Motivic Homotopy Theory

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Abstract

These are my (live) TeX'd notes for the motivic homotopy theory seminar in Bonn, WiSe 2025/26. The abstract is:

The goal of motivic homotopy theory, as introduced by Morel and Voevodsky, is to bring homotopical techniques into the world of algebraic geometry. The fundamental idea is to replace manifolds by smooth schemes over a base, so that the affine line $\mathbb{A}^1$ plays the role of the interval in usual homotopy theory. We aim to give the participant a feel for this category by first discussing Hoyois' proof of the Hopkins–Morel isomorphism. This passes through several motivic versions of fundamental homotopical constructions such as the identification of the Steenrod algebra for mod p cohomology and the Landweber exact functor theorem, and provides a strong connection between algebraic cobordism and motivic cohomology[1]. In the second half of the seminar[2], we discuss a more recent take on the theory of motivic spectra that in fact does away with the $\mathbb{A}^1$-homotopy invariance entirely. This compromise is motivated by compatibility with algebraic K-theory, and we will see how to set up a motivic analogue of Snaith's theorem that provides a universal property for algebraic K-theory over all base schemes, while passing through a discussion of orientations in this new setting.

My notation and language is not always consistent with the speakers' choices. I also occassionally added some parts which were not included in the actual talks; such parts will always be indicated by a star like Lemma*.

Feel free to send me feedback. :-)

Contents

[1] This result has received renewed attention in recent years: over a base field of positive characteritic p we only understand the motivic Steenrod algebra (and therefore also algebraic cobordism) after inverting p. It is not known how to complete this picture to the characteristic (see [AE25, CF25] for recent innovations in this direction, and cf. the second half of the syllabus for the refined context in which op. cit. plays out), but it should be the foundational computation for setting up the hypothetical prismatic stable homotopy category.

[2] We closely follow these lecture notes and strongly recommend that the speaker use these as a main reference for the structure of the talk due to the fact that some results are improved and re-proved between references.

1 Setting Up (Fabio Neugebauer)

Let S be a scheme, the base scheme. It'll be useful to assume some properties like Noetherianity, finite-dimensionality, etc. throughout the talk, so let us just already do it here. Let $\mathbf{Sm}_S$ denote the category of smooth schemes of finite type over S. The smoothness is particularly relevant for the purity theorem but one can also arrive at the same result by changing not only $\mathbf{Sm}_S$ but also the Grothendieck topology that we will introduce.

TALK 1
16.10.2025

1.1 Setup of Motivic Spaces

Let's compare motivic homotopy theory to classical homotopy theory.

$$\mathbf{Mfld} \xmapsto{\text{coherently contract } \mathbb{R}^1} \mathcal{S} \xmapsto{\text{invert } S^1} \mathbf{Sp}$$

$$\mathbf{Sm}_S \xmapsto[\text{coherently contract } \mathbb{A}^1_S]{} \mathbf{Spc}(S) \xmapsto[\text{invert } \mathbb{P}^1]{} \mathcal{SH}(S)$$

Let's make this precise. We wish to define $\mathbf{Spc}(S) = L_{\mathbb{A}^1}\mathbf{Sh}_?(\mathbf{Sm}_S)$ as the ∞-category of motivic spaces.

We need to take sheaves because $\mathbf{Sm}_S$ behaves too badly. What Grothendieck topology do we take? One could try the Zariski topology but this has too few covers. One could try the étale topology which has enough geometry but infinite cohomological dimension. There is something in between:

$$\text{Zar} < \text{Nisnevich} < \text{étale}$$

In the Nisnevich topology the upshot is that descent is essentially some version of excision. We will not formally define the Nisnevich topology but will state a characterization of Nisnevich sheaves:

Theorem 1.1. A presheaf $F : \mathbf{Sm}_S^{\mathrm{op}} \to \mathcal{S}$ is a **Nisnevich sheaf** if and only if:

(i) $F(\emptyset) \simeq *$,

(ii) Let

$$\begin{array}{ccc} W & \longrightarrow & V \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p} \\ U & \xrightarrow[i]{} & X \end{array}$$

be a **Nisnevich square** in $\mathbf{Sm}_S$, i.e. it is cartesian, i is an open immersion, p is étale and $p^{-1}(X - U) \to X - U$ is an isomorphism. Then, the square

$$\begin{array}{ccc} F(X) & \longrightarrow & F(V) \\ \big\downarrow & & \big\downarrow \\ F(U) & \longrightarrow & F(W) \end{array}$$

is a pullback square.

Corollary 1.2. The Yoneda embedding $\mathbf{Sm}_S \hookrightarrow \mathbf{Sh}^{\mathrm{Nis}}(\mathbf{Sm}_S)$ sends Nisnevich squares to pushouts.

Proof. By the Yoneda Lemma the pullback square in 1.1(ii) becomes

$$\begin{array}{ccc} \mathrm{Map}_{\mathbf{Sh}(\mathbf{Sm}_S)}(\text{よ}X, F) & \longrightarrow & \mathrm{Map}_{\mathbf{Sh}(\mathbf{Sm}_S)}(\text{よ}V, F) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Map}_{\mathbf{Sh}(\mathbf{Sm}_S)}(\text{よ}U, F) & \longrightarrow & \mathrm{Map}_{\mathbf{Sh}(\mathbf{Sm}_S)}(\text{よ}W, F) \end{array}$$

for all $F \in \mathbf{Sh}(\mathbf{Sm}_S)$. Thus, $\text{よ}X \simeq \text{よ}U \amalg_{\text{よ}W} \text{よ}V$ as desired. □

Definition 1.3. A presheaf $F \in \mathbf{PSh}(\mathbf{Sm}_S)$ is called **$\mathbb{A}^1$-invariant** if for all $X \in \mathbf{Sm}_S$ the map

$$\mathrm{pr}^* : F(X) \to F(X \times \mathbb{A}^1)$$

is an equivalence.

Definition 1.4. We denote by $\mathbf{Spc}(S) \subseteq \mathbf{PSh}(\mathbf{Sm}_S)$ is the full subcategory of $\mathbb{A}^1$-invariant Nisnevich sheaves. This is the ∞-category of **motivic spaces**.

Proposition 1.5. The inclusion $\mathbf{Spc}(S) \to \mathbf{PSh}(\mathbf{Sm}_S)$ preserves filtered colimits and admits a finite product-preserving left adjoint

$$L_{\mathrm{mot}} = \operatorname*{colim}_n (L_{\mathrm{Nis}} \to L_{\mathbb{A}^1} L_{\mathrm{Nis}} \to \cdots)$$

with $L_{\mathbb{A}^1} F \simeq \mathrm{colim}_{[n] \in \Delta^{\mathrm{op}}} F(X \times \Delta^n)$.

Proof. These exist for formal reasons but you need the explicit formula to prove the finite product-preservation. Essentially, the key step is to use that sifted colimits commute with finite limits in $\mathcal{S}$. □

Remark 1.6. This functor L_{mot} is not left-exact. In fact, $\mathbf{Spc}(S)$ is not an ∞-topos.

Definition 1.7. A map $H : X \times \mathbb{A}^1 \to Y$ in $\mathbf{Sm}_S$ is called **$\mathbb{A}^1$-homotopy**.

For any $a : *_{\mathbf{Spc}(S)} \simeq S \to \mathbb{A}^1$ we get $H_a : X \to X \times \mathbb{A}^1 \xrightarrow{H} Y$ and for all $a, b \in \mathbb{A}^1(S)$ we have $H_a \simeq H_b$ in $\mathbf{Spc}(S)$. The first map is always an equivalence as a section of $\mathrm{pr} : X \times \mathbb{A}^1 \to X$, in particular it always has the same inverse, so it is always the same map (up to equivalence).

1.2 Motivic Spheres

Definition 1.8. We write $\mathbb{G}_m$ for the pointed S-scheme $(\mathbb{A}^1 - \{0\}, 1)$.

Observation 1.9. The squares

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \times \mathbb{A}^1 \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{A}^1 \times \mathbb{G}_m & \longrightarrow & \mathbb{A}^2 - \{0\} \end{array} \qquad \begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

are Nisnevich squares. So we obtain pushout squares

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\ \downarrow & \ulcorner & \downarrow \\ \mathbb{G}_m & \longrightarrow & \mathbb{A}^2 - \{0\} \end{array} \qquad \begin{array}{ccc} \mathbb{G}_m & \longrightarrow & * \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & \mathbb{P}^1 \end{array}$$

in $\mathbf{Spc}(S)$ after contracting $\mathbb{A}^1$. We deduce $\mathbb{A}^2 - \{0\} \simeq \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m)$ and $\mathbb{P}^1 \simeq \Sigma\mathbb{G}_m$ in $\mathbf{Spc}(S)_*$. For the first one, you still need to play around a little bit.

Definition 1.10. For integers $d \geq j \geq 0$ the **motivic sphere** are $S^{d,j} = S^{d-j} \wedge \mathbb{G}_m^j \in \mathbf{Spc}(S)_*$.

So $\mathbb{P}^1 \simeq S^{2,1}$ and $\mathbb{A}^2 - \{0\} \simeq S^{3,2}$ by 1.9.

Proposition 1.11. There is an equivalence $S^{2n-1,n} \simeq \mathbb{A}^n - \{0\}$.

In particular,

$$S^{2n,n} \simeq S^1 \wedge S^{2n-1,n} \simeq \operatorname{cofib}(\mathbb{A}^n - \{0\} \to \mathbb{A}^n) = \mathbb{A}^n/(\mathbb{A}^n - \{0\}),$$

i.e. contract the boundary of a disc which should be a sphere.

