$.

Theorem 1.16. Let $(\mathscr{C}, \mathscr{E})$ be a geometric setup admitting finite products and $I, P \subset \mathscr{E}$ be a suitable decomposition. Let $D : \mathscr{C}^{\mathrm{op}} \to \mathbf{CMon}(\mathbf{Cat}_\infty)$ be a functor. Suppose:

(i) For $j : U \to X$ in I the functor j^* admits a left adjoint $j_!$ such that the following are satisfied:

(a) Base change: Given a pullback

$$\begin{tikzcd} U' \arrow[r, "i"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "f"] \\ U \arrow[r, "j"'] & X \end{tikzcd}$$

the Beck-Chevalley map $i_! g^* \Rightarrow f^* j_!$ is an equivalence.[6]

(b) Projection: The preferred map $j_!(j^*(-) \otimes (-)) \Rightarrow (-) \otimes j_!(-)$ is an equivalence.

(ii) For $p : Y \to X$ in P the functor p^* admits a right adjoint p_* such that the following are satisfied:

(a) Base change: Given a pullback

$$\begin{tikzcd} Y' \arrow[r, "q"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "f"] \\ Y \arrow[r, "p"'] & X \end{tikzcd}$$

the Beck-Chevalley map $f^* p_* \Rightarrow q_* g^*$ is an equivalence.

(b) Projection: The preferred map $(-) \otimes p_*(-) \Rightarrow p_*(p^*(-) \otimes (-))$ is an equivalence.

(iii) For a pullback square

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "p \in P"] \\ U \arrow[r, "j \in I"'] & X \end{tikzcd}$$

the Beck-Chevalley map $j_! q_* \Rightarrow p_* i_!$ is an equivalence.

Then, D extends to a 3-functor formalism $D : \mathbf{Span}(\mathscr{C}, \mathscr{E}) \to \mathbf{Cat}_\infty$.

One observation is that for this to have any chance to be true, we will have $j^* \simeq j^!$.

There are some extended ways to look at this for which we also obtain uniqueness then [CLL25, DK25]. These solve some really fundamental problems because Liu-Zheng's work is really hard to understand, as the combinatorics going on in their papers is too involved.

Example 1.17. We get a 3FF on $\mathbf{LCH}$ given by $X \mapsto \mathbf{Sh}(X, \mathcal{D}\mathbb{Z})$.

You can still get a 6FF which can be done using Verdier duality. Thomas is not aware of a different way than this to obtain the 6FF on $\mathbf{LCH}$.

[6] Note that this is a property!

1.5 Extending 6FF

This is the last part that we forced Thomas to speak about.

Definition 1.18. Let D be a 3FF on $(\mathscr{C}, \mathscr{E})$ and $\mathscr{C}$ be a site. Then, D is **sheafy** if $D^* : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ is a sheaf.

Proposition 1.19. Let D be a sheafy 3FF on $(\mathscr{C}, \mathscr{E})$ and $\mathscr{C}$ be subcanonical. Let $\mathscr{E}' \subseteq \mathbf{Sh}(\mathscr{C})$ be those maps $f : T \to R$ such that for all $X \in \mathscr{C}$ and all maps $X \to R$ the map $T \times_R X \to X$ lies in $\mathscr{E}$. Then,

(i) $(\mathscr{C}, \mathscr{E}) \to (\mathbf{Sh}(\mathscr{C}), \mathscr{E}')$ is a map of geometric setups,

(ii) D extends to a sheafy 3-functor formalism D' on $(\mathbf{Sh}(\mathscr{C}), \mathscr{E}')$,[7]

(iii) if D is a presentable 6FF, then so is D'.

Remark 1.20. The condition on $\mathscr{E}'$ is essentially that it consists of those maps which locally lie in $\mathscr{E}$. That's one of the main points for the above results. These decompositions into open immersions and proper maps may not be possible but only possible locally.

In practice, $\mathscr{E}'$ is often chosen much larger and indeed, there are ways of enlargening $\mathscr{E}'$ which often is quite specific to the 3FF we consider.

Remark* 1.21. There is another extension result allowing 'stacky' maps [HM24, Theorem 3.4.11] which was not mentioned in this talk. It might feature in future talks.

2 Kernel Categories 1 (Jonah Epstein)

TALK 2
23.10.2025

2.1 Recollection on Enriched and $(\infty, 2)$-Categories

There are several ways to set this up; we follow Gepner-Haugseng [GH15].

Recall that a colored operad/multicategory $\mathscr{M}$ consists of objects and for $X_1, \cdots, X_n, Y \in \mathscr{M}$ a set of multimorphisms $\mathscr{M}(X_1, \cdots, X_n; Y)$ together with identity, composition and associativity assumptions.

Example 2.1. Let S be a set. Then, there is a multicategory $\mathscr{O}_S$ defined as follows:

(i) Objects: $S \times S$,

(ii) Maps: We have

$$\mathscr{O}_S\left((X_0, Y_1), (X_1, Y_2), \cdots, (X_{n-1}, Y_n); (X_0, Y_n)\right) = \begin{cases} * & Y_i = X_i \quad \text{for all } i \\ \varnothing & \text{else.} \end{cases}$$

This multicategory is supposed to encode composition.

Definition 2.2. An **enriched category** with objects S over a monoidal category $\mathscr{V}$ is an $\mathscr{O}_S$-algebra in $\mathscr{V}$.

So this is a map $\mathscr{O}_S \to \mathscr{V}$ and the intuition is that on objects we have $(X, Y) \mapsto \mathrm{Hom}^{\mathscr{V}}(X, Y)$ and on multimorphisms

$$\left(((X, Y), (Y, Z)) \to (X, Z)\right) \mapsto \left(\mathrm{Hom}^{\mathscr{V}}(X, Y) \otimes \mathrm{Hom}^{\mathscr{V}}(Y, Z) \to \mathrm{Hom}^{\mathscr{V}}(X, Z)\right).$$

This now generalizes to the ∞-world. If we only want a set of objects, then we can take $\mathscr{O}_S$ and directly use the ∞-version of the above definition [GH15, Definition 2.2.17]. For spaces, Gepner-Haugseng define a generalized ∞-operad Δ_S^{op} [GH15, after Remark 2.4.4].

[7] Recall that a contravariant functor starting from an ∞-topos is a *sheaf* if it is limit-preserving.

Definition* 2.3. Let $S \in \mathcal{S}$. Consider $\Delta^{\mathrm{op}} \to \mathcal{S}$, $[n] \mapsto S^{\times n}$. It can be checked to satisfy the Rezk-Segal conditions [GH15, after Remark 2.4.4], so it unstraightens to a double ∞-category[8] $\Delta_S^{\mathrm{op}} \to \Delta^{\mathrm{op}}$.

So we want to consider the following definition:

Definition 2.4. Let $S \in \mathcal{S}$ and $\mathscr{V}$ be a monoidal ∞-category. A **$\mathscr{V}$-enriched ∞-category** with space of objects S is an Δ_S^{op}-algebra in $\mathscr{V}$.

To define the ∞-category of enriched ∞-categories, we want one for all possible $S \in \mathcal{S}$.

There exists a cartesian fibration $\mathbf{Alg}(\mathscr{V}) \to \mathbf{Op}_\infty^{\mathrm{ns,gen}}$ whose fiber over $\mathscr{O}$ is $\mathbf{Alg}_{\mathscr{O}}(\mathscr{V})$.

Definition 2.5.

(i) The pullback

$$\begin{tikzcd}
\mathbf{Alg}_{\mathrm{cat}}(\mathscr{V}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbf{Alg}(\mathscr{V}) \arrow[d] \\
\mathcal{S} \arrow[r, "\Delta_-^{\mathrm{op}}"'] & \mathbf{Op}_\infty^{\mathrm{ns,gen}}
\end{tikzcd}$$

is the **∞-category of categorical algebras** in $\mathscr{V}$.

(ii) We let $\mathbf{Enr}_{\mathscr{V}} \subseteq \mathbf{Alg}_{\mathrm{cat}}(\mathscr{V})$ be the reflective subcategory where fully faithful and essentially surjective functors are inverted. This is the ∞-category of **$\mathscr{V}$-∞-categories**.

For (ii) we really want to impose some condition forcing $S \simeq \mathscr{C}^{\mathrm{core}}$ for a $\mathscr{V}$-enriched category $\mathscr{C}$. That's why we don't take $\mathbf{Alg}_{\mathrm{cat}}(\mathscr{V})$ but rather $\mathbf{Enr}_{\mathscr{V}}$.

We can transfer enrichments.

Construction 2.6. Let $\alpha : \mathscr{V} \to \mathscr{W}$ be lax monoidal. Then, there is a functor

$$\tau_\alpha : \mathbf{Enr}_{\mathscr{V}} \to \mathbf{Enr}_{\mathscr{W}},\ X \mapsto X,\ \mathrm{Map}^{\mathscr{V}}(X,Y) \mapsto \alpha\left(\mathrm{Map}^{\mathscr{V}}(X,Y)\right).$$

Remark* 2.7. If $\mathscr{V}$ is a presentably monoidal ∞-category, then $\mathrm{Map}_{\mathscr{V}}(\mathbb{1}_{\mathscr{V}}, -) : \mathscr{V} \to \mathcal{S}$ is lax monoidal [GH15, Example 4.3.20], so we can transfer enrichments to obtain the underlying ∞-category in this case. Does some variant work if $\mathscr{V}$ is not presentable?

Remark 2.8. If $\mathscr{V}$ is closed monoidal, then, $\mathscr{V}$ is enriched over itself with mapping objects $\mathrm{Map}^{\mathscr{V}}(X,Y) = \underline{\mathrm{Map}}_{\mathscr{V}}(X,Y)$.

Comment.* It seems like Gepner-Haugseng need to pass to a different model to show this [GH15, Corollary 7.4.10]. In Lurie's model of enriched ∞-categories this is also discussed in [HM24, Example C.1.12]. □

Example 2.9.

(i) $\mathbf{Cat}_\infty$ is self-enriched.

(ii) $\mathbf{Span}(\mathscr{C})$ is self-enriched with $\underline{\mathrm{Map}}_{\mathbf{Span}(\mathscr{C})}(X,Y) = X \times Y$.

Definition 2.10. We write $\mathbf{Cat}_{(\infty,2)} = \mathbf{Enr}_{\mathbf{Cat}_\infty}$ as the ∞-category of $(\infty,2)$-categories.