1.3 Base Change

Let $f : T \to S$ be a map of schemes. We get a functor $\mathbf{Sm}_S \to \mathbf{Sm}_T$, $X \mapsto T \times_S X$ which gives rise to an adjunction

$$\mathbf{PSh}(\mathbf{Sm}_S) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathbf{PSh}(\mathbf{Sm}_T)$$

by left Kan extension. This passes to motivic spaces:

$$\mathbf{Spc}(S) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathbf{Spc}(T)$$

such that

$$\begin{array}{ccc} \mathbf{Spc}(S) & \xrightarrow{f^*} & \mathbf{Spc}(T) \\ \uparrow{\scriptstyle L_{\mathrm{mot}}} & & \uparrow{\scriptstyle L_{\mathrm{mot}}} \\ \mathbf{PSh}(\mathbf{Sm}_S) & \xrightarrow[f^*]{} & \mathbf{PSh}(\mathbf{Sm}_T) \end{array}$$

commutes (which is checked on right adjoints).

Remark 1.12. The functor f^* preserves finite products.

Proof. It preserves finite products on the scheme level and so also on presheaves by abstract nonsense [Lur09, Proposition 6.1.5.2]. Since the motivic localizations preserve products, this also descends to motivic spaces. □

Proposition 1.13 (Nisnevich Separation)**.** Let $\{f_i : U_i \to S\}_i$ be a Nisnevich cover. Then, the family $\{f_i^* : \mathbf{Spc}(S) \to \mathbf{Spc}(U_i)\}_i$ is conservative.

1.4 Motivic Thom Spaces

The concept of Thom spaces allows you to study vector bundles despite having contracted $\mathbb{A}^1$.

Definition 1.14.

(i) Let $\mathbf{Vect}_S$ denote the 1-category of vector bundles over S and isomorphisms.

(ii) The **Thom space functor** is $\mathrm{Th} : \mathbf{Vect}_S \to \mathbf{Spc}(S)_*$, $E \mapsto E/(E - \{0\})$.

Or rather: $\mathrm{Th}(E) \simeq E/(E - X)$.

Proposition 1.15. The functor Th is symmetric monoidal.

Proof. We begin by factoring Th into lax symmetric monoidal functors.[3]

$$\mathbf{Vect}_S \longrightarrow \mathrm{Ar}(\mathbf{Sm}_S)^{\mathbf{Day}} \longrightarrow \mathrm{Ar}(\mathbf{Spc}(S))^{\mathbf{Day}} \xrightarrow{\text{cofib}} \mathbf{Spc}(S)_*$$

$$E \longmapsto (E - \{0\} \to E)$$

Now symmetric monoidality is a property, so it suffices to check for $E, E' \in \mathbf{Vect}_S$ that the map

$$\mathrm{Th}(E) \wedge \mathrm{Th}(E') \to \mathrm{Th}(E \wedge E')$$

is an equivalence. WLOG, E, E' are trivial by Nisnevich separation in which case we get

$$(\mathbb{A}^1/\mathbb{A}^1 - \{0\})^{\wedge n} \simeq (S^{2,1})^{\wedge n} \simeq S^{2n,n} \simeq \mathbb{A}^n/\mathbb{A}^n - \{0\}.$$

□

1.5 Motivic Spectra

Definition 1.16. The category $(\mathcal{SH}(S), \otimes)$ is the initial presentably symmetric monoidal category under $\mathbf{Spc}(S)_*$, i.e. it comes with a functor $\Sigma^\infty : \mathbf{Spc}(S)_* \to \mathcal{SH}(S)$, on which tensoring with $\mathbb{S}^{d,j} = \Sigma^\infty S^{d,j}$ defines an equivalence $\Sigma^{d,j} = \mathbb{S}^{d,j} \otimes - : \mathcal{SH}(S) \to \mathcal{SH}(S)$.

Existence on $\mathbf{CAlg}(\mathbf{Pr}^L)_{\mathscr{C}/}$ with inverting some compact objects is a formal thing by Robalo [Rob13].

Construction 1.17. Let $f : T \to S$ be a map of schemes. Then, we obtain a symmetric monoidal left adjoint sitting in the square

$$\begin{array}{ccc} \mathbf{Spc}(S)_* & \xrightarrow{f^*} & \mathbf{Spc}(T) \\ {\scriptstyle \Sigma^\infty}\downarrow & & \downarrow{\scriptstyle \Sigma^\infty} \\ \mathcal{SH}(S) & \xrightarrow[\exists ! f^*]{} & \mathcal{SH}(T) \end{array}$$

by definition of $\mathcal{SH}(S)$.

Remark 1.18.

(i) Since $\Sigma^{1,0} = \Sigma$ we get that $\mathcal{SH}(S)$ is stable.

(ii) It suffices to invert $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$.

Theorem 1.19. Let

$$\mathbf{Spc}(S)_*[(\mathbb{P}^1)^{-1}] = \mathrm{colim}\left(\mathbf{Spc}(S)_* \xrightarrow{(-)\wedge\mathbb{P}^1} \mathbf{Spc}(S)_* \xrightarrow{(-)\wedge\mathbb{P}^1} \mathbf{Spc}(S)_* \to \cdots\right)$$

in $\mathbf{Pr}^L$.[4] Then, this is an idempotent algebra in $\mathbf{Mod}_{\mathbf{Spc}(S)_*}(\mathbf{Pr}^L)$. Then, the preferred map $\mathbf{Spc}(S)_*[(\mathbb{P}^1)^{-1}] \to \mathcal{SH}(S)$ is an equivalence in $\mathbf{CAlg}(\mathbf{Pr}^L)$.

Proof. This is a categorification of the group completion theorem. Robalo proved that we need to check that $\mathbb{P}^1$ is symmetric, i.e. the cyclic permutation $(1\ 2\ 3) : (\mathbb{P}^1)^{\wedge 3} \to (\mathbb{P}^1)^{\wedge 3}$ is equivalent to the identity [Rob13].

We use $(\mathbb{P}^1)^{\wedge 3} \simeq \mathrm{Th}(\mathbb{A}^3)$. We factor the matrix for $(1\ 2\ 3)$ into elementary matrices $E(a)$ over $\mathbb{Z}$ with $a \in \mathbb{Z}$ which is possible since $\det(1\ 2\ 3) = 1$. Then,

$$\mathbb{A}^1 \times \mathbb{A}^3 \to \mathbb{A}^3,\ (t, x) \mapsto E(ta)(X)$$

is a homotopy to the identity. □

[3] Here, [1] obtains the monoidal structure via max.

[4] I.e. you take the limit of the right adjoints.

1.6 Motivic Thom Spectrum

The symmetric monoidal composite

$$\mathbf{Vect}_S \xrightarrow{\text{Th}} \mathbf{Spc}(S)_* \xrightarrow{\Sigma^\infty} \mathcal{SH}(S).$$

sends all $E \in \mathbf{Vect}_S$ to $\otimes$-invertible objects. On trivial bundles those are spheres where it is true. In general we check this locally by Nisnevich separation. So it factors the group completion

$$J : K(S) = (\mathbf{Vect}_S)^{\text{gp}} \to \mathcal{SH}(S)$$

as a symmetric monoidal functor. Via $K(S) \to \mathbb{Z}$ we can pick out the rank and define $K(S)^0$.

Definition 1.20. The **motivic Thom spectrum** is

$$\text{MGL} = \text{colim}(J : K(S)^0 \to \mathcal{SH}(S)) \in \mathbf{CAlg}(\mathcal{SH}(S)).$$

Remark* 1.21.

(i) This is a bit inaccurate, it only has Thom isomorphisms for bundles over S instead of over all S-schemes. One needs some more parametrizations to make this work properly.

Indeed, let us recall the classical setting [ACB19, Proposition 3.16]. Say some ring spectrum E is MU-oriented, so the composite BU $\to$ Pic $\mathbb{S} \to$ Pic E is nullhomotopic. Any virtual degree 0 vector bundle is represented by a map $f : X \to \text{BU}$. So the composite

$$X \xrightarrow{f} \text{BU} \xrightarrow{J} \text{Pic}\,\mathbb{S} \xrightarrow{\text{Ind}_{\mathbb{S}}^{E}} \text{Pic}\,E$$

is nullhomotopic as well. This gives

$$E^\bullet(\text{Th}(f)) \cong E^\bullet(\text{Th}(f) \otimes E) \cong E^\bullet\left(\text{colim}(\text{Ind}_{\mathbb{S}}^{E} \circ J \circ f)\right) \cong E^\bullet(X),$$

i.e. the Thom isomorphism. The first isomorphism is by adjunction.

Now the same holds in the motivic setting but $K(S)^0$ are only virtual degree 0 vector bundles over S. We want vector bundles over arbitrary S-schemes, so we need some notion parametrizing over S-schemes.

(ii) Here is a more down-to-earth description. Let $f : X \to S$ be a smooth S-scheme. Then, there is a pullback functor $f^* : \mathcal{SH}(S) \to \mathcal{SH}(X)$ essentially by functoriality of $\mathbf{Sm}_{/-}$. It admits a left adjoint $f_\sharp : \mathcal{SH}(X) \to \mathcal{SH}(S)$. Then, we define

$$\text{MGL}_S = \underset{f \in \mathbf{Sm}_S}{\text{colim}}\, f_\sharp\, \text{colim}\left(J_X : K(X)^0 \to \mathcal{SH}(X)\right) = \underset{f \in \mathbf{Sm}_S}{\text{colim}}\, f_\sharp\, \text{Th}_X(J)$$

as **Voevodsky's algebraic cobordism spectrum**. More information about it is e.g. contained in [BH21, Section 16].

1.7 Homotopy t-Structure

Definition 1.22. Let $\mathcal{X} \in \mathbf{Sh}^{\text{Nis}}(\mathbf{Sm}_S)$. Then, $\underline{\pi}_0(\mathcal{X}) \in \mathbf{Sh}(\mathbf{Sm}_S)$ is defined as the sheafification of $U \mapsto \pi_0(\mathcal{X}(U))$. Similarly, $\underline{\pi}_n(\mathcal{S}, x) \in \mathbf{Sh}(\mathbf{Sm}_S)$ for $(\mathcal{X}, x) \in \mathbf{Sh}(\mathbf{Sm}_S)_*$.