Definition 2.11. A map $f : Y \to X$ is an $(\infty,2)$-category $\mathscr{C}$ is **left adjoint** if there exists a map $g : X \to Y$ and 2-morphisms $\eta : \mathrm{id}_Y \Rightarrow gr$ and $\varepsilon : fg \Rightarrow \mathrm{id}_X$ called **(co-)unit** such that the diagrams

$$\begin{tikzcd}
f \arrow[r, Rightarrow, "f\eta"] \arrow[dr, equal] & fgf \arrow[d, Rightarrow, "\varepsilon f"] \\
& f
\end{tikzcd}
\qquad
\begin{tikzcd}
g \arrow[r, Rightarrow, "\eta g"] \arrow[dr, equal] & gfg \arrow[d, Rightarrow, "g\varepsilon"] \\
& g
\end{tikzcd}$$

commute.

[8] I.e. a generalized non-symmetric ∞-operad $\Delta_S^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ which is a coCartesian fibration.

2.2 Category of Kernels

Throughout the entire talk, assume that $(\mathscr{C}, \mathscr{E})$ is a geometric setup with finite products.

Definition 2.12. Let $D : \mathbf{Span}(\mathscr{C}, \mathscr{E}) \to \mathbf{Cat}_\infty$ be a 3FF.

(i) If $\mathscr{E} = \text{all}$, let $\mathscr{K}_D = \tau_D(\mathbf{Span}(\mathscr{C})) \in \mathbf{Cat}_{(\infty,2)}$ be the **$(\infty, 2)$-category of kernels**.

(ii) Let $S \in \mathscr{C}$ and put $\mathscr{C}_\mathscr{E} \subset \mathscr{C}$ be the subcategory spanned by E. Then, $(\mathscr{C}_{\mathscr{E}/S}, E)$ is a geometric setup and there is a map of geometric setups $(\mathscr{C}_{\mathscr{E}/S}, \text{all}) \to (\mathscr{C}, \mathscr{E})$ which thus induces a 3FF

$$D_S : \mathbf{Span}\left(\mathscr{C}_{\mathscr{E}/S}\right) \to \mathbf{Span}(\mathscr{C}, \mathscr{E}) \to \mathbf{Cat}_\infty$$

which hence allows us to define

$$\mathscr{K}_{D,S} = \tau_{D_S}\left(\mathbf{Span}(\mathscr{C}_{\mathscr{E}/S})\right) \in \mathbf{Cat}_{(\infty,2)},$$

the **$(\infty, 2)$-category of kernels**.

Concretely, $\mathscr{K}_{D,S}$ is the following $(\infty, 2)$-category:

- Objects: Maps $X \to S$ in $\mathscr{E}$,
- Maps: $\mathrm{Fun}_S(Y, X) = D(X \times_S Y)$,
- Composition: Let $N \in D(X \times_S Y), M \in D(Y \times_S Z)$. Consider $X \times_S Y \times_S Z$ along with all the possible projections. Then, $M \circ N = (\pi_{13})_!(\pi_{12}^* M \otimes \pi_{23}^* N) \in D(X \times_S Z)$.

Note the self-symmetry. It suggests the following.

Remark 2.13 ([HM24, Proposition 4.1.4]). There is an equivalence $\mathscr{K}_{D,S}^{\mathrm{op}} \simeq \mathscr{K}_{D,S}$.

Proposition 2.14 ([HM24, Proposition 4.1.5]). The functor D_S from 2.12 splits as

$$\mathbf{Span}(\mathscr{C}_{\mathscr{E}/S}) \xrightarrow{\Phi_{D,S}} \mathscr{K}_{D,S} \xrightarrow{\Psi_{D,S}} \mathbf{Cat}_{(\infty,2)}$$

where the functors are given as follows:

- The functor $\Phi_{D,S}$ is id on objects and on morphisms we start with $f = [Y \leftarrow Z \to X]$ which induces a map $f' : Z \to X \times_S Y$ and we put $\Phi_{D,S}(f) = f'_!(\mathbb{1}) \in D(X \times_S Y)$.
- Put $\Psi_{D,S}(X) = D(X)$ on objects and on maps let $M \in \mathrm{Fun}_S(Y, X) = D(X \times_S Y)$, then we put
$$\Psi_{D,S}(M) = (\pi_1)_! \left(M \otimes \pi_2^*(-)\right) : D(Y) \to D(X).$$

This M is often called *kernel* of the Fourier-Mukai transform $(\pi_1)_!(M \otimes \pi_2^*(-))$.

Remark* 2.15. Kevin Lin's answer in https://mathoverflow.net/questions/9834 is a great way to motivate this terminology. Recall that a classical Fourier transform is a function $g(y) = \int f(x)e^{2\pi ixy}\, dx$. Here, M resembles $f(x)$ and $e^{2\pi ixy}$ resembles $\pi_2^*(-)$. Integration along the fiber comes from the pushforward. The term $f(x)$ is typically called *integral kernel* and kernel here does not have the meaning from algebra but rather stands for the central object in this integration – in German *Kern* (der Sache). See https://mathoverflow.net/questions/24098.

Remark 2.16 ([HM24, Section 4.2]). This construction is functorial in D and S.

2.3 Descent

Recall that for a sieve $\mathscr{U} \subseteq \mathscr{C}_{/U}$ a functor $\mathcal{F} : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ *satisfies descent* if $\mathcal{F}(U) \xrightarrow{\simeq} \lim_{V \in \mathscr{U}^{\mathrm{op}}} \mathcal{F}(V)$.

Definition 2.17. A functor $\mathcal{F} : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ satisfies **universal descent** along $\mathscr{U}$ if it has descent along all pullbacks

$$\begin{tikzcd} f^*\mathscr{U} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{U} \arrow[d] \\ \mathscr{C}_{/U'} \arrow[r] & \mathscr{C}_{/U} \end{tikzcd}$$

on $\mathscr{U}$.

If $D : \mathbf{Span}(\mathscr{C}, \mathscr{E}) \to \mathbf{Cat}_\infty$ is a 3FF, then we write $D^* : \mathscr{C}^{\mathrm{op}} \to \mathbf{Span}(\mathscr{C}, \mathscr{E}) \to \mathbf{Cat}_\infty$.

Proposition 2.18 ([HM24, Proposition 4.3.1, 4.3.3])**.**

(i) Let D be a 3FF and $\mathscr{U}$ be a sieve for $X \in \mathscr{C}$. Suppose that D^* satisfies universal descent along $\mathscr{U}$. Then,

$$\mathscr{K}_{D,X} \to \lim_{U \in \mathscr{U}^{\mathrm{op}}} \mathscr{K}_{D,U}$$

is fully faithful.

(ii) Let $\mathscr{U}$ be a sieve on $X \in \mathscr{C}$ and suppose $\mathscr{E} =$ all. Suppose that D^* satisfies universal descent along $\mathscr{U}$. Then,

$$\operatorname*{colim}_{U \in \mathscr{U}} U \xrightarrow{\simeq} X$$

in $\mathscr{K}_D$.

2.4 Suave/Prim Objects and Morphisms

We obtain interesting interactions between $*$ and $!$.

Definition 2.19. Let D be a 3FF on $(\mathscr{C}, \mathscr{E})$. Fix a map $f : X \to S$ in $\mathscr{E}$ and $P \in D(X)$.

(i) We say that P is **f-suave** if it is a left adjoint morphism in $\mathscr{K}_{D,S}$. We write $\mathrm{SD}_f(P) \in D(X)$ for the right adjoint, called **f-suave dual** of P.

(ii) We say that P is **f-prim** if it is a right adjoint morphism in $\mathscr{K}_{D,S}$. We write $\mathrm{PD}_f(P) \in D(X)$ for the left adjoint, called **f-prim dual** of P.

Remark 2.20. Let $\mathrm{id}_X : X \to X$. Then, $P \in D(X)$ is (id_X-)suave if and only if it is prim if and only if it is dualizable.

Proof.* Each of those three conditions corresponds to the existence of $Q \in \mathrm{Fun}_X(X, X) \simeq D(X)$ with commuting diagrams

$$\begin{tikzcd} P \arrow[r, Rightarrow] \arrow[dr, equal] & P \otimes Q \otimes P \arrow[d, Rightarrow] \\ & P \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[r, Rightarrow] \arrow[dr, equal] & Q \otimes P \otimes Q \arrow[d, Rightarrow] \\ & Q \end{tikzcd}$$

so we win. □

Lemma 2.21 ([HM24, Lemma 4.4.5])**.** Let D be a 6FF. Let $(f : X \to S) \in \mathscr{E}$ and $P \in D(X)$. Then, P is f-suave if and only if the natural map

$$\pi_1^* \underline{\mathrm{Map}}_{D(X)}(P, f^! \mathbb{1}) \otimes \pi_2^* P \to \underline{\mathrm{Map}}_{D(X \times_S X)}(\pi_1^* P, \pi_2^! P)$$

becomes an equivalence after applying $\mathrm{Map}_{D(X)}(\mathbb{1}, \Delta^!(-))$. Then, $\mathrm{SD}_f(P) \simeq \underline{\mathrm{Map}}_{D(X)}(P, f^! \mathbb{1})$.

Here, $\underline{\mathrm{Map}}$ is one of the six functors. There is a similar criterion for primness [HM24, Lemma 4.4.6]. In practice it's often not so hard that these two objects are equivalent but rather that it comes from this natural map.

Heyer-Mann [HM24, Section 4.4] show a bunch of additional stuff about these such as locality on source and target. These are maybe meant to be introduced when we really need to apply them.

Lemma 2.22 ([HM24, Lemma 4.4.18]).

(i) Suppose that $(\Delta_f)_!\mathbb{1} \in D(X \times_S X)$ is compact, then f-suave objects in $D(X)$ are compact.

(ii) Suppose that $\mathbb{1} \in D(S)$ is compact, then f-prim objects in $D(X)$ are compact.

Remark* 2.23. You can also deduce some general duality-type statement and relation between suave and prim duals [HM24, Lemma 4.4.17, 4.4.19]. We see a special case of it in 2.26.

2.5 Suave & Prim Maps

We have just discussed suave and prim objects. Now we discuss suave and prim maps.

Definition 2.24. Let D be a 3FF. Let $f : Y \to X$ be a map in $\mathscr{E}$.

(i) Then, f is **D-suave** if $\mathbb{1} \in D(Y)$ is f-suave. We call $\omega_f = \mathrm{SD}_f(\mathbb{1}) \in D(Y)$ the **dualizing complex**.

(ii) Then f is **D-prim** if $\mathbb{1} \in D(Y)$ is f-prim. We call $\delta_f = \mathrm{PD}_f(\mathbb{1}) \in D(Y)$ the **codualizing complex**.

(iii) A D-suave map f is **D-smooth** if ω_f is invertible.

The dualizing complex is relatively common in geometry but as of now this is not really the case for the codualizing complex.

Lemma 2.25 ([HM24, Lemma 4.5.4, 4.5.5]). Let D be a 6FF and let $\pi_1, \pi_2 : Y \times_X Y \to Y$.

(i) Then, $f : Y \to X$ is D-suave if and only if $\pi_1^* f^! \mathbb{1}_{D(X)} \to \pi_2^! \mathbb{1}_{D(Y)}$ is an equivalence. In this case, $\omega_f \simeq f^! \mathbb{1}_{D(X)}$.

(ii) Then, $f : Y \to X$ is D-prim if and only if $f_!(\pi_2)_* \Delta_! \mathbb{1}_{D(Y)} \to f_* \mathbb{1}_{D(Y)}$ is an equivalence. In this case, $\delta_f \simeq (\pi_2)_* \Delta_! \mathbb{1}_{D(Y)}$.

Now, finally the interaction of $*$ and !; they are related by a twist given suaveness/primness conditions.

Proposition 2.26 ([HM24, Corollary 4.5.11]). Let D be a 6FF.

(i) If f is D-suave, then $\omega_f \otimes f^* \simeq f^!$ and $f^* \simeq \underline{\mathrm{Map}}_{D(Y)}(\omega_f, f^!)$.

(ii) If f is D-prim, then $f_!(\delta_f \otimes -) \simeq f_*$ and $f_! \simeq f_* \underline{\mathrm{Map}}_{D(Y)}(\delta_f, -)$.

Theorem 2.27 (General base change, [HM24, Lemma 4.5.13]). Let

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"'] & X \end{tikzcd}$$

be a pullback diagram in $\mathscr{C}_{\mathscr{E}}$.

(i) If g is D-suave, then the natural maps

$$g^* f_* \stackrel{\simeq}{\Rightarrow} f'_* g'^*, f'_! g'^! \stackrel{\simeq}{\Rightarrow} g^* f_!, f'^* g^! \stackrel{\simeq}{\Rightarrow} g'^! f^*, g'^* f^* \stackrel{\simeq}{\Rightarrow} f'^! g^*$$

are equivalences.

(ii) If g is D-prim, then the natural maps

$$f^* g_* \stackrel{\simeq}{\Rightarrow} g'_* f^*, g'_! f'^! \stackrel{\simeq}{\Rightarrow} f^! g_!, g_! f'_* \stackrel{\simeq}{\Rightarrow} f_* g'_!, f_! g'_* \stackrel{\simeq}{\Rightarrow} g_* f'_!$$

are equivalences.

Proposition 2.28 ([HM24, Corollary 4.5.18])**.** Let $(f : X \to S) \in \mathscr{E}$.

(i) If f is D-suave, then every dualizable object $P \in D(X)$ is f-suave and $\mathrm{SD}_f(P) \simeq P^\vee \otimes \omega_f$.

(ii) If Δ_f is D-suave, then every f-suave object $P \in D(X)$ is dualizable and in this case, $\mathrm{SD}_f(P) \simeq P^\vee \otimes \omega_{\Delta_f}^{-1}$.

There are some more results in [HM24, Section 4.5] and it's best to have them introduced when we actually need them. All results of this talk are essentially proved by pure abstract nonsense.

3 Category of Kernels 2 (Maria Stroe)

3.1 Étale & Proper Maps

TALK 3
06.11.2025

Recall (2.26): Let D be a 6FF on $(\mathscr{C}, \mathscr{E})$ with $(f : Y \to X) \in \mathscr{E}$.

(i) If f is D-suave, then $\omega_f \otimes f^* = \mathrm{SD}_f(\mathbb{1}) \otimes f^* \simeq f^! : D(X) \to D(Y)$.

(ii) If f is D-prim, then $f_!(\delta_f \otimes -) \simeq f_* : D(Y) \to D(X)$.

We wish to discuss notions that trivialize these twists (3.3) – that's the point of étale and proper maps.

Definition 3.1. Let D be a 3FF on $(\mathscr{C}, \mathscr{E})$ and let $(f : Y \to X) \in \mathscr{E}$ be a truncated map.

(i) We say that f is **D-étale** if it is D-suave and Δ_f is D-étale or an equivalence.

(ii) We say that f is **D-proper** if it is D-prim and Δ_f is D-proper or an equivalence.

We can make this inductive definition by the inductive nature of truncatedness: f is n-truncated if Δ_f is $(n-1)$-truncated and (-2)-truncated if it is an equivalence.

Remark* 3.2. Suppose that f is n-truncated. Then, f is D-étale if $f, \Delta_f, \Delta_{\Delta_f}, \cdots$ are all D-suave (eventually, we get an equivalence).

The next result shows that twists are trivialized in the étale/proper setting.

Proposition 3.3. Let D be a 6FF on $(\mathscr{C}, \mathscr{E})$ and $(f : Y \to X) \in \mathscr{E}$ such that Δ_f is D-étale. Then, there exists a preferred natural transformation $f^! \Rightarrow f^*$ of functors $D(X) \to D(Y)$ such that TFAE:

(i) f is D-étale.

(ii) $f^! \mathbb{1}_X \to f^* \mathbb{1}_X$ is an equivalence in $D(Y)$.

(iii) $f^! \Rightarrow f^*$ is an equivalence of functors $D(X) \to \mathcal{D}(Y)$

Proof. Induction on truncatedness n of f. By induction hypothesis suppose that $\Delta_f^! \simeq \Delta_f^*$.

Let's discuss the base step, so say that $g : X \to Y$ is an equivalence. By base change on

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g^{-1}"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

we get $g^! \simeq g^*$.

Then, we construct

$$f^! \simeq \Delta_f^* \pi_2^* f^! \simeq \Delta_f^! \pi_2^* f^! \Rightarrow \Delta_f^! \pi_1^! f^* \simeq f^*$$

with

$$\begin{tikzcd} Y \arrow[dr, "\Delta_f"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, equal] & & \\ & Y \times_X Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\ & Y \arrow[r, "f"'] & X \end{tikzcd}$$

Here, we use functoriality and an explicit natural transformation $\pi_2^* f^! \Rightarrow \pi_1^! f^*$ which we now construct. For this, we use the two ingredients:

- Base change: $\pi_{1!}\pi_2^* \overset{\simeq}{\Rightarrow} f^* f_!$,
- $f_! \dashv f^!, \pi_{1!} \dashv \pi_1^!$.

So we can write

$$\pi_2^* f^! \overset{\eta_{\pi_1}}{\Longrightarrow} \pi_1^! \pi_{1!} \pi_2^* f^! \overset{\simeq}{\Rightarrow} \pi_1^! f^* f_! f^! \overset{\varepsilon_f}{\Rightarrow} \pi_1^! f^*.$$

Now let's start the proof.

(ii) $\Longrightarrow$ (i): We want to show that f is D-suave. Consider the map

$$\pi_2^* f^! \mathbb{1}_X \to \pi_1^! f^* \mathbb{1}_X \simeq \pi_1^! \mathbb{1}_Y.$$

By (ii) it is an equivalence after applying $\Delta_f^!$, so we are done by [HM24, Lemma 4.5.4].

(i) $\Longrightarrow$ (iii): From suave base change (2.27) we get that $\pi_2^* f^! \Rightarrow \pi_1^! f^*$ is an equivalence by suave base change and applying $\Delta_f^!$ to this implies $f^! \overset{\simeq}{\Rightarrow} f^*$.

□

There is a similar result for properness by replacing $(-)^!, (-)^*$ by $(-)_!, (-)_*$ [HM24, Lemma 4.6.4(ii)].

Lemma 3.4. Let D be a 3FF on $(\mathcal{C}, \mathcal{E})$.

(i) Then, D-étale maps are stable under base change and composition, and if $f, g \in \mathcal{E}$ and fg, Δ_f are D-étale, then g is D-étale.

(ii) The D-étaleness of a truncated map is D^*-local on the target. If D is compatible with small colimits, then we can check D-étaleness of f on a universal D^*-cover of D-étale maps in $\mathcal{C}_{\mathcal{E}}$ of the source.

(ii') An $f \in \mathscr{E}$ is D-proper if it is so on a universal D-proper D^*-cover on the source such that $f_!$ commutes with $\mathscr{U}^{\mathrm{op}}$-indexed limits.

Proof.

(i) Let's only do base change. Suppose that $f : Y \to X$ is D-étale and that $f' : Y' \to X'$ is a base change of f. Induction on n. Since f is n-truncated, also f' is n-truncated.

- f' is D-suave: D-suaveness is preserved under base change.
- $\Delta_{f'}$ is D-étale: Note that $\Delta_{f'}$ is $(n-1)$-truncated and a pullback of Δ_f, so by induction hypothesis $\Delta_{f'}$ is D-étale.

(ii) Assume that f is n-truncated and locally D-étale on some universal D^*-cover $\mathscr{U}$ of the target. Then, f is D-suave since D-suaveness is stable under base change. We're left to show that Δ_f is D-étale. Consider the diagram

$$\begin{tikzcd} V \arrow[r, "\Delta_{f_U}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V \times_U V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & U \arrow[d] \\ Y \arrow[r, "\Delta_f"'] & Y \times_X Y \arrow[r, "\pi"'] & X \end{tikzcd}$$

where $\pi = f \circ \pi_1 = f \circ \pi_2$. By assumption, Δ_{f_U} is D-étale, so Δ_f is locally D-étale on the universal D^*-cover $\pi^*\mathscr{U}$.

□

3.2 Descendability and Exceptional Descent

Goal: Suave and prim maps are good sources of $*$-covers and !-covers. To begin, let's recall some descent statements.