Theorem 1.23. Let k be a perfect field. If $\mathcal{X} \in \mathbf{Spc}(k)_*$, then $\underline{\pi}_i(\mathcal{X}) \in \mathbf{Spc}(S)$ (and all its deloopings) are $\mathbb{A}^1$-invariant for $i \geq 1$.

Definition 1.24. Let $\mathcal{SH}(S)_{\geq 0} \subseteq \mathcal{SH}(S)$ be the category spanned by

$$\{\Sigma^k_{\mathbb{G}_m} \Sigma^\infty_+ X : k \in \mathbb{Z}, X \in \mathbf{Sm}_S\}$$

which you close up under extensions and colimits.

It's formal to obtain that $\mathcal{SH}(S)_{\geq 0}$ is the connective part of a t-structure, the **homotopy t-structure**.

Theorem 1.25 (Morel)**.** Let k be a field and $E \in \mathcal{SH}(k)$.

(i) We have $E \in \mathcal{SH}(S)_{\geq d}$ if and only if $\underline{\pi}_{p,q}(E) = \underline{\pi}_0(\Omega^\infty \Sigma^{-p,-q} E) = 0$ for all $p - q < d$.

(ii) We have $E \in \mathcal{SH}(S)_{\leq d}$ if and only if $\underline{\pi}_{p,q}(E) = 0$ for all $p - q > d$.

There are many interesting t-structures such as the Chow t-structure but this is quite close to the one on **Sp**.

Remark 1.26.

(i) For $f : S \to T$ the functor $f^* : \mathcal{SH}(T) \to \mathcal{SH}(S)$ is t-exact.

(ii) The Betti realization functor $\mathrm{Be}_{\mathbb{R}} : \mathcal{SH}(\mathbb{R}) \to \mathbf{Sp}$, $(X \in \mathbf{Sm}_{\mathbb{R}}) \mapsto X(\mathbb{R})^{\mathrm{an}}$ is t-exact.

1.8 Stable Stems

Fabio is ending with this because I forced him to. For this part let $S = \operatorname{Spec} k$.

Definition 1.27. The **stable stems** are $\pi_i(\mathbb{S})_j = \left[\Sigma^\infty S^i, \Sigma^\infty \mathbb{G}_m^j\right]_{\mathcal{SH}(k)}$.

Example 1.28.

(i) Take $\eta : \mathbb{A}^2 - \{0\} \to \mathbb{P}^1$ which gives $[\eta] \in \pi_0(\mathbb{S})_{-1}$.[5]

(ii) For $a \in k^\times$ we have $a : * \to \mathbb{G}_m$ yielding $[a] \in \pi_0(\mathbb{S})_1$.

Theorem 1.29 (Morel)**.** There is an isomorphism $\pi_0(\mathbb{S})_\bullet \cong \mathbb{Z}\langle [a], [\eta] \rangle / \text{relations} = K^{\mathrm{MW}}(k)_\bullet$.

This is **Milnor-Witt K-theory** and was already defined before this motivic story! We get **Milnor K-theory** via $K^M(k)_\bullet = K^{\mathrm{MW}}(k)_\bullet / [\eta]$ which computes K_0, K_1, K_2 of algebraic K-theory.

2 Slice Filtration & K-Theory (Qi Zhu)

2.1 Beilinson's Dream

TALK 2
23.10.2025

We all have dreams but your dream is not relevant for this talk.[6] Beilison's dream takes the center stage.

Motivated by questions about the zeta function ζ as well as Grothendieck's vision of a category of motives, Beilinson and Lichtenbaum conjectured the existence of motivic cohomology in the 80s [BK25, Introduction]. It is lousely supposed to satisfy a number of desiderata, among others:

(i) It gives rise to an analog of Atiyah-Hirzebruch spectral sequence (2.22).

(ii) It is essentially described by higher Chow groups (2.30).

(iii) There should be a certain range of support (2.32).

(iv) There is a close relation to étale cohomology (2.34).

We will discuss all of those in this talk and therefore see some of the historic motivations for motivic homotopy theory.

[5] To see that this is non-trivial, one can for example use Betti realization which gives the classical Hopf map.
[6] Sorry.

2.2 Motivic K-Theory Spectrum

2.2.1 Thomason-Trobaugh K-Theory

Recall that for a ring A one can define its algebraic K-theory as $K(A) = \left(\mathbf{Proj}_A^{\mathrm{fg,core}}\right)^{\mathrm{gp}}$. As so often in algebraic geometry, we can extend this to (nice) schemes.

Theorem 2.1. Let S be a regular Noetherian scheme of finite dimension. Then, there exists a motivic space $K \in \mathbf{Spc}(S)$, the so-called **Thomason-Trobaugh K-theory** such that for every $\operatorname{Spec} A \in \mathbf{Sm}_S$ we have $K(\operatorname{Spec} A) \simeq K(A)$.

Proof Idea. One can make the assignment $F : \mathbf{Sm}_S^{\mathrm{op}} \to \mathcal{S}$, $X \mapsto K(\mathcal{O}_X(X))$ functorial. The Thomason-Trobaugh K-theory is $K = L_{\mathrm{mot}}F$.

It remains to show that $F \to L_{\mathrm{mot}}F = K$ is an equivalence on affines. By working with localization formulas – namely [Bac21, Theorem 2.21] and the sheafification formula – it suffices to show that F on is $\mathbb{A}^1$-invariant and Nisnevich-local on affine schemes. This translates to properties of algebraic K-theory, namely $\mathbb{A}^1$-invariance $K(A[t]) \simeq K(A)$ and a Nisnevich descent condition. These are non-trivial properties of the K-theory of regular rings [Bac21, Theorem 2.25]. □

Recall $K(A) \simeq K_0(A) \times \mathrm{BGL}(A)^+$. This can also be extended to schemes.

Construction 2.2. There are presheaves

$$\mathrm{GL}_n : \mathbf{Sm}_S^{\mathrm{op}} \to \mathbf{Grp},\ X \mapsto \mathrm{GL}_n(\mathcal{O}_X(X)) \quad \text{and} \quad \mathrm{GL} = \operatorname*{colim}_n \mathrm{GL}_n\,.$$

Taking classifying spaces sectionwise yields $\mathrm{BGL} \in \mathbf{PSh}(\mathbf{Sm}_S)$.

Fact 2.3 ([Bac21, Theorem 2.28])**.** Let S be a regular Noetherian scheme of finite dimension. Then,

$$K \simeq L_{\mathrm{mot}}(\mathbb{Z} \times \mathrm{BGL}) \in \mathbf{Spc}(S).$$

2.2.2 Algebraic K-Theory Spectrum

Recall from Talk 1 that $\mathcal{SH}(S) \simeq \lim\left(\cdots \xrightarrow{\Omega_{\mathbb{P}^1}} \mathbf{Spc}(S) \xrightarrow{\Omega_{\mathbb{P}^1}} \mathbf{Spc}(S)\right)$. We want to construct a motivic analog of KU, i.e. a motivic spectrum KGL representing algebraic K-theory. To do this we first construct the representing motivic spaces.

Construction 2.4. Let X be an S-scheme. Denote by $\mathbf{Vect}(X)$ the 1-category of vector bundles on X. Then,[7]

$$K(\mathbf{Vect}(X)) = \left(\mathbf{Vect}(X)^{\oplus,\mathrm{core}}\right)^{\mathrm{gp}}$$

is the **direct sum K-theory** of X.

Lemma 2.5. Let S be a regular Noetherian scheme of finite dimension. Then, $K \simeq L_{\mathrm{mot}}K(\mathbf{Vect}(-))$.

Proof. By the Serre-Swan theorem, there is a symmetric monoidal functor $\mathbf{Proj}^{\mathrm{fg}}_{\mathcal{O}_X(X)} \to \mathbf{Vect}(X)$ which is an equivalence on affines. In particular, this induces a map

$$K(\mathcal{O}_-(-)) \to K(\mathbf{Vect}(-))$$

in $\mathbf{PSh}(\mathbf{Sm}_S)$ which is an equivalence on affines. So this is a Zariski equivalence. Thus,

$$K \simeq L_{\mathrm{mot}}K(\mathcal{O}_-(-)) \simeq L_{\mathrm{mot}}L_{\mathrm{Zar}}K(\mathcal{O}_-(-)) \simeq L_{\mathrm{mot}}L_{\mathrm{Zar}}K(\mathbf{Vect}(-)) \simeq L_{\mathrm{mot}}K(\mathbf{Vect}(-))$$

where we use $L_{\mathrm{mot}}L_{\mathrm{Zar}} \simeq L_{\mathrm{mot}}$ which follows from the Nisnevich topology being finer than the Zariski topology. □

[7] We perform everything in $\mathbf{Cat}_\infty$.

Remark 2.6. The definition of K via the direct sum K-theory is more general than the Thomason-Trobaugh K-theory – it doesn't require these regularity conditions on S.

Observation 2.7. The construction of $K(\mathbf{Vect}(X))$ was naturally as a functor to **CGrp** which in particular forgets to $\mathcal{S}_*$. In other words, we can obtain natural basepoints via $0 \in K(\mathbf{Vect}(X))$, so we obtain lifts $K(\mathbf{Vect}(-)) \in \mathbf{PSh}(\mathbf{Sm}_S)_*$ and $L_{\mathrm{mot}}K(\mathbf{Vect}(-)) \in \mathbf{Spc}(S)_*$.

Construction 2.8. Consider the tautological line bundle $\gamma = \mathcal{O}_{\mathbb{P}^1}(-1) \in \mathbf{Vect}(\mathbb{P}^1)$. External tensor product of vector bundles yields a natural[8] additive functor $- \otimes \gamma : \mathbf{Vect}(X) \to \mathbf{Vect}(X \times \mathbb{P}^1)$ which induces a map of commutative monoids $\gamma : K(\mathbf{Vect}(X)) \to K(\mathbf{Vect}(X \times \mathbb{P}^1))$. Similarly, there is a map $1 : K(\mathbf{Vect}(X)) \to K(\mathbf{Vect}(X \times \mathbb{P}^1))$ for the trivial line bundle $1 \in \mathbf{Vect}(\mathbb{P}^1)$. Since $K(\mathbf{Vect}(X))$ is grouplike, we can form the difference $\gamma - 1$.

The following is a motivic version of Bott periodicity.

Theorem 2.9 (Motivic Bott Periodicity)**.**

(i) The map $\gamma - 1$ assembles into a map

$$\gamma - 1 : K(\mathbf{Vect}(-)) \to \Omega_{\mathbb{P}^1} K(\mathbf{Vect}(-))$$

in $\mathbf{PSh}(\mathbf{Sm}_S)_*$.

(ii) The induced map

$$\gamma - 1 : L_{\mathrm{mot}}K(\mathbf{Vect}(-)) \to L_{\mathrm{mot}}\Omega_{\mathbb{P}^1}K(\mathbf{Vect}(-)) \to \Omega_{\mathbb{P}^1}L_{\mathrm{mot}}K(\mathbf{Vect}(-))$$

is an equivalence.

Proof.

(i) In 2.8 we constructed a map[9]

$$K(\mathbf{Vect}(-)) \to \Omega_{\mathbb{P}^1_+} K(\mathbf{Vect}(-))$$

in $\mathbf{PSh}(\mathbf{Sm}_S)_*$ which by adjunction corresponds to a map $\mathbb{P}^1_+ \otimes K(\mathbf{Vect}(-)) \to K(\mathbf{Vect}(-))$. We want to produce a map $\mathbb{P}^1 \otimes K(\mathbf{Vect}(-)) \to K(\mathbf{Vect}(-))$. Via the cofiber sequence $*_+ \to \mathbb{P}^1_+ \to \mathbb{P}^1$, we need to show that the composite

$$K(\mathbf{Vect}(-)) \simeq *_+ \otimes K(\mathbf{Vect}(-)) \longrightarrow \mathbb{P}^1_+ \otimes K(\mathbf{Vect}(-)) \longrightarrow K(\mathbf{Vect}(-))$$

is nullhomotopic. But this is induced by the restriction of $\gamma - 1$ to $* = S$ which is 0 since $\gamma|_S = 1$ is the trivial bundle. Adjoining again, we have successfully constructed a map $K(\mathbf{Vect}(-)) \to \Omega_{\mathbb{P}^1}K(\mathbf{Vect}(-))$.

(ii) We only discuss this in the case S is Noetherian, regular and of finite-dimensional, although this is true in general [Bac21, Footnote 12]. Then, by 2.5 this is the statement $K(X) \simeq K(X_+ \wedge \mathbb{P}^1)$ for Thomason-Trobaugh K-theory. This follows from the so-called *projective bundle formula* [Wei13, Theorem V.1.5].[10]

□

[8] Note that here naturality is still 1-categorical and hence can be checked by hand.

[9] That evaluation on $- \times \mathbb{P}^1$ corresponds to $\Omega_{\mathbb{P}^1_+}$ follows via a Yoneda argument.

[10] I think you use that the cofiber sequence $\mathbb{P}^1 \to X \times \mathbb{P}^1 \to X_+ \wedge \mathbb{P}^1$ induces a fiber sequence on K-theory by the Yoneda Lemma. Then, $K_\bullet(X \times \mathbb{P}^1) = K_\bullet(\mathbb{P}^1_X) \cong K_\bullet(X)[z]/z^2$ by the projective bundle formula. This z part is killed by $K(\mathbb{P}^1)$.

Mimicking the topological counterpart, we define:

Definition 2.10. The **(motivic) algebraic K-theory spectrum** is the object

$$\mathrm{KGL} = \mathrm{KGL}_S = ((K, K, \cdots); \gamma - 1 : K \to \Omega_{\mathbb{P}^1} K) \in \mathcal{SH}(S)$$

where we write $K = L_{\mathrm{mot}} K(\mathbf{Vect}(-))$ here for brevity.

Remark 2.11.

(i) So $\Omega^\infty \mathrm{KGL} \simeq K$ with which we have constructed a motivic spectrum representing K. In particular, this recovers algebraic K-theory in case S is regular, Noetherian and of finite dimension (2.5).

(ii) In general, KGL represents Weibel's homotopy K-theory KH, an $\mathbb{A}^1$-invariant approximation to K-theory. We need non $\mathbb{A}^1$-invariant K-theory to fix this defect and it will be a focus towards the end of this seminar.

Remark 2.12. Essentially since everything worked analogous to the topological counterpart one deduces that the complex Betti realization is $\mathrm{Be}_{\mathbb{C}}\, \mathrm{KGL}_{\mathbb{C}} \simeq \mathrm{KU}$ [Ban05, Lemma 4.23]. It turns out that $\mathrm{Be}_{\mathbb{R}}\, \mathrm{KGL} \simeq 0$ as opposed to $\mathrm{Be}_{\mathbb{R}}\, \mathrm{KGL} \simeq \mathrm{KO}$, answering a question from Thomas during the talk [Ban05, Lemma 4.24].

Corollary 2.13.

(i) Let $n \in \mathbb{Z}$, then $\Sigma^{2n,n}\, \mathrm{KGL} \simeq \mathrm{KGL}$.

(ii) Let S be regular, Noetherian and finite-dimensional. For $X \in \mathbf{Sm}_S$ we have

$$[\Sigma^{p,q}\Sigma^\infty_+ X, \mathrm{KGL}_S] \cong \begin{cases} K_{p-2q}(X) & p \geq 2q, \\ 0 & \text{else.} \end{cases}$$

Proof.

(i) We use $\mathcal{SH}(S) \simeq \lim\left(\cdots \xrightarrow{\Omega_{\mathbb{P}^1}} \mathbf{Spc}(S) \xrightarrow{\Omega_{\mathbb{P}^1}} \mathbf{Spc}(S)\right)$. By motivic Bott periodicity (2.9) both $\Sigma^{2n,n}\, \mathrm{KGL}$ and KGL are given by $(\cdots, K, K)$.

(ii) By Bott periodicity from (a), we can shift so far to assume $q = 0$. If $p \geq 0$, then we use adjunctions and the Yoneda Lemma to compute

$$[\Sigma^{p,0}\Sigma^\infty_+ X, \mathrm{KGL}_S] \cong [\Sigma^\infty(S^p \wedge X_+), \mathrm{KGL}_S] \cong [S^p \wedge X_+, K]_* \cong K_p(X).$$

On the other hand,

$$[\Sigma^{-p,0}\Sigma^\infty_+ X, \mathrm{KGL}_S] \cong [\Sigma^{p,p}\Sigma^\infty_+ X, \Sigma^{2p,p}\, \mathrm{KGL}_S] \cong [\mathbb{G}_m^{\wedge p} \wedge X_+, K]_*.$$

Now, there is a cofiber sequence

$$X_+ \longrightarrow (\mathbb{G}_m^{\times p} \times X)_+ \longrightarrow \mathbb{G}_m^{\wedge p} \wedge X_+$$

in $\mathbf{Spc}(S)_*$.[11] Now, let us only consider $p = 1$, for $p \geq 2$ we argue by induction. In that case, we obtain an exact sequence

$$K_1(\mathbb{G}_m \times X) \twoheadrightarrow K_1(X) \longrightarrow [\mathbb{G}_m \wedge X_+, K]_* \longrightarrow K_0(\mathbb{G}_m \times X) \xrightarrow{\sim} K_0(X)$$

where the first arrow is surjective and the last arrow is an isomorphism by *Bass' fundamental theorem*. Thus, the middle term must be 0 by exactness.

∎

In particular, for $X = S$ we have $\pi_{p,q}\, \mathrm{KGL}_S \cong \begin{cases} K_{p-2q}(S) & p \geq 2q, \\ 0 & \text{else.} \end{cases}$

[11] If we omit the added basepoints of the first two terms, then this is a cofiber sequence in $\mathbf{Spc}(S)$.

2.3 The Slice Filtration

We know wish to construct a motivic version of the Whitehead filtration.

2.3.1 Axiomatic Approach to Slice Filtrations

Due to the bigrading on motivic spectra, there are multiple imaginable filtrations that generalize the Whitehead filtration. As such, we will begin by giving a general procedure for such constructions. Let me introduce the following ad hoc name. Drew calls this *slice filtration* [Hea19, Definition 2.1].

Definition 2.14 ([GRSOsr12, Section 2.1])**.** Let $\mathscr{C} \in \mathbf{CAlg}(\mathbf{Pr}^L_{\mathrm{st}})$, compactly generated by a set of objects $\mathcal{T}$. Let $\{\mathscr{C}_i\}_{i\in\mathbb{Z}}$ be a family of full subcategories of $\mathscr{C}$. Then, $\{\mathscr{C}_i\}_{i\in\mathbb{Z}}$ is a **slice setup** of $\mathscr{C}$ if the following conditions are satisfied.

(i) For $i \in \mathbb{Z}$ we have $\mathscr{C}_{i+1} \subseteq \mathscr{C}_i$.

(ii) Each $\mathscr{C}_i$ is generated under colimits and extensions by a set of compact objects $\mathcal{K}_i$.

(iii) We have $\mathbb{1} \in \mathscr{C}_0$.

(iv) Each $t \in \mathcal{T}$ is contained in some $\mathscr{C}_i$.

(v) If $c_0 \in \mathscr{C}_0$ and $c_n \in \mathscr{C}_n$, then $c_0 \otimes c_n \in \mathscr{C}_n$.