Definition 3.5 (Descent data)**.** Let $\mathscr{C}, \mathscr{V} \in \mathbf{Cat}_\infty$ and $\mathscr{U} \subseteq \mathscr{C}_{/U}$ be a sieve and $\mathcal{F} : \mathscr{C}^{\mathrm{op}} \to \mathscr{V}$. Then, we write

$$\mathrm{Desc}(\mathscr{U}, \mathcal{F}) = \lim_{V \in \mathscr{U}^{\mathrm{op}}} \mathcal{F}(V).$$

We say that $\mathcal{F}$ **descends along** $\mathscr{U}$ if $\mathcal{F}(U) \xrightarrow{\simeq} \mathrm{Desc}(\mathscr{U}, \mathcal{F})$.

Definition 3.6. Let D be a 6FF on $(\mathscr{C}, \mathscr{E})$.

(i) We say that $\mathscr{U} \subseteq (\mathscr{C}_{\mathscr{E}})_{/U}$ is a $D^!$**-cover** if it is generated by a small family $\{U_i \to U\}_i$ and $D^!$ descends along $\mathscr{U}$.

(ii) We say that $\mathscr{U}$ is a **universal** $D^!$**-cover** if for every $V \to U$ in $\mathscr{C}$ the family $\{U_i \times_U V \to V\}$ generates a $D^!$-cover.

(iii) A map $f : Y \to X$ is a **(universal)** $D^!$**-cover** if the sieve generated by f is a (universal) $D^!$-cover.

Lemma 3.7. Let D be a 6FF on $(\mathscr{C}, \mathscr{E})$ and $(f : Y \to X) \in \mathscr{E}$ be such that $D(X)$ has all countable limits and colimits. If f is D-suave and $f^* : D(X) \to D(Y)$ is conservative, then f is a universal !-cover and $*$-cover.[9]

[9] We need limits for ! and colimits for $*$.

Proof. One can view $D^!$-descent as

$$D^!(X) \xrightarrow{\simeq} \lim_{[n]\in\Delta} D^!(Y_n)$$

with $Y_n = Y^{\times_X (n+1)}$. We use Lurie's Beck-Chevalley condition [Lur17, Corollary 4.7.5.3]:

(1) The ∞-category $D(X)$ admits geometric realizations of $f^!$-split simplicial objects and these geometric realizations are preserved by $f^!$.

(2) Let $\alpha : [m] \to [n]$, then

$$\begin{tikzcd} D^!(Y_m) \arrow[r, "d^{0!}_{(m)}"] \arrow[d, "\alpha^!"'] \arrow[dr, phantom, "\lrcorner", very near start] & D^!(Y_{m+1}) \arrow[d, "\alpha^!"] \\ D^!(Y_n) \arrow[r, "d^{0!}_{(n)}"'] & D^!(Y_{n+1}) \end{tikzcd}$$

is left adjointable where $d^0 : Y_{m+1} \simeq Y_m \times_X Y \to Y_m \times_X X$.

Here is why:

(1) By assumption, $D(X)$ has all countable colimits. By D-suaveness, $f^!(-) \simeq \omega_f \otimes f^*(-)$ and f^* is a left adjoint, so it preserves colimits. Thus, $f^!$ preserves colimits.

(2) By suave base change $d^0_{(n)!}\alpha'^! \xRightarrow{\simeq} \alpha^! d^0_{(m)!}$ (see 2.27).

By (1) & (2) we get that $D^!(X) \to \lim_{n\in\Delta} D^!(Y_n)$ has a fully faithful left adjoint. Moreover, $f^* \simeq \underline{\mathrm{Map}}_{D(Y)}(\omega_f, f^!)$, so $f^!$ (see 2.26) is also conservative. $\square$

There is a similar criterion for prim maps but it is slightly more involved. To discuss this, we will take a detour first.

3.3 Mathew's Notion of Descent

Let us take a quick detour to Mathew's descendability notion [Mat16].

Definition 3.8. Let $(\mathscr{C}, \otimes, \mathbb{1})$ be a symmetric monoidal stable ∞-category and $A \in \mathbf{CAlg}(\mathscr{C})$. We write $\langle A \rangle \subseteq \mathscr{C}$ for the thick $\otimes$-ideal generated by A.

(i) We say that A is **descendable** if $\mathbb{1}_{\mathscr{C}} \in \langle A \rangle$.

(ii) Suppose furthermore that $\mathscr{C}$ is presentable and that $\otimes$ commutes with all colimits in both variables. Then, $\mathscr{C}$ is called a **stable homotopy theory**.

(iii) Let $\mathscr{C}$ be a stable homotopy theory. Then, $A \to B$ **admits descent** if B is descendable in $\mathbf{CAlg}(\mathbf{Mod}_A(\mathscr{C}))$.

In particular, A being descendable means that $\mathbb{1} \to A$ admits descent.

Example 3.9.

(i) Let $R \in \mathbf{CRing}$, then $\mathcal{D}(R)$ is a stable homotopy theory.

(ii) Let X be a scheme or a prestack. Then, $\mathbf{QCoh}_X$ is a stable homotopy theory.

Definition 3.10. Let $\mathscr{C}$ be an ∞-category with finite colimits.

(i) A filtered diagram $F : I \to \mathscr{C}$ is **ind-constant** if it lies in the essential image of $\mathscr{C} \to \mathrm{Ind}(\mathscr{C})$.

(ii) A simplicial object $M_\bullet : \Delta^{\mathrm{op}} \to \mathscr{C}$ is **ind-constant** if $\mathbb{Z}_{\geq 0} \to \mathscr{C}$, $n \mapsto \lim_{m \in \Delta_{\leq n}} M_n$ is ind-constant.

There is a dual notion for *pro-constant cofiltered diagrams* and *pro-constant cosimplicial objects*.

Definition 3.11. Let $A \in \mathbf{CAlg}(\mathscr{C})$. We write

$$\mathrm{CB}^\bullet(A) = (A \rightrightarrows A \cdots)$$

for the **cobar resolution**.

Proposition 3.12 ([Mat16, Proposition 3.20])**.** An object $A \in \mathbf{CAlg}(\mathscr{C})$ is descendable if and only if $\mathrm{CB}^\bullet(A)$ defines a constant pro-object on $\{\mathrm{Tot}_n \mathrm{CB}^\bullet(A)\}_n$ which converges to $\mathbb{1}$ and $(\mathbb{1}_{\mathscr{C}})_n \to \{\mathrm{Tot}_n \mathrm{CB}^\bullet(A)\}_n$ is a pro-isomorphism.

Proposition 3.13 ([Mat16, Proposition 3.22])**.** Let $\mathscr{C}$ be a stable homotopy theory and A be descendable. Then, the adjunction

$$\mathscr{C} \underset{U}{\overset{A\otimes -}{\rightleftarrows}} \mathbf{Mod}_A(\mathscr{C})$$

is comonadic. In particular,

$$\mathscr{C} \xrightarrow{\simeq} \mathrm{Tot}\left(\mathbf{Mod}_A(\mathscr{C}) \rightrightarrows \mathbf{Mod}_{A\otimes A}(\mathscr{C}) \cdots\right).$$

3.4 Back to Six Functor Formalisms

One can state Mathew's notions slightly more generally, namely in a stable monoidal ∞-category $\mathscr{C}$ instead of in $\mathbf{CAlg}(\mathscr{C})$.

Lemma 3.14 ([HM24, Lemma 4.7.4])**.** Let D be a stable 6FF and $(f : Y \to X) \in \mathscr{E}$ such that $D(X)$ has all countable colimits and limits. Suppose that f is D-prim and that $f_*\mathbb{1} \in D(X)$ is descendable.

(i) Then, f is a universal D^*-cover and $D^!$-cover.

(ii) Every $f^!$-split simplicial object in $D(X)$ is ind-constant and every f^*-split cosimplicial object in $D(X)$ is pro-constant.

Proof Idea. We want to show that $f_! f^! \in \mathrm{Fun}(D(X), D(X))$ is descendable (but not in the algebra category). This can be reduced into descendability of $f_*\mathbb{1}$. □

4 Six-Functor Formalism on Condensed Anima (Gabriel Ong)

We will work light today, i.e. set $\kappa = \aleph_1$. Some set-theoretic technicalities go away then. TALK 4 13.11.2025

4.1 Condensed Math

We want topological algebra to behave better categorically.

Example 4.1. The category **TopAb** is not an abelian category, but **Cond**(**Ab**) is.

We try to rebuild ordinary algebra in condensed land. Instead of sets, abelian groups, rings, ... consider condensed sets, condensed abelian groups, condensed rings and so on. But really you should look at *analytic rings*. These date back to old ideas like Johnstone's observation that **CHaus** $\to$ **Set** is monadic.

Definition 4.2.

(i) Let **ProFin** be the full subcategory of the 1-category of topological spaces spanned by sequential limits of finite sets.[10]

(ii) It becomes a site with covers the finitely jointly surjective families.

Definition 4.3. A **condensed anima** X is a hypersheaf of anima on **ProFin**. This gives rise to an ∞-category **Cond(An)**.

Definition 4.4.

(i) A **surjection** of condensed anima is an effective epimorphism.

(ii) A **quasicompact** condensed anima is an object if all covers[11] admits a finite subcover.

(iii) A condensed anima X is **quasiseparated** if for $Y, Z \to X$ with Y, Z qc also $Y \times_X Z$ is qc.

For $X \in \mathbf{Top}$ consider $\mathrm{Hom}_{\mathbf{Top}}(-, X) : \mathbf{ProFin}^{\mathrm{op}} \to \mathbf{Set}$. This is a condensed set.

Proposition 4.5.

(i) This construction gives a fully faithful embedding

$$(\text{metrizable compactly generated spaces}) \hookrightarrow \mathbf{Cond}(\mathbf{An}).$$

(ii) This restricts to an equivalence

$$(\text{metrizable compact Hausdorff spaces}) \simeq \mathbf{Cond}(\mathbf{Set})^{\mathrm{qcqs}}.$$

We shall briefly remark that metrizable spaces are already compactly generated.

4.2 Six Functor Formalism for Λ-Sheaves

Recall that singular cohomology $H^i(X; \mathbb{Z})$ can be written as sheaf cohomology $H^i(X; \underline{\mathbb{Z}})$ for nice enough spaces.

Definition 4.6. Let S be a finite set and Λ be a ring. We denote by $\Lambda(S) = \prod_{x \in S} \Lambda$ the set of Λ-valued continuous functions.