Observation 2.15. Let $i_q : \mathscr{C}_q \hookrightarrow \mathscr{C}$. Since it is closed under colimits, it admits a right adjoint $r_q : \mathscr{C} \to \mathscr{C}_q$. We put $f_q = i_q \circ r_q : \mathscr{C} \to \mathscr{C}$.

Definition 2.16. Let $\mathscr{C} \in \mathbf{CAlg}(\mathbf{Pr}^L_{\mathrm{st}})$ be equipped with a slice setup and $c \in \mathscr{C}$. We wish to construct a map $f_{q+1}c \to f_q c$. By adjunction, it corresponds to a map $f_{q+1}c \to c$ which can be taken as the counit of $i_{q+1} \dashv r_{q+1}$. Then, we obtain a filtration

$$\begin{array}{ccccccccc} \cdots \longrightarrow & f_1 c & \longrightarrow & f_0 c & \longrightarrow & f_{-1}c & \longrightarrow \cdots \longrightarrow & c \\ & \downarrow & & \downarrow & & \downarrow & & \\ & s_1 c & & s_0 c & & s_{-1} c & & \end{array}$$

is the **slice tower** with **slices** $s_n c = \operatorname{cofib}(f_{n+1}c \to f_n c)$.

In the definition of slice setups (2.14) the conditions (i), (ii) are to get the filtration running, condition (iii) is in some sense a normalization condition, condition (iv) ensures that the induced slice filtrations are exhaustive and condition (v) gives some good compatibilities with multiplicative structures [Hea19, Section 2].

Example 2.17.

(i) For $\mathscr{C} = \mathbf{Sp}$ with $\mathcal{K}_q = \{\mathbb{S}^m : m \geq q\}$ leading to $\mathscr{C}_q \simeq \mathbf{Sp}_{\geq q}$ yields the classical Whitehead tower in $\mathbf{Sp}$.

(ii) Let $\mathcal{K}_q = \{\Sigma^{a,b}\Sigma^\infty_+ X : X \in \mathbf{Sm}_S,\ b \geq q\} \subseteq \mathcal{SH}(S)$. We denote by $\Sigma^{q,q}\mathcal{SH}(S)^{\mathrm{eff}} \subseteq \mathcal{SH}(S)$ be the localizing subcategory generated by $\mathcal{K}_q$. Then, the filtration

$$\cdots \hookrightarrow \Sigma^{q+1,q+1}\mathcal{SH}(S)^{\mathrm{eff}} \hookrightarrow \Sigma^{q,q}\mathcal{SH}(S)^{\mathrm{eff}} \hookrightarrow \cdots$$

defines a slice setup. Associated to it is **(Voevodsky's) slice filtration**.

By construction, $\Sigma^{q,q}\mathcal{SH}(S)^{\mathrm{eff}} \simeq \Sigma^{q,q}(\Sigma^{0,0}\mathcal{SH}(S)^{\mathrm{eff}})$. We also write $\mathcal{SH}(S)^{\mathrm{eff}} = \Sigma^{0,0}\mathcal{SH}(S)^{\mathrm{eff}}$.

Remark 2.18. For the motivic slice filtration you can check $f_q \simeq \Sigma^{q,q} f_0 \Sigma^{-q,-q} : \mathcal{SH}(S) \to \mathcal{SH}(S)$ explicitly, following Bachmann [Bac21, Exercise 3.4].

Remark 2.19. This is not the focus of the talk but certainly the axiomatic construction leads to numerous additional interesting filtration.

(i) Taking $\mathcal{K}_q = \{\Sigma^{q+i,i}\Sigma^\infty_+ X : X \in \mathbf{Sm}_S,\ i \in \mathbb{Z}\} \subseteq \mathcal{SH}(S)$ yields the **homotopy t-structure**.

(ii) Taking $\mathcal{K}_q = \{\Sigma^{2a,a}\Sigma^\infty_+ X : X \in \mathbf{Sm}_S,\ a \geq q\} \subseteq \mathcal{SH}(S)$ we obtain the **very effective slice filtration**. There are also cellular versions of the slice and very effective slice filtrations [Hea19, Section 4].

(iii) In $\mathbf{Sp}^{C_2}$ we define the following slice cells:

- $S^{2q,q\sigma}$ of dimension $2q$,
- $S^{2q-1,q\sigma}$ of dimension $2q-1$,
- $S^q \otimes (C_2)_+$ of dimension q.

Take $P^q\mathbf{Sp}^{C_2} \subseteq \mathbf{Sp}^{C_2}$ be the full subcategory generated under extensions and colimits of slice cells of dimension $\geq q$. This gives rise to the **Hill-Hopkins-Ravenel slice filtration** for $\mathbf{Sp}^{C_2}$. Ignoring cells of the second form gives Ullman's **regular slice filtration**. See [Hea19, Section 5.2]. There is also a version for more general G – on the other hand, C_2 seems fitting in the context of motivic homotopy theory which was also the purpose of [Hea19].

2.3.2 Motivic Examples

Definition 2.20.

(i) The **effective algebraic K-theory spectrum** is the motivic spectrum $\mathrm{kgl} = f_0\,\mathrm{KGL}$.

(ii) The **motivic cohomology spectrum** is the motivic spectrum $H\mathbb{Z} = s_0\,\mathrm{KGL}$.

Remark 2.21. The first definition of motivic cohomology is due to Voevodsky in the mid 90s as certain derived functors of so-called *motivic complexes* of sheaves $\mathbb{Z}(q)$, realizing Beilinson's dream. Afterwards, one may ask for alternative descriptions of this object. It was Voevodsky's first conjecture about his slice filtration [Voe02b, Conjecture 1] that $s_0\,\mathrm{KGL}$ is one such candidate. Here are some other ways of producing $H\mathbb{Z}$ over perfect fields.

(i) Classically singular chains defines a right adjoint $i^* : \mathbf{Sp} \to \mathbf{Ch}(\mathbf{Ab})$ and we can define $H\mathbb{Z} = i_* i^* \mathbb{S}$. Motivically, one can mimic this construction by replacing $\mathbf{Sp}$ with $\mathcal{SH}(k)$ and $\mathbf{Ch}(\mathbf{Ab})$ by the stable ∞-category of motives $\mathbf{DM}(k)$.

(ii) Classically, one can construct $H\mathbb{Z}$ as an infinite loop space via Eilenberg-MacLane spaces which can be viewed as $\mathrm{SP}^\infty(S^n)$ via the Dold-Thom theorem. This also be realized motivically over characteristic 0, i.e. one writes out a sequential $\mathbb{P}^1$-spectrum via Eilenberg MacLane spaces realized through symmetric products.

More naively attempting to take Eilenberg-MacLane objects in the ∞-topos $\mathbf{Sh}_{\mathrm{Nis}}(\mathbf{Sm}_k)$ and then applying $L_{\mathbb{A}^1}$ does not work – this only gives an S^1-spectrum and not an $\mathbb{P}^1$-spectrum.

(iii) Classically, $H\mathbb{Z} \simeq \tau_{\leq 0}\mathbb{S} \simeq \pi_0\mathbb{S}$. Motivically, $H\mathbb{Z} \simeq s_0\mathbb{1}$ is one of Voevodsky's original conjectures about his slice filtration which was shown by Levine. In fact, this combined with another conjecture, also proved by Levine, yields the first conjecture [Voe02c, Lev08].

Remark 2.22. The spectral associated to the slice filtration of KGL is a motivic version of the Atiyah-Hirzebruch spectral sequence [BL99, Voe02c], namely for $X \in \mathbf{Sm}_k$ there is a strongly convergent spectral sequence

$$E_2^{p,q} = H\mathbb{Z}^{p-q,-q}(X) \Rightarrow K_{-p-q}(X).$$

This is the one of the starting points of Hahn-Raksit-Wilson's even filtration and the related motivic filtrations [HRW24].

Lemma 2.23. There are equivalences $f_n \mathrm{KGL} \simeq \Sigma^{2n,n} \mathrm{kgl}$ and $s_n \mathrm{KGL} \simeq \Sigma^{2n,n} H\mathbb{Z}$.

Proof. Via 2.18 we compute

$$\begin{aligned} f_n \mathrm{KGL} &= \Sigma^{n,n} f_0 \Sigma^{-n,-n} \mathrm{KGL} \\ &\simeq \Sigma^{2n,n} \Sigma^{-n,0} f_0 \Sigma^{-n,-n} \mathrm{KGL} \\ &\simeq \Sigma^{2n,n} f_0 \Sigma^{-n,0} \Sigma^{-n,-n} \mathrm{KGL} \\ &\simeq \Sigma^{2n,n} f_0 \mathrm{KGL} \\ &= \Sigma^{2n,n} \mathrm{kgl}\,. \end{aligned}$$

and

$$\begin{aligned} s_n \mathrm{KGL} &= \mathrm{cofib}(f_{n+1} \mathrm{KGL} \to f_n \mathrm{KGL}) \\ &\simeq \mathrm{cofib}(\Sigma^{2(n+1),n+1} \mathrm{kgl} \to \Sigma^{2n,n} \mathrm{kgl}) \\ &\simeq \Sigma^{2n,n} \mathrm{cofib}(\Sigma^{2,1} \mathrm{kgl} \to \mathrm{kgl}) \\ &\simeq \Sigma^{2n,n} \mathrm{cofib}(f_1 \mathrm{KGL} \to f_0 \mathrm{KGL}) \\ &= \Sigma^{2n,n} H\mathbb{Z}. \end{aligned}$$

□

2.4 Motivic Cohomology & Historic Theorems

2.4.1 Motivic Cohomology Groups

Much of motivic homotopy theory was developed to understand motivic cohomology. Let us introduce some notation before discussing a number of results in the field.