Proposition 4.7 ([HM24, Construction 3.5.16, Lemma 3.5.12])**.**

(i) The functor $\mathbf{Fin}^{\mathrm{op}} \to \mathbf{CAlg}(\mathbf{Pr}^L_{\mathrm{st}})$, $S \mapsto \mathbf{Mod}_{\Lambda(S)}$ extends to a 6FF on **ProFin** given by $\lim_i S_i \mapsto \mathrm{colim}_i \mathbf{Mod}_{\Lambda(S_i)}$.

(ii) This 6FF satisfies hyperdescent.

Proof.

(i) Use monadicity and construction from suitable decompositions with $E = P = (\text{all})$ and $I = (\text{equiv})$.

(ii) We do this for static Λ. More generally, run Lurie's faithfully flat descent [Lur18, Appendix D]. It's enough to show that for every surjection $S' \twoheadrightarrow S$ between profinite sets that $\Lambda(S) \to \Lambda(S')$ is faithfully flat. Pass to presentations and check this termwise.

[10] These are light profinite sets!

[11] These are defined by surjections from (i).

□

There are two ways to extend to stacks.

(i) New !-able maps to those locally !-able in the site.

(ii) More generally, take E' to be bigger than this locally !-able class.

Here is an example from scheme theory.

Example 4.8. Let $\mathscr{C} = \mathbf{AffSch}$. Then, $\mathbb{P}^1_{\mathbb{Z}}$ is a stack but pullback over $\operatorname{Spec} k \to \operatorname{Spec} \mathbb{Z}$ gives $\mathbb{P}^1_k$ but $\mathbb{P}^1_k \to \operatorname{Spec} k$ is not an affine map.

This suggests that (i) is often not so useful.[12]

Theorem 4.9 ([HM24, Theorem 3.4.11])**.** Let $D_0 : \mathbf{Span}(\mathscr{C}, \mathscr{E}) \to \mathbf{Pr}^L$ be a 6FF with (hyper)descent for (hyper)subcanonical $\mathscr{C}$. Then, there exists some (minimal choice of) $\mathscr{E}'$ which is

- $*$-local on the target,
- !-local on the source or target,
- tame, i.e. every map $f : Y \to X$ in $\mathscr{E}'$ with $X \in \mathscr{C}$ is !-locally on the source in $\mathscr{E}$.

such that D_0 extends uniquely to $D : \mathbf{Span}(\mathscr{X}, \mathscr{E}') \to \mathbf{Cat}_\infty^\times$.

Applied to $D(-,\Lambda) : \mathbf{Span}(\mathbf{ProFin}) \to \mathbf{Cat}_\infty$ we deduce:

Theorem 4.10. There is a collection of maps $\mathscr{E}'$ in $\mathbf{Cond}(\mathbf{An})$ uniquely extending the 6FF on $\mathbf{ProFin}$ to $D : \mathbf{Span}(\mathbf{Cond}(\mathbf{An}), \mathscr{E}') \to \mathbf{Cat}_\infty^\times$ where:

(i) *-local: An $f : X \to Y$ lies in $\mathscr{E}'$ if and only if for every representable S there exists $S \to Y$ we have $X \times_Y S \to S$ is in $\mathscr{E}'$.

(ii) !-local: Membership in $\mathscr{E}'$ can be checked on after composition or pullback with a universal !-cover.

(iii) Tame: Maps $f : X \to S$ are in $\mathscr{E}'$ if and only if there exists a !-cover of X and the composition of f with any map in the cover is in $\mathscr{E}$.

Definition 4.11. The **6FF for Λ-sheaves on $\mathbf{Cond}(\mathbf{An})$** is the one from the theorem (4.10). The maps in $\mathscr{E}'$ are called **Λ-fine**.

Definition 4.12. Let Λ be a ring and $f : X \to *$ for $X \in \mathbf{Cond}(\mathbf{An})$.

(i) The **Λ-valued cohomology** is $\Gamma(X, \Lambda) = f_* \mathbb{1}_X \in \mathbf{Mod}_\Lambda$.

(ii) The **compactly supported Λ-valued cohomology** is $\Gamma_c(X, \Lambda) = f_! \mathbb{1}_X \in \mathbf{Mod}_\Lambda$ for Λ-fine f.

Similarly, cohomology of sheaves.

[12] However, one could first try to enlarge $\mathscr{C}$ and then apply (i).

4.3 Poincaré Duality

As usual in the stacky world, an open/closed immersion of condensed anima is a map which after pulling back along a representable is an open/closed subset.

Lemma 4.13. Let $\Lambda \in \mathbf{CRing}$.

(i) Every open immersion of condensed anima is étale.

(ii) Every map from a profinite set to a qs condensed set is proper. In particular, every closed immersion of condensed anima is proper.

Proof.

(i) By locality reduce to profinite sets $f : X \to Y$. Reduce further to f being an inclusion of clopens. Any open is a finite disjoint union of clopens in the light setting. The functors have an explicit description here, f_* is the restriction, check explicitly with unit and counit.

(ii) Check after pulling back to a profinite set. Consider $f : S \to X$ and $T \to X$. Show that $S \times_X T \to T$ is proper. Consider the subset $S \times_X T \subseteq X \times T$. The source and target are profinite and by conditions from last time we can deduce that this is proper.

$\square$

Proposition 4.14. Let X be a metrizable compact Hausdorff space and

$$U \xrightarrow{j \text{ open}} X \xleftarrow{i \text{ closed}} Z$$

with $X = U \sqcup Z$ in **Set**. Then,

$$j_! \mathbb{1}_U \longrightarrow \mathbb{1}_X \longrightarrow i_* \mathbb{1}_Z$$

is a fiber sequence in $D(X, \Lambda)$.

Proof. Base change shows $j^* i_* \mathbb{1}_Z \simeq 0$. So $j^* \operatorname{fib}(\mathbb{1}_X \to i_* \mathbb{1}_Z) \simeq j^* \mathbb{1}_X \simeq \mathbb{1}_U$. Adjunction gives a map

$$j_! \mathbb{1}_U \to \operatorname{fib}(\mathbb{1}_X \to i_* \mathbb{1}_Z).$$

Since this was a disjoint decomposition, (j^*, i^*) is a conservative family of functors [HM24, Lemma 4.8.5]. So we are done. $\square$

Lemma 4.15. Let $f : [0,1] \to *$ be the projection.

(i) Then, f is proper.

(ii) Let X be a metrizable locally compact Hausdorff space and let $P : X \times [0,1] \to X$. Then, P^* is fully faithful.

Proof.

(i) Consider $g : \{0,1\}^{\mathbb{N}} \twoheadrightarrow [0,1]$. Then, g is proper and $f \circ g : \{0,1\}^{\mathbb{N}} \to *$ is proper. By cancellation we need to show that $g_* \mathbb{1}_{\{0,1\}^{\mathbb{N}}}$ is descendable.

For $n > 0$ let C_n be the disjoint union of 2^n closed intervals, e.g. $C_1 = [0,1/2] \amalg [1/2,1]$, with $\{0,1\}^{\mathbb{N}} \cong \lim_n C_n$. Write $h_n : C_n \to [0,1]$. Then, $g_* \mathbb{1}_{\{0,1\}^{\mathbb{N}}} \simeq \operatorname{colim}_n h_{n*} \mathbb{1}_{C_n}$, so it's enough to show that $h_{n*} \mathbb{1}_{C_n}$ is descendable. The pushforward of $\mathbb{1}$ on this closed cover is descendable in this setting.

$\square$

Theorem 4.16. Let $f : X \to *$ be the projection from a manifold X. Then, ω_X is locally equivalent to $\Lambda[-n]$ where n is the local dimension.

Proof. By locality, it's enough to consider $X = \mathbb{R}^n$. Can further reduce to $X = \mathbb{R}$. Consider $j : \mathbb{R} \xrightarrow{\sim} (0,1) \hookrightarrow [0,1]$. We have $f_* j_! \mathbb{1}_{\mathbb{R}} \simeq \Gamma_c(\mathbb{R}, \Lambda)$ is the fiber of $\Gamma([0,1], \Lambda) \to \Gamma(\partial[0,1], \Lambda)$ which is $\Lambda[-1]$. Thus, $\omega_X \simeq \Lambda[-n]$ by locality. $\square$

Corollary 4.17. Let X be a manifold and $\Lambda \in \mathbf{CRing}$. Then, $\Gamma(X, \omega_X) \simeq \Gamma_c(X, \Lambda)^\vee$.

Proof. We have $\Gamma(X, \omega_X) \simeq f_* f^! \mathbb{1} \simeq f_* \underline{\mathrm{Map}}_X(\mathbb{1}_X, f^! \mathbb{1}_*) \simeq \mathrm{Map}_*(f_! \mathbb{1}_X, \mathbb{1}_*) \simeq \Gamma_c(X, \Lambda)^\vee$. $\square$

5 Applications to (Smooth) Representation Theory (Qi Zhu)

Surely, you know representation theory. Now, you just have to be smooth about it!

TALK 5
20.11.2025

5.1 Smooth Representation Theory through Condensed Anima

5.1.1 Condensed Anima & Classifying Stacks

Recall from last talk that **Cond(An)** $= \mathbf{Sh}^{\mathrm{hyp}}(\mathbf{ProFin})$ and the six functor formalism of condensed anima.

Recollection 5.1. Let $\Lambda \in \mathbf{CAlg}$, then the universal property of Ind yields a diagram

$$\begin{tikzcd} * \arrow[r, "\Lambda"] \arrow[d, hook, "\langle 1 \rangle"'] & \mathbf{CAlg} \\ \mathbb{F}^{\mathrm{op}} \arrow[ur, dashed] \arrow[d, hook] & \\ \mathbf{ProFin}^{\mathrm{op}} \arrow[uur, dotted, bend right, "(S_i)_i \mapsto \mathrm{colim}_i \prod_{x \in S_i} \Lambda"'] & \end{tikzcd}$$

and postcomposing with $\mathbf{Mod}_{(-)}$ gives $D(-, \Lambda) : \mathbf{ProFin}^{\mathrm{op}} \to \mathbf{Cat}_\infty$. This can be extended to $\mathbf{Cond}(\mathbf{An})^{\mathrm{op}}$ and then to a six functor formalism $D(-, \Lambda) : \mathbf{Span}(\mathbf{Cond}(\mathbf{An}), \Lambda\text{-fine}) \to \mathbf{Cat}_\infty$ [HM24, Construction 3.5.16].

The ∞-topos $\mathbf{Cond}(\mathbf{An})$ contains $\mathbf{Top}$ but also a homotopical direction – in particular, it allows us to form classifying stacks of topological groups. We will use this observation to study smooth representation theory of locally profinite groups.

Definition 5.2. A **locally profinite group** is a Hausdorff, locally compact, totally disconnected topological group.

So compact locally profinite groups are precisely the profinite groups.

Example 5.3. This includes profinite groups like Galois groups of (infinite) field extensions $\mathrm{Gal}(L/K)$ or the Morava stabilizer group $\mathbb{G}$, but also discrete groups, $\mathbb{Q}_p$ and p-adic Lie groups such as $\mathrm{GL}_n(\mathbb{Q}_p)$.

Let G be a locally profinite group, then it is in particular a group object in $\mathbf{Cond}(\mathbf{An})$. If it acts on some $X \in \mathbf{Cond}(\mathbf{An})$, then we can form the stacky quotient

$$X /\!\!/ G = \underset{[n] \in \Delta^{\mathrm{op}}}{\mathrm{colim}}\, G^{\times n} \times X \in \mathbf{Cond}(\mathbf{An}).$$

We will in particular care about the classifying stacks $* /\!\!/ G$. Indeed, it gives information about representation theory as follows!

5.1.2 Representation Theory

Let's define smooth representation theory!

Definition 5.4. Let G be a locally profinite group, $\Lambda \in \mathbf{CRing}$ and V be a continuous G-representation. It is **smooth** if $\mathrm{Stab}_G(v) \subseteq G$ is open for every $v \in V$. We write $\mathbf{Rep}_\Lambda(G)^\heartsuit$ for the 1-category of smooth G-representations and

$$\mathbf{Rep}_\Lambda(G) = \mathcal{D}\left(\mathbf{Rep}_\Lambda(G)^\heartsuit\right) \quad \text{and} \quad \widehat{\mathbf{Rep}}_\Lambda(G) = \widehat{\mathbf{Rep}_\Lambda(G)}$$

for its unbounded derived category and the left t-completion thereof.

Theorem 5.5 ([HM24, Proposition 5.1.12])**.** Let $\Lambda \in \mathbf{CRing}$ and G be a locally profinite group. There, there is a natural t-exact equivalence $D(* /\!\!/ G, \Lambda) \simeq \widehat{\mathbf{Rep}}_\Lambda(G)$.

Proof Idea. The proof strategy is by *derived descent from abelian descent*.

1. One first develops some general abstract nonsense to discuss the question for which $X \in \mathbf{Cond}(\mathbf{An})$ the ∞-category $D(X, \Lambda)$ is the (left t-completion[13] of the) derived category of its heart. This turns out to be true for $* /\!\!/ G$, so $D(* /\!\!/ G, \Lambda) \simeq \widehat{\mathcal{D}}\left(D(* /\!\!/ G, \Lambda)^\heartsuit\right)$ [HM24, Example 5.1.2, Proposition 5.1.8].

   Thus, it suffices to prove $D(* /\!\!/ G, \Lambda)^\heartsuit \simeq D(* /\!\!/ G, \Lambda)^\heartsuit$. In other words, it suffices to study the relevant abelian descent data to obtain derived descent.

2. To perform abelian descent one notices $D(G^n, \Lambda) \simeq \mathcal{D}\left(\mathbf{Mod}^\heartsuit_{\Lambda_c(G^n)}\right)$ where we denote by $\Lambda_c(G^n) \subseteq \Lambda_c(G^n)$ locally constant functions $G^n \to \Lambda$ with compact support [HM24, Lemma 5.1.9].[14] Writing out the descent diagram for $* /\!\!/ G$ and noting that we are working in 1-categories, we obtain that $D(* /\!\!/ G, \Lambda)^\heartsuit$ is the limit of

$$\mathbf{Mod}^\heartsuit_\Lambda \rightrightarrows \mathbf{Mod}^\heartsuit_{\Lambda_c(G)} \overset{\pi_2^*}{\underset{\pi_1^*}{\overset{m^*}{\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}}}} \mathbf{Mod}^\heartsuit_{\Lambda_c(G\times G)},$$

   i.e. abelian descent.

   At this point, writing out an equivalence $\mathbf{Rep}_\Lambda(G)^\heartsuit \to D(* /\!\!/ G, \Lambda)^\heartsuit$ is a 1-categorical problem which can be handled by hand [HM24, Proposition 5.1.12].

$\square$

Remark 5.6 ([HM24, Corollary 5.1.14, Remark 5.1.15])**.** Let $\varphi : H \to G$ be a map of locally profinite groups. This induces an adjunction

$$D(* /\!\!/ G, \Lambda) \underset{f_*}{\overset{f^*}{\rightleftarrows}} D(* /\!\!/ H, \Lambda)$$

which can be described in terms of smooth representations.

(i) The pullback f^* is the derived functor of taking a G-representation to its underlying H-representation. It is called **restriction/inflation** depending on whether f is injective or surjective.

[13] This part is automatic [HM24, Lemma 3.5.14].

[14] This result is stated for disjoint unions of profinite sets. To apply it to the locally profinite G we note that by van Dantzig's theorem there exists a compact open subgroup $K \leq G$, so we obtain a disjoint union decomposition $G = \bigsqcup_{[g]\in G/K} gK$.

(ii) If φ is the inclusion of a closed subgroup, then f_* is the right derived functor of smooth induction $R\operatorname{Ind}_H^G$. It φ is a topological quotient map with kernel U, then f_* is the right derived functor of taking U-fixed points $R(-)^U$, also denoted $(-)^U$.

(iii) The symmetric monoidal structure corresponds to the underlying tensor product of Λ-modules with diagonal G-action.

5.2 Six Functors in Representation Theory

We have already described some of the six operations. Now, we shall also describe the !-functor and discuss some of the six functor formulaic features.

5.2.1 !

Let $\Lambda \in \mathbf{CRing}$ and G be a locally profinite group, then natural maps such as $* /\!\!/ G \to *$ need not be Λ-fine, but we want shriekability to study six functor phenomena like being suave/prim. We fix this by posing mild conditions.

Definition 5.7. Let $\Lambda \in \mathbf{CRing}$.

(i) Let G be a profinite group. We call

$$\operatorname{cd}_\Lambda G = \sup\left\{n : H^n(G, V) \neq 0 \text{ for some } V \in \mathbf{Rep}_\Lambda(G)^\heartsuit\right\} \in \mathbb{N} \cup \{\infty\}$$

the **Λ-cohomological dimension** of G.

(ii) We say that a locally profinite group G has **locally finite Λ-cohomological dimension** if there exists an open profinite subgroup $K \leq G$ such that $\operatorname{cd}_\Lambda K < \infty$.

Many p-adic Lie groups satisfy this condition [HM24, Example 5.2.2].

Lemma 5.8. Let $\Lambda \in \mathbf{CRing}$.

(i) Let G be a locally profinite group and $H \leq K \leq G$ be compact subgroups with open K and (closed) H. The map $* /\!\!/ K \to * /\!\!/ G$ is Λ-étale and $* /\!\!/ H \to * /\!\!/ K$ is Λ-proper.

(ii) Let G be a profinite group with $\operatorname{cd}_\Lambda G < \infty$. Then, $* /\!\!/ G \to *$ is Λ-proper.

(iii) Let $H \to G$ be a map of locally profinite groups with locally finite Λ-cohomological dimension. Then, $* /\!\!/ H \to * /\!\!/ G$ is Λ-fine.

Proof.

(i) First note that $* \to * /\!\!/ G$ is a $*$-cover since $* \to * /\!\!/ G$ is an effective epimorphism[15] and D is sheafy. Thus, we need to check that the pullback[16] $G/K \to *$ is Λ-étale [HM24, Lemma 4.6.3(ii)].[17] This can be checked on open covers [HM24, Corollary 4.8.4(i)] but G/K is discrete, so it reduces to $* \to *$ being Λ-étale.

Similarly, for Λ-properness, we need to check that $K/H \to *$ is Λ-proper. This is true because K/H is a profinite set [HM24, Lemma 4.8.2(ii)].

(ii) We apply backwards 2-out-of-3 [HM24, Corollary 4.7.5] on

[15] This means that it is equivalent to its Čech nerve, which can be checked by hand.

[16] To compute the pullback we use the delooping $\Omega(* /\!\!/ G) \simeq G$, some pullback pastings and the LES associated to fiber sequences [NSS15, Definition 2.26].

[17] This pullback is truncated, so in particular, the map $* /\!\!/ K \to * /\!\!/ G$ is truncated.

$$* \xrightarrow{g} * /\!\!/ G \xrightarrow{f} *$$

so we need to show that g is Λ-prim, $fg = \mathrm{id}_*$, that f is truncated, Λ-proper and that $g_*\mathbb{1} \in D(* /\!\!/ G, \Lambda)$ is descendable. The first part follows from (i), the second part is clear. Truncatedness follows from $\Omega B \simeq \mathrm{id}$ [Lur09, Lemma 7.2.2.1]. and that $g_*\mathbb{1}$ is descendable requires the finite cohomological dimension [HM24, Proposition 5.2.5].

(iii) Since the shriekable maps are right cancellative (by definition of geometric setups), it suffices to check that $* /\!\!/ G \to *$ (and $* /\!\!/ H \to *$) is Λ-fine. This can be checked after restriction to $* /\!\!/ K$ for some compact open subgroup $K \leq G$ with $\mathrm{cd}_\Lambda K < \infty$.

Indeed, such $K \leq G$ exists by locally finite Λ-cohomological dimension and (i) shows that $* /\!\!/ K \to * /\!\!/ G$ is Λ-suave. It is furthermore $*$-conservative since this is just the restriction of a representation. Thus, the map is a universal !-cover and Λ-fine maps can be tested !-locally on the source. This then follows from (ii).

□

In particular, those maps $* /\!\!/ H \to * /\!\!/ G$ are shriekable, so we should describe the shrieks.

Construction 5.9. Let $\Lambda \in \mathbf{CRing}$ and let $H \leq G$ be a closed subgroup of a locally profinite group.

(i) For $V \in \mathbf{Rep}_\Lambda(H)^\heartsuit$ we set $\text{c-Ind}_H^G(V)$ as the set of elements $f : G \to V$ such that

  (a) f is locally constant,
  (b) $f(hg) = hf(g)$ for all $h \in H, g \in G$,
  (c) the image of $\operatorname{supp} f$ in $H \setminus G$ is compact.