Definition 2.24. Let $E \in \mathcal{SH}(S)$ and $X \in \mathbf{Sm}_S$, then $E^{p,q}(X) = [\Sigma^\infty_+ X, \Sigma^{p,q} E]$ is the **bigraded cohomology theory** represented by E.

Remark 2.25. So $\pi_{p,q} E \cong [\Sigma^{p,q} \Sigma^\infty_+ S, E] \cong E^{-p,-q}(S)$.

Definition 2.26. The bigraded cohomology theory represented by $H\mathbb{Z}$ is called **motivic cohomology** and is denoted by

$$H^{p,q}(X) = H^{p,q}(X; \mathbb{Z}) = H^p(X; \mathbb{Z}(q)) = H\mathbb{Z}^{p,q}(X).$$

The cohomology theory associated to $H\mathbb{Z}/n$ is also called motivic cohomology.

2.4.2 Higher Chow Groups

We first construct an algebro-geometric version of singular homology.

Construction 2.27 (Bloch)**.** Let $X \in \mathbf{Sm}_S$.

(i) We write $Z^d(X) = \mathbb{Z}\{x \in X : \operatorname{codim}(\overline{\{x\}} \subseteq X) = d \iff \dim \mathcal{O}_{X,x} = d\}$.

(ii) If $i : Y \hookrightarrow X$ is a closed immersion, then $c = \sum_n a_n x_n \in Z^d(X)$ is in **good position with respect to i** if the components of $Y \cap \overline{\{x_n\}}$ have codimension $\geq d$ on Y for every n. We write $Z^d(X)_i \subseteq Z^d(X)$ for such cycles. One can construct a pullback map $i^* : Z^d(X)_i \to Z^d(Y)$ [MVW06, Definition 17A.6].

(iii) Let $z^d(X, n) \subseteq Z^d(X \times \Delta^n)$ consists of those cycles in good position with respect to all faces $X \times \Delta^i \subseteq X \times \Delta^n$. Then, we put

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i^* : z^d(X, n) \to z^d(X, n-1).$$

This yields a chain complex and we write $\mathrm{CH}^d(X, n) = H_n(z^d(X, \bullet), \partial)$ for the **higher Chow groups**.

Observation 2.28. Flat maps preserve codimensions of subschemes [Bac21, Remark 4.13], so it induces pullback maps on z^d which descends to pullbacks of higher Chow groups CH^d. In other words, $\mathrm{CH}^d(-, n)$ is contravariantly functorial in flat maps.

Remark 2.29. This generalizes classical Chow groups $\mathrm{CH}^d(X) \cong \mathrm{CH}^d(X, 0)$ [Bac21, Example 4.11].

There are certainly many exciting things to be said about CH^d [Bac21, Section 4.3, 4.4] but we will focus on the connection to motivic cohomology.

Theorem 2.30 ([Voe02a])**.** Let $X \in \mathbf{Sm}_k$ and $p, q \in \mathbb{Z}$. Then, there are natural isomorphisms

$$H^{p,q}(X) \cong \mathrm{CH}^q(X, 2q - p).$$

This paper [Voe02a] is a 5-page paper with 100 citations!

Remark 2.31. Besides connecting two seemingly disjoint objects, we can extract many interesting consequences.

(i) We have argued that $\mathrm{CH}^d(-, n)$ is functorial in flat maps (2.28(ii)) but motivic cohomology $H^{p,q}(-)$ is functorial in all maps of schemes, so this functoriality transfers to $\mathrm{CH}^d(-, n)$.

(ii) Since $H^{p,q}(-)$ is represented by a motivic spectrum, we deduce that $\mathrm{CH}^d(-, n)$ is $\mathbb{A}^1$-invariant and satisfies Nisnevich descent.

(iii) By construction, $\mathrm{CH}^d(X, n) = 0$ for $n < 0$. Thus, $H^{p,q}(X) \cong \mathrm{CH}^q(X, 2q - p) = 0$ for $p > 2q$.

This result also allows us to compute weight 0 and weight 1 motivic cohomology after further higher Chow group computations [Bac21, Exercise 4.2, Theorem 4.14]. One obtains

$$H^{\bullet,0}(X) \cong \mathbb{Z}^{\pi_0 X}[0] \quad \text{and} \quad H^{p,1}(X) \cong \begin{cases} \mathrm{Pic}(X) & p = 2, \\ \mathcal{O}_X(X)^\times & p = 1, \\ 0 & \text{else.} \end{cases}$$

Conjecture 2.32 (Beilinson-Soulé Vanishing Conjecture)**.** Is $H^{p,q}(X) \cong 0$ for $p < 0$?

2.4.3 Bloch-Kato Conjecture

Let ℓ be an integer invertible in k. The Kummer exact sequence yields a connecting homomorphism $\partial : k^\times \to H^1_{\text{ét}}(k, \mu_\ell)$. Very briefly, via multiplicativity of étale cohomology, the definition of Milnor K-theory (with Artin reciprocity) and ℓ-torsion of $H^\bullet_{\text{ét}}(k, \mu_\ell^{\otimes \bullet})$, this induces a map

$$\begin{array}{ccc} k^\times \otimes \cdots \otimes k^\times & \xrightarrow{\partial^n} & H^n_{\text{ét}}(k, \mu_\ell^{\otimes n}) \\ \downarrow & \overset{\partial^n}{\nearrow} & \\ K_n^M(k) & & \\ \downarrow & \overset{\partial_n}{\nearrow} & \\ K_n^M(k)/\ell & & \end{array}$$

the so-called **Galois symbol** or **norm residue map**.

Theorem 2.33 (Norm Residue Theorem/(Motivic) Bloch-Kato Conjecture)**.** Let ℓ be an integer invertible in k. The map $\partial_n : K_n^M(k)/\ell \to H^n_{\text{ét}}(k, \mu_\ell^{\otimes n})$ is an isomorphism.

For $\ell = 2$ this was first conjectured by Milnor and as such is the *Milnor conjecture*. For $n = 2$ this is the *Merkurjev-Suslin theorem*, as this case was first proven by them and the first major advance in the resolution of this theorem. Voevodsky first proved the Milnor conjecture which earned him a fields medal and he later went on to prove the entire theorem with ideas from Rost.

Theorem 2.34 (Beilinson-Lichtenbaum Conjecture, Rost-Voevodsky)**.** Let $X \in \mathbf{Sm}_k$ and $\ell \in \mathbb{Z}$ be invertible in k. Then, $H^{p,q}(X, \mathbb{Z}/\ell) \cong H^p_{\text{ét}}(X, \mu_\ell^{\otimes q})$ for $p \leq q$.

This implies the norm residue theorem. Indeed, we first mention:

Theorem 2.35 (Nesterenko-Suslin '90, Totaro '92)**.** We have $\mathrm{CH}^d(k, n) \cong \begin{cases} 0 & n < d, \\ K_d^M(k) & n = d. \end{cases}$

So combining this result (2.35) with Levine's comparison of $H^{p,q}$ and CH^d (see 2.30) we see that the Milnor K-group term from the norm residue theorem (2.33) is identified with the motivic cohomology term from the Beilinson-Lichtenbaum conjecture (2.34).

Combining the Bloch-Kato conjecture with the motivic Atiyah-Hirzebruch spectral sequence also gave the resolution of the Quillen-Lichtenbaum conjecture which related étale cohomology to algebraic K-theory.

The resolution of this conjecture was a huge leap in the development of motivic homotopy theory. It required motivic versions of Spanier-Whitehead duality and the Steenrod algebra. Especially the latter is a focus point of this seminar.

3 $7 \leq n$ Functors for $\mathcal{SH}$ (Lucas Piessevaux)

We will discuss a number of functors on $\mathcal{SH}$. In fact, there are at least seven relevant ones: $f_\sharp, f_*, f^*, f_!, f^!, \otimes, \underline{\mathrm{Map}}$. TALK 3 30.10.2025

Recall from the previous two talks for a (qcqs) scheme S that $\mathbf{Spc}(S) = L_{\mathrm{mot}}\mathbf{PSh}(\mathbf{Sm}_S)$ and $\mathcal{SH}(S) = \mathbf{Spc}(S)_*[(\mathbb{P}^1)^{\otimes -1}]$. This contains examples like KH and $\mathbb{Z}(n)^{\mathrm{mot}}$ which is mysterious but in nice enough cases contains the examples $H^i_{\text{ét}}(-; \mu_\ell^{\otimes j})$ for $i \leq j$ and $R\Gamma_{\mathrm{Zar}}(-, W_r\Omega^i_{\log})$.

Our goals today are:

- $\mathcal{SH}^* : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{CAlg}(\mathbf{Pr}^L_{\mathrm{st}})$ with $\mathscr{E} = $ (lft),

- motivic properties: $\mathbb{A}^1$-invariance, gluing, Thom twists.

In particular, this is not supposed to be an exercise in the theory of Heyer-Mann [HM24] but rather we want to apply the formalism to prove motivic properties!

3.1 The Functors 1-4 and 7: $(\sharp, *, \otimes)$

We begin with the easier functors, namely the $*$'s, $\otimes$, $\underline{\mathrm{Map}}$ and $\sharp$ which are functors 1-4 and number 7.

3.1.1 Closed Monoidality and Push-Pull $(*, \otimes)$

- $\otimes$: Equipping $\mathbf{Spc}(S)$ with a cartesian symmetric monoidal structure, we obtain a symmetric monoidal structure on $\mathbf{Spc}(S)_*$ given by the smash product. This gives rise to $\mathcal{SH}(S) \in \mathbf{CAlg}(\mathbf{Pr}^L_{\mathrm{st}})$.[12]
- $*$: For $f : T \to S$ we get $T \times_S - : \mathbf{Sm}_S \to \mathbf{Sm}_T$ which preserves $\mathbb{A}^1_X \to X$ and Nisnevich squares. So it induces an adjunction

$$\mathbf{Spc}(S) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathbf{Spc}(T)$$

where f_* is induced by restriction and f^* is obtained via left Kan extension (and localization). On representables we obtain $f^* X_+ \simeq (T \times_S X)_+$, whence f^* is symmetric monoidal and $f^* \mathbb{P}^1_S \simeq \mathbb{P}^1_T$.