  It becomes a smooth G-representation via the right translation action on the domain.

(ii) The functor c-Ind_H^G is exact, so we denote its derived functor by

$$\text{c-Ind}_H^G : \widehat{\mathbf{Rep}}_\Lambda(H) \to \widehat{\mathbf{Rep}}_\Lambda(G).$$

This is the **compact induction functor**.

Proposition 5.10 ([HM24, Lemma 5.4.2, Proposition 5.4.4])**.** Let $\Lambda \in \mathbf{CRing}$ and $H \leq G$ be a closed subgroup in a locally profinite group with locally finite Λ-cohomological dimension.

(i) Then, $f_! : D(* /\!\!/ H, \Lambda) \to D(* /\!\!/ G, \Lambda)$ is t-exact.

(ii) The diagram

$$\begin{tikzcd} \widehat{\mathbf{Rep}}_\Lambda(H) \arrow[r, "\text{c-Ind}_H^G"] \arrow[d, "\simeq"'] & \widehat{\mathbf{Rep}}_\Lambda(G) \arrow[d, "\simeq"] \\ D(* /\!\!/ H, \Lambda) \arrow[r, "f_!"'] & D(* /\!\!/ G, \Lambda) \end{tikzcd}$$

  commutes.

Remark 5.11. In fact, $\widehat{\mathbf{Rep}} \simeq \mathbf{Rep}$ in this setting [HM24, Proposition 5.3.10].

5.2.2 Suave & Prim in Representation Theory

Let us describe suave and prim objects and hence recover notions of duality.

Definition 5.12. Let $\Lambda \in \mathbf{CRing}$ and let G be a locally profinite group with $f : * /\!\!/ G \to *$.

(i) Let $V \in D(* /\!\!/ G, \Lambda)$. We write $V^G = \Gamma(* /\!\!/ G, V) = f_* V$ for the **derived invariants** of V.

(ii) Suppose that G has locally finite Λ-cohomological dimension. An object $V \in D(* /\!\!/ G, \Lambda)$ is called **admissible** if $V^K \in \mathbf{Mod}_\Lambda$ is dualizable for all compact open $K \leq G$ with $\mathrm{cd}_\Lambda K < \infty$.

(iii) Suppose that G is a profinite group with $d = \mathrm{cd}_\Lambda G < \infty$. We say that it is **$\Lambda$-Poincaré** (of dimension d) if $f_* : D(* /\!\!/ G, \Lambda) \to \mathbf{Mod}_\Lambda$ preserves dualizable objects.

(iv) A locally profinite group is **locally Λ-Poincaré** (of dimension d) if it admits an open profinite subgroup which is Λ-Poincaré (of dimension d).

Lemma 5.13 ([HM24, Lemma 5.3.11])**.** Let $\Lambda \in \mathbf{CRing}$ and G be a locally profinite group with $i_K : K \hookrightarrow G$ a compact open subgroup with $\mathrm{cd}_\Lambda K < \infty$. Let $V \in D(* /\!\!/ G, \Lambda)$. The following are equivalent:

(i) V is dualizable,

(ii) $i_K^* V$ is dualizable in $D(* /\!\!/ K, \Lambda)$,

(iii) the underlying Λ-module of V is dualizable.

Proof.

(i) $\implies$ (iii): The implication (i) $\implies$ (iii) is because $D(* /\!\!/ G, \Lambda) \to \mathbf{Mod}_\Lambda$ is symmetric monoidal.

(ii) $\implies$ (i): Let $V^\vee = \underline{\mathrm{Map}}_{D(*/\!\!/G,\Lambda)}(V, \mathbb{1})$. It suffices to check that $V \otimes V^\vee \to \underline{\mathrm{Map}}_G(V, V)$ is a G-equivariant equivalence. To do so, consider the following commutative diagram:

$$\begin{tikzcd} i_K^*(V \otimes V^\vee) \arrow[rr] \arrow[d, "\simeq"'] & & i_K^* \underline{\mathrm{Map}}_G(V, V) \arrow[d, "\simeq"] \\ i_K^* V \otimes i_K^*(V^\vee) \arrow[r, "\simeq"'] & i_K^* V \otimes (i_K^* V)^\vee \arrow[r, "\simeq"'] & \underline{\mathrm{Map}}_K(i_K^* V, i_K^* V) \end{tikzcd}$$

The left map is an equivalence since i_K^* is symmetric monoidal. The lower right map is an equivalence by assumption (ii). For the remaining equivalences, we consider the projection formula

$$i_{K!} \circ (i_K^* V \otimes -) \overset{\simeq}{\Rightarrow} V \otimes i_{K!}(-)$$

whose two-fold right adjoints form an equivalence

$$i_K^* \underline{\mathrm{Map}}_G(V, -) \overset{\simeq}{\Rightarrow} \underline{\mathrm{Map}}_K(i_K^* V, i_K^* -).$$

This explains the bottom left and the right equivalence. In particular, the top arrow must be an equivalence. We conclude with conservativity of i_K^*.[18]

[18]On models, we are just forgetting an action but the map being an equivalence can be tested underlying.

(iii) $\implies$ (ii): Since $f_K : * /\!/ K \to *$ is Λ-proper (5.8(ii)) we conclude that the f_K-prim and dualizable objects in $D(* /\!/ K, \Lambda)$ agree [HM24, Lemma 4.6.3(iii)]. So (iii) means that q^*V is prim where $q : * \to * /\!/ K$ and we need to show that V is f_K-prim. But q is Λ-prim (5.8(i)) and $q_*\mathbb{1}$ is descendable [HM24, Proposition 5.2.5].[19] So V is prim [HM24, Corollary 4.7.5].

□

In special settings there are more checkable conditions for admissibility [HM24, Remark 5.3.13]. Another finiteness condition is compactness which will thus naturally show up in our arguments below. Let us briefly state it here.

Lemma 5.14 ([HM24, Corollary 5.3.4])**.** Consider $\Lambda \in$ **CRing** and a profinite group G with $\mathrm{cd}_\Lambda G < \infty$. Then, $\mathbb{1} \in D(* /\!/ G, \Lambda)$ is compact.

Proof. By 5.8(ii) the map $f : * /\!/ G \to *$ is Λ-proper, so we can compute

$$\mathrm{RHom}_{D(*/\!/G,\Lambda)}(\mathbb{1}, -) \simeq f_*\underline{\mathrm{Map}}_G(\mathbb{1}, -) \simeq f_* \simeq f_!$$

which commutes with colimits as a left adjoint. Here, the first equivalence follows by passing to left adjoints. Now we can pass to the underlying spectrum and then apply Ω^∞ to obtain the underlying space and both of these passages commute with filtered colimits. □

Proposition 5.15 ([HM24, Proposition 5.3.14, 5.3.19])**.** Let $\Lambda \in$ **CRing** and G be a profinite group with locally finite Λ-cohomological dimension. Let $V \in D(* /\!/ G, \Lambda)$ and $f : * /\!/ G \to *$.

(i) The object V is f-prim if and only if it is compact.

  (a) In this case, $\mathrm{PD}_f(V) \simeq \underline{\mathrm{Map}}_G(V, \Lambda_c(G))$.

  (b) If $K \le G$ is a compact open subgroup with $\mathrm{cd}_\Lambda K < \infty$ and $V \in D(* /\!/ K, \Lambda)$ is dualizable, then $\mathrm{PD}_f(\text{c-Ind}_K^G V) \simeq \text{c-Ind}_K^G V^\vee$.

(ii) The object V is f-suave if and only if it is admissible. In this case, $\mathrm{SD}_f(V) \simeq \underline{\mathrm{Map}}_G(V, f^!\mathbb{1})$.

(iii) The map $* /\!/ G \to *$ is Λ-suave if and only if G is locally Λ-Poincaré.

Proof. Let's start by recalling a classical result from smooth representation theory that we will use.

Lemma [HM24, Lemma 5.3.7]**.** For $V \in D(* /\!/ G, \Lambda)$ we have $\mathrm{colim}_{\substack{K \le G \text{ open} \\ \mathrm{cd}_\Lambda K < \infty}} V^K \simeq V$.

The fun thing is that you can also recover this result via a 6FF argument [HM24, Lemma 5.3.7].

(i) Note that $\Lambda \in \mathbf{Mod}_\Lambda$ is compact. This implies that every f-prim object is compact in $D(* /\!/ G, \Lambda)$ [HM24, Lemma 4.4.18(ii)]. So onto the converse.

Claim. Let

$$\mathcal{G} = \{i_{K!}\mathbb{1} : i_K : * /\!/ K \to * /\!/ G,\ K \le G \text{ compact open with } \mathrm{cd}_\Lambda K < \infty\}.$$

Then, $\mathcal{G}$ consists of compact and f-prim objects and generates $D(* /\!/ G, \Lambda)$.

[19] This uses $\mathrm{cd}_\Lambda K < \infty$.

Proof. We have seen that $f_K : * /\!/ K \to *$ is Λ-prim (5.8(ii)), i.e. $\mathbb{1} \in D(* /\!/ K, \Lambda)$ is f_K-prim. Moreover, i_K is Λ-suave (5.8(i)), so $i_{K!}\mathbb{1}$ is f-prim [HM24, Lemma 4.4.9(ii)].

Furthermore, $\mathbb{1}$ is compact by 5.14. Since $i_{K!} \dashv i_K^! \simeq i_K^* \dashv i_{K*}$ by Λ-étaleness of i_K (see 5.8(i)), it admits a right adjoint who commutes with (filtered) colimits and hence preserves compact objects. So $i_{K!}\mathbb{1}$ is compact.

To see that $\mathcal{G}$ is generating we observe

$$P^K = f_{K*}i_K^*P \simeq f_*i_{K*}\underline{\mathrm{Map}}_K(\mathbb{1}, i_K^*P) \simeq f_*\underline{\mathrm{Map}}_K(i_{K!}\mathbb{1}, P) \simeq \mathrm{RHom}_{D(*/\!/G,\Lambda)}(i_{K!}\mathbb{1}, P)$$

where the third equality is general 6FF nonsense [HM24, Proposition 3.2.2]. By the result discussed in the beginning of the proof, we conclude. □

Denote by $\langle \mathcal{G} \rangle \subseteq D(* /\!/ G, \Lambda)$ the full subcategory generated by $\mathcal{G}$ under (co-)fibers and retracts. Since primness is closed under these operations [HM24, Corollary 4.4.13] we get $\langle \mathcal{G} \rangle \subseteq \mathrm{Prim}(* /\!/ G)$. On the other hand, $\mathrm{Ind}(\langle \mathcal{G} \rangle) \simeq D(* /\!/ G, \Lambda)$ since $\mathcal{G}$ consists of compact generators [Lur09, Proposition 5.3.5.11]. Passing to compact objects yields $\langle \mathcal{G} \rangle \simeq D(* /\!/ G, \Lambda)^\omega$.