Proposition 3.1. Given $f : T \to S$ there exists an adjunction

$$\mathcal{SH}(S) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{SH}(T)$$

such that:

(i) $f^* \Sigma^{\infty - n}_{\mathbb{P}^1} X_+ \simeq \Sigma^{\infty-n}_{\mathbb{P}^1} (T \times_S X)_+$,

(ii) f^* is strong symmetric monoidal and strongly cocontinuous.[13]

Proof. The composite

$$\begin{array}{ccc} \mathbf{Spc}(S)_* & \longrightarrow & \mathbf{Spc}(T)_* \\ & & \downarrow \\ & & \mathcal{SH}(T) \end{array}$$

sends $\mathbb{P}^1_S \mapsto \Sigma^\infty_{\mathbb{P}^1} \mathbb{P}^1_T$, so the universal property of stabilization induces the adjunction. It moreover gives rise to the formula in (i). For (ii) the strong symmetric monoidality is inherited from the unstable setting and strong cocontinuity follows from $\Sigma^{\infty-n}_{\mathbb{P}^1}(T \times_S X)_+ \in \mathcal{SH}(T)^\omega$, meaning that it sends a family of compact generators to compact objects. □

Remark 3.2. Note that the formula 3.1(i) uniquely determines f^* since $\mathcal{SH}(S)$ is generated by these $\Sigma^{\infty-n}_{\mathbb{P}^1} X_+$.

[12] In particular, $\Sigma^\infty_+ X \otimes \Sigma^\infty_+ Y \simeq \Sigma^\infty_+ (X \times Y)$.
[13] I.e. preserves compact objects.

3.1.2 Forgetful Functor $\sharp$

Onto the 7th functor.

If $f : T \to S$ is a smooth map, then we have a functor

$$\mathbf{Sm}_T \to \mathbf{Sm}_S,\ (X \to T) \mapsto (X \to T \to S).$$

By left Kan extension we obtain an adjunction

$$\mathbf{PSh}(\mathbf{Sm}_T) \underset{f^*}{\overset{f_\sharp}{\rightleftarrows}} \mathbf{PSh}(\mathbf{Sm}_S).$$

Remark 3.3.

(i) One can check on representables that for smooth f, this f^* agrees with the f^* defined in the push-pull section.

(ii) Given $X \in \mathbf{Sm}_T$ we have $f_\sharp(X \to T) = (X \to T \to S) \in \mathbf{PSh}(\mathbf{Sm}_S)$ on representables by left Kan extension. In that regard, we are just forgetting.

Now, the smooth projection formula (SPF).

Proposition 3.4 (SPF)**.** Given a smooth $f : T \to S$ there exists an adjunction

$$\mathbf{Spc}(T)_* \underset{f^*}{\overset{f_\sharp}{\rightleftarrows}} \mathbf{Spc}(S)_*$$

of $\mathbf{Spc}(S)_*$-modules where we view $\mathbf{Spc}(T)_* \in \mathbf{Mod}_{\mathbf{Spc}(S)_*}$ via f^*.

Proof. The $\mathbf{Spc}(S)_*$-linearity is an equivalence $f_\sharp(X \otimes f^*Y) \simeq f_\sharp X \otimes Y$, i.e. the smooth projection formula. Note that all functors involved commute with colimits and Σ^{-n}, so we may check the formula on representables. There, it's asking for $X \times_T (T \times_S Y) \cong X \times_S Y$ in $\mathbf{Sch}_S$ which is pullback pasting. □

Corollary 3.5. Given a smooth $f : T \to S$ there exists an adjunction

$$\mathcal{SH}(T) \underset{f^*}{\overset{f_\sharp}{\rightleftarrows}} \mathcal{SH}(S)$$

with $f_\sharp \Sigma^{\infty-n}_{\mathbb{P}^1} X_+ \simeq \Sigma^{\infty-n}_{\mathbb{P}^1} X_+$.

Proof. Basechanging the $\mathbf{Spc}(S)_*$-algebra $\mathcal{SH}(S)$ we obtain an $\mathcal{SH}(S)$-linear adjunction

$$\mathbf{Spc}(T)_* \otimes_{\mathbf{Spc}(S)_*} \mathcal{SH}(S) \underset{f^*}{\overset{f_\sharp}{\rightleftarrows}} \mathcal{SH}(S)$$

but the left side is seen to be

$$\mathbf{Spc}(T)_* \otimes_{\mathbf{Spc}(S)_*} \mathbf{Spc}(S)_*[(\mathbb{P}^1_S)^{\otimes -1}] \simeq \mathbf{Spc}(T)_*[(f^*\mathbb{P}^1_S)^{\otimes -1}] \simeq \mathcal{SH}(T).$$

The formula is as before. □

Remark* 3.6. The equivalence $f_\sharp \Sigma^{\infty-n}_{\mathbb{P}^1} X_+ \simeq \Sigma^{\infty-n}_{\mathbb{P}^1} X_+$ uniquely determines $f_\sharp$.

The next result (smooth base change) involves Beck-Chevalley maps, one of which is a mate of the other. To check that both are equivalences, it suffices to check that one of them is. This will be true for many results in this talk and we will only write down one of the transformations later.

Proposition 3.7 (SBC)**.** Given a pullback square

$$\begin{array}{ccc} T' & \xrightarrow{g} & S' \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ T & \xrightarrow[f]{} & S \end{array}$$

with smooth p, q the BC transformations

$$q_\sharp g^* \Rightarrow f^* p_\sharp \quad \text{and} \quad f^* p_* \Rightarrow q_* g^*$$

are equivalences.

Proof. It suffices to check $q_\sharp g^* \Rightarrow f^* p_\sharp$, then its mate automatically becomes an equivalence by abstract nonsense. All functors here commute with colimits and desuspensions, so we are going to evaluate on X_+ with $X \in \mathbf{Sm}_{S'}$. We need to show $T' \times_{S'} X \simeq T \times_S X$ as T-schemes. This follows from the pasting of the pullback squares

$$\begin{array}{ccc} T' \times_{S'} X & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ T' & \longrightarrow & S' \\ \downarrow & \lrcorner & \downarrow \\ T & \longrightarrow & S \end{array}$$

so we are done. □

Emma: *This is just parametrized cocompleteness of presheaf topoi.*

Corollary 3.8. Let $j : U \hookrightarrow X$ be an open immersion. Then,

$$\begin{array}{ccc} U & = & U \\ \| & \lrcorner & \downarrow{\scriptstyle j} \\ U & \underset{j}{\hookrightarrow} & X \end{array}$$

is a pullback, so the smooth base change formula tells us $\mathrm{id} \simeq j^* j_\sharp$ and $j^* j_* \simeq \mathrm{id}$, so $j_\sharp$ and j_* are fully faithful.

3.2 Motivic Properties

3.2.1 Homotopy Invariance

Warning 3.9. The assignment $S \mapsto \mathcal{SH}(S)$ does not invert $\mathbb{A}^1$-equivalences. What does is the functor

$$\mathbf{Sm}_S \to \mathcal{SH}(S),\ (f : X \to S) \mapsto \Sigma^\infty_{\mathbb{P}^1} X_+.$$

Sven remarks that this the classical analog is $\mathbf{Sh}(*) \not\simeq \mathbf{Sh}(\mathbb{R})$.

Lemma 3.10. Let $p : E \to S$ be an affine bundle, then p^* is fully faithful.

Proof. Since p is smooth, the adjunction $p_\sharp \dashv p^*$ exists. Consider $\varepsilon : p_\sharp p^* \Rightarrow \mathrm{id}$. All functors commute with colimits, so this is determined by the value on representables. Evaluate on $\Sigma^\infty_{\mathbb{P}^1} X_+$ with $X \in \mathbf{Sm}_S$. Then, we need to check $\Sigma^\infty_{\mathbb{P}^1}(E \times_S X)_+ \xrightarrow{\simeq} \Sigma^\infty_{\mathbb{P}^1} X_+$ in $\mathcal{SH}(S)$. We may first trivialize E and assume $E = \mathbb{A}^n_S$ by Nisnevich descent and then this is $\mathbb{A}^1$-invariance. □

3.2.2 Thom Twists

Recall that for a finite locally free sheaf $\mathcal{E} \to S$ we take the cofiber sequence

$$(\mathbb{V}(E) \setminus S)_+ \longrightarrow \mathbb{V}(E)_+ \longrightarrow \mathrm{Th}_S(E)$$

as the defining sequence for $\mathrm{Th}_S(E)$.

Lemma 3.11. This refines to an assignment

$$\begin{array}{ccc} \mathbf{Vect}_S^{\mathrm{core}} & \xrightarrow{\mathrm{Th}_S(-)} & \mathrm{Pic}(\mathcal{SH}(S)) \\ \downarrow & \nearrow & \\ K(S) & & \end{array}$$

where $K(S)$ is Q-construction K-theory.

Proof. By Nisnevich separation (1.13) of $\mathcal{SH}$ we can always descend to the affine case $S = \operatorname{Spec} R$, so $P \subseteq R^{\oplus n}$. Consider the SES

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

and write $\pi : T \to S$ as the moduli of splittings of this SES. This is an affine bundle because locally its fibers are of the form $\underline{\hom}_S(E'', E')$. So π^* is fully faithful. □

Definition 3.12. We write $K(S) \to \mathrm{Pic}(\mathcal{SH}(S)) \simeq \mathrm{Aut}_{\mathcal{SH}(S)}(\mathcal{SH}(S))$, $\mathcal{E} \mapsto \langle \mathcal{E} \rangle$.