(a) This follows from the general prim dual formula [HM24, Lemma 4.4.6] while using the c-Ind to understand $\Delta_!$ from that formula.

(b) The map $f_K : * /\!/ K \to *$ is Λ-proper (5.8(ii)). So, the dualizables agree with the f_K-prims in $D(* /\!/ K, \Lambda)$ [HM24, Lemma 4.6.3(iii)] which in particular means that f_K-prim duality is the usual duality. Moreover, $h_! = \text{c-Ind}_K^G$ commutes with prim duality [HM24, Lemma 4.4.9]. So

$$\mathrm{PD}_f(\text{c-Ind}_K^G V) \simeq \text{c-Ind}_K^G \left(\mathrm{PD}_{f_K}(V)\right) \simeq \text{c-Ind}_K^G V^\vee$$

as desired.

(ii) We use

Lemma [HM24, Corollary 4.4.15]**.** Let D be a 6FF on some geometric setup $(\mathscr{C}, \mathscr{E})$ and $f : X \to S$ be a map in $\mathscr{E}$. Let $(Q_i)_{i \in I}$ be a family of objects in $D(X)$. Assume that the Q_i are f-prim and $D(X \times_S X)$ is generated by $\pi_1^*Q_i \otimes \pi_2^*Q_j$. Then, $P \in D(X)$ is f-suave if and only if $f_*\underline{\mathrm{Map}}(Q_i, P)$ is dualizable for all Q_i.

We take the family $(Q_i)_{i \in I} = \mathcal{G}$ from (i). We have seen there that its consists of Λ-prim objects and moreover,

$$\pi_1^*i_{K!}\mathbb{1} \otimes \pi_2^*i_{K'!}\mathbb{1} \simeq i_{(K \times K')!}\mathbb{1}$$

generates $D(* /\!/ (G \times G), \Lambda)$ by the same argument as in (i). We have also seen in the proof of (i) that $f_*\underline{\mathrm{Map}}(i_{K!}\mathbb{1}, V) \simeq V^K$, so the only if part of the statement translates to admissibility. The suave dual formula is an instance of the general formula [HM24, Lemma 4.4.5].

(iii) Suppose first that G is locally Λ-Poincaré. Let $H \leq G$ be a compact open Λ-Poincaré subgroup. As in the proof of 5.8(iii) we see that $* /\!/ H \to * /\!/ G$ is a universal !-cover, so it suffices to show that $* /\!/ H \to *$ is Λ-suave [HM24, Lemma 4.5.8(i)]. So WLOG G is Λ-Poincaré.

We need to show that $\mathbb{1} \in D(* /\!/ G, \Lambda)$ is Λ-suave, i.e. admissible by (ii). In other words, we need that $V^K = f_{K*}\mathbb{1}$ is dualizable for every compact open $K \leq G$ with $\mathrm{cd}_\Lambda K < \infty$. For this, we note $f_{K*}\mathbb{1} \simeq f_*i_{K*}\mathbb{1}$ and f_* preserves dualizables because G is Λ-Poincaré. On the other hand, i_K is Λ-proper (5.8(i)), so $i_{K*}\mathbb{1} \simeq i_{K!}\mathbb{1}$. Now $\mathbb{1}$ is compact by 5.14 and $i_{K!}$ preserves compacts as demonstrated in the proof of (i). On the other hand, compacts and dualizables agree (5.16).

Conversely, suppose that $* /\!\!/ G \to *$ is Λ-suave. Since G has locally finite Λ-cohomological dimension, it has a compact open subgroup K with $\mathrm{cd}_\Lambda K < \infty$. Moreover, being Λ-suave is the same as admissibility by (ii), so $* /\!\!/ K \to *$ is still Λ-suave. So WLOG G is profinite with $\mathrm{cd}_\Lambda G < \infty$. Since $\mathbb{1} \in D(* /\!\!/ G, \Lambda)$ is f-suave, i.e. admissible, the object $f_* i_{K*}\mathbb{1} \simeq f_{K*}\mathbb{1}$ is dualizable in $\mathbf{Mod}_\Lambda$ for every compact open $K \le G$. On the other hand, $i_{K*}\mathbb{1} \simeq i_{K!}\mathbb{1}$ generate the dualizables in $D(* /\!\!/ G, \Lambda)$ under (co-)fibers and retractions as demonstrated in (i). So f_* preserves dualizables.

□

This prim duality is also called *Bernstein–Zelevinsky duality* and it is an example of a statement that is really terrible to prove by writing down formulas but follows formally from six functor nonsense! Just from the formulas, it's not clear that this formula for the prim duality is interesting and it's hard to get this explicit prim duality formula on compact inductions by only playing around with the formulas. With 6FF nonsense it's not that bad!

Corollary 5.16. Let $\Lambda \in \mathbf{CRing}$ and G be a profinite group with $\mathrm{cd}_\Lambda G < \infty$.

(i) Then, $D(* /\!\!/ G, \Lambda)$ is compactly generated.

(ii) An object is compact if and only if it is dualizable.

Proof.

(i) We have seen this in the proof of 5.15(i), it is compactly generated by what we called $\mathcal{G}$.

(ii) By 5.15(i) the compact objects agree with the f-prim objects where $f : * /\!\!/ G \to *$. So we need to show that f-primality agrees with dualizability. But $* /\!\!/ G \to *$ is Λ-proper (5.8(ii)) and in this setting we are done [HM24, Lemma 4.6.3(iii)].

□

Example 5.17 ([HM24, Example 5.3.21, 5.3.22])**.** Let p be a prime.

(i) Let Λ be a $\mathbb{Z}[1/p]$-algebra and G be locally pro-p. Then, G is locally Λ-Poincaré.

(ii) Let Λ be a $\mathbb{Z}/p^n$-algebra and G be a p-adic Lie group. Then, G is locally Λ-Poincaré.

In each case one can give explicit descriptions of the dualizing complex and so suave duality (5.15(iii)) recovers Poincaré duality in these settings. This is not really a new proof of Poincaré duality because it relies on results from classical representation theory which are close to Poincaré duality.

5.3 What the Hecke?

What the heck is a Hecke algebra?

They show up in various areas of mathematics. Frankly, I know neither of the motivations but https://www.math.columbia.edu/~martinez/Notes/introtohecke.pdf seems useful.

Definition 5.18. Let Λ be a field with $\mathrm{char}\,\Lambda = p > 0$ and $K \le G$ be a compact open subgroup of a locally profinite group with $V \in \mathbf{Rep}_\Lambda(K)^\heartsuit$. Then, $\mathcal{H}(G, K, V) = \mathrm{End}_G(\text{c-Ind}_K^G V)$ is the associated **Hecke algebra**.

Fact 5.19 ([HM24, Remark 5.5.1]).

(i) There is an isomorphism

$$\mathcal{H}(G,K,V) \xrightarrow{\sim} \{f : G \to \mathrm{End}_\Lambda(V) : f \text{ is } K\text{-}K\text{-linear}, \operatorname{supp} f \text{ compact}\}.$$

(ii) Under this identification there is an involutive anti-isomorphism of algebras

$$\iota : \mathcal{H}(G,K,V) \xrightarrow{\sim} \mathcal{H}(G,K,V^*),\ \iota(T)(g) = (T(g^{-1}))^*.$$

There are more refined derived versions of this construction by taking derived endomorphisms instead of the underived version [HM24, Remark 5.5.1].

Definition 5.20. Let $\Lambda \in \mathbf{CRing}$ and G be a locally profinite group with a compact open subgroup $K \leq G$ with $\mathrm{cd}_\Lambda K < \infty$.

(i) We denote by $\mathcal{H}_K$ the $\mathbf{Mod}_\Lambda$-enriched ∞-category whose objects are the dualizable objects in $\mathbf{Rep}_\Lambda(K)$ and whose mapping objects are

$$\mathcal{H}_K(V,W) = \mathrm{RHom}_G(\text{c-Ind}_K^G V, \text{c-Ind}_K^G W) \in \mathbf{Mod}_\Lambda.$$

(ii) We denote by $\mathcal{H}_K^\bullet = \mathcal{H}_K(\mathbb{1},\mathbb{1}) \in \mathbf{Alg}_{\mathbb{E}_1}(\mathbf{Mod}_\Lambda)$ and **derived Hecke algebra** of weight $\mathbb{1}$.

Theorem 5.21 ([HM24, Proposition 5.5.4, 5.5.6]). Let $\Lambda \in \mathbf{CRing}$ and G be a locally profinite group with a compact open subgroup $K \leq G$ with $\mathrm{cd}_\Lambda K < \infty$.

(i) Prim duality PD on $\mathrm{Prim}(* /\!\!/ G)$ induces an involutive equivalence

$$\mathcal{H}_K^{\mathrm{op}} \xrightarrow{\simeq} \mathcal{H}_K,\ V \mapsto V^\vee = \mathrm{RHom}_\Lambda(V,\Lambda)$$

of $\mathbf{Mod}_\Lambda$-enriched ∞-categories.

(ii) Let Λ be a field with $\operatorname{char} \Lambda = p > 0$ and G be a p-adic Lie group with a p-torsionfree compact open subgroup $I \leq G$. Then, (i) induces an anti-involution $\mathrm{Inv} : (\mathcal{H}_I^\bullet)^{\mathrm{op}} \xrightarrow{\simeq} \mathcal{H}_I^\bullet$ which coincides with Schneider–Sorensen's anti-involution $\mathrm{Inv}_{\mathrm{SS}}$ [HM24, Remark 5.5.1].

It seems like previously this was only defined for fields of positive characteristic Λ and you need to work a little to write down these maps. Prim duality immediately yields a map and works for all $\Lambda \in \mathbf{CRing}$.

A fruitful plan of developing new mathematics seems to be: Find/Take any six functor formalism and try to specialize all of the general abstract 6FF notions that we have learned to the example. Anyhow, the next goal of the seminar will be to carry out this plan on the category of topological spaces.

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