Example 3.13. We have $\mathcal{O}_{\mathbb{P}^1} \mapsto [2](1) = \Sigma^{2,1}$.

3.2.3 Purity

This is the ur-theorem of $\mathcal{SH}$ and the reason Morel-Voevodsky setup $\mathcal{SH}$ the way they do. Their insight is that their definition allows us to perform things like deformation to the normal cone.

Definition 3.14.

(i) A **smooth closed pair** (X, Z) is a closed immersion $Z \hookrightarrow X$ in $\mathbf{Sm}_S$.

(ii) A **map of smooth closed pairs** $(X', Z') \to (X, Z)$ is a map $X' \to X$ that induces a pullback square

$$\begin{array}{ccc} Z' & \longrightarrow & Z \\ \downarrow & \lrcorner & \downarrow \\ X' & \longrightarrow & X \end{array}$$

on closed subschemes.

(iii) A map $f : (X', Z') \to (X, Z)$ is **weakly excisive** if

$$\begin{array}{ccccc} Z' & \longrightarrow & X' & \longrightarrow & X'/(X' \setminus Z') \\ \downarrow & & & & \downarrow \\ Z & \longrightarrow & X & \longrightarrow & X/(X \setminus Z) \end{array}$$

is cocartesian in $\mathbf{Spc}(S)$.

Example 3.15. Nisnevich squares and zero sections of affine bundles are examples.

The point is that there is a slightly larger class of squares than the Nisnevich squares which are sent to cocartesian squares including things like blow-up squares.

Construction 3.16. Let (X, Z) be a smooth closed pair.

(i) The blowup

$$\begin{array}{ccc} E = p^{-1}(Z) & \hookrightarrow & \mathrm{Bl}_Z(X) \\ \downarrow & \lrcorner & \downarrow p \\ Z & \xhookrightarrow[i]{} & X \end{array}$$

is a map of smooth closed pairs.

(ii) Let $D_Z X = \mathrm{Bl}_{Z\times 0}(X \times \mathbb{A}^1) \setminus \mathrm{Bl}_{Z\times 0}(X \times 0)$ be the **deformation to the normal cone**. This has a closed immersion $Z \times \mathbb{A}^1 \hookrightarrow D_Z X$ which forms a smooth closed pair.

Blowups are universal smooth closed pairs such that the exceptional divisor is an effective Cartier divisor.

Here is the idea: The scheme $D_Z X$ is a family over $\mathbb{A}^1$ with generic fiber X and special fiber the normal cone $N_Z X$ of Z in X, as suggested by its nomenclature. Pictorially:

$$\begin{array}{ccccc} X & \hookrightarrow & D_Z X & \hookleftarrow & N_Z X \\ \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbb{A}^1 & \longleftarrow & 0 \end{array}$$

This induces a cospan $(X, Z) \to (D_Z X, Z \times \mathbb{A}^1) \leftarrow (N_Z X, Z)$.

Theorem 3.17. These maps are weakly excisive.

Proof Sketch. Assume $(X, Z) = (\mathbb{V}(\mathcal{E}), Z)$. Then, $(\mathrm{Bl}_Z X, p^{-1}(Z)) = (\mathbb{V}(\mathcal{O}_{\mathbb{P}_Z(\mathcal{E})}(1)), \mathbb{P}_Z(\mathcal{E}))$. Then,

$$\begin{array}{ccccc} \mathbb{P}_Z(\mathcal{E}) & \longrightarrow & \mathbb{V}(\mathcal{O}_{\mathbb{P}_Z(\mathcal{E})}(1)) & \longrightarrow & \mathbb{V}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))/\mathbb{V}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \setminus \mathbb{P}(\mathcal{E}) \\ \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\ Z & \longrightarrow & \mathbb{V}(\mathcal{E}) & \longrightarrow & \mathbb{V}(\mathcal{E})/(\mathbb{V}(\mathcal{E}) \setminus Z) \end{array}$$

Now, use pasting. The right square is cocartesian in $\mathbf{PSh}(\mathbf{Sm}_S)$ and the left square is cocartesian in $L_{\mathbb{A}^1}\mathbf{PSh}(\mathbf{Sm}_S)$. □

The above is hard, so we are sketchy. Applying weak excisiveness, we have constructed a zig-zag of equivalences giving rise to:

Corollary 3.18 (Purity)**.** Let (X, Z) be a smooth closed pair. Then, there is an equivalence

$$\frac{X}{X \setminus Z} \simeq \frac{N_Z X}{N_Z X \setminus Z} = \mathrm{Th}_Z(N_Z X)$$

in $\mathcal{SH}(S)$.

3.2.4 Localization/Gluing

We want to study what happens in the setting

$$U \xhookrightarrow{\text{open}} X \xhookleftarrow{\text{closed}} Z$$

which gives rise to a stable recollement

$$\mathcal{SH}(U) \rightleftarrows \mathcal{SH}(X) \rightleftarrows \mathcal{SH}(Z).$$

Recall that the term $K(\mathbf{Perf}(X)_Z)$ shows up when trying to compute the K-theory of open-closed decompositions. On the other hand, $\mathrm{KH} \not\simeq K$ and in particular $\mathbb{A}^1$-invariance yields $\mathrm{KH}(\mathbf{Perf}(X)_Z) \simeq \mathrm{KH}(\mathbf{Perf}(Z))$.

Remark 3.19. The small étale site ét_S is a topological/nil invariant. Moreover, $(\mathbf{Sm}_S^{\mathrm{Nis}}, L_{\mathbb{A}^1})$ is also topologically invariant.

Theorem 3.20. Given an open-closed decomposition $U \xhookrightarrow{j} X \xhookleftarrow{i} Z$ we get a stable recollement

$$\mathcal{SH}(U) \underset{j_\sharp}{\overset{j^*}{\rightleftarrows}} \mathcal{SH}(X) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathcal{SH}(Z).$$

Lemma 3.21. Let $E \in \mathbf{Spc}(S)$. Then, the square

$$\begin{array}{ccc} j_\sharp j^* E & \longrightarrow & E \\ \downarrow & \ulcorner & \downarrow \\ U & \longrightarrow & i_* i^* E \end{array}$$

is cocartesian.

Construction 3.22. Given $X \in \mathbf{Sm}_S$ and $t : Z \to X_Z$. We set

$$\Phi_S(X,t) : \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}, \; \Phi_S(X,t)(Y) = \begin{cases} \mathrm{Hom}_S(Y,X) \times_{\mathrm{Hom}_S(Y_Z,X_Z)} \{Y_Z \to Z \xrightarrow{t} X_Z\} & Y_Z \neq \varnothing, \\ * & Y_Z = \varnothing \end{cases}$$

as the moduli space of maps into X that factor through Z on the special fiber.

Remark 3.23. There is an equivalence $\Phi_S(X,t) \simeq \left(X \amalg_{X_U} U\right) \times_{i_* X_Z} S$ in $\mathbf{PSh}(\mathbf{Sm}_S)$.

Lemma 3.24. Up to L_{Nis} the presheaf $\Phi_S(X,t)$ is invariant under étale neighbourhoods of $t(Z)$.

Proof. Consider

$$\begin{array}{ccccc} Z & \xrightarrow{t'} & X'_Z & \longrightarrow & X' \\ \| & & \downarrow & & \downarrow p \\ Z & \xrightarrow[t]{} & X_Z & \longrightarrow & X \end{array}$$

with étale p, then $\Phi_S(p)$ is a Nisnevich equivalence. These are presheaves of sets. Let's show that it's an effective epimorphism. Let $Y \to X$ be a class in $\Gamma(Y; \Phi_S(X,t))$ and set

$$Y \xleftarrow{q \text{ étale}} Y' = X' \times_X Y \xrightarrow{g} X'$$

then

$$\begin{array}{ccc} q^{-1}(Y_U) & \longrightarrow & Y' \\ \downarrow & \lrcorner & \downarrow q \\ Y_U & \hookrightarrow & Y \end{array}$$

is a Nisnevich square. Note that $f|_{Y'}$ is lifted by g and that $f|_{Y_U}$ lifts trivially. Check the same for the diagonal. □

Lemma 3.25. Up to $L_{\mathbb{A}^1}$ the presheaf $\Phi_S(X, t)$ is invariant under affine bundles.

Proof. Given a vector bundle $E \to S$ consider $t : S \to \mathbb{V}(E)$ and t_Z. We'd like to show $\Phi_S(\mathbb{V}(E), t_Z) \simeq S$. Consider

$$\mathbb{A}^1 \times \Phi_S(\mathbb{V}(E), t_Z) \to \Phi_S(\mathbb{V}(E), t_Z),\ (a, f) \mapsto af.$$

This is a homotopy between id and

$$\Phi_S(\mathbb{V}(E), t_Z) \longrightarrow S \xrightarrow{t} \Phi_S(\mathbb{V}(E), t_Z).$$

□

Proof of 3.20. Everything commutes with colimits,[14] so we can reduce to representables. The claim is equivalent to $X \amalg_{X_U} U \to i_* X_Z$ to be a motivic equivalence for all $X \in \mathbf{Sm}_S$. By universality of colimits this is equivalent to $(X \amalg_{X_U} U) \times_{i_* X_Z} S \to S$ is a motivic equivalence, i.e. $\Phi_S(X, t) \simeq S$ for every t.

- Use Nisnevich descent to reduce to S affine.
- Replace $t(Z)$ by an étale neighbourhood $Z \hookrightarrow V \hookrightarrow X$ such that $p : V \to S$ is étale at Z with restriction $t_V : Z \to V_Z$.
- Pick it small enough for it to admit $h : V \hookrightarrow \mathbb{V}(E)$ such that h is étale at $t(Z)$.
- Use $\mathbb{A}^1$-invariance.

□

Corollary 3.26. The functor $i_* : \mathcal{SH}(Z) \to \mathcal{SH}(X)$ is fully faithful.

Proof. We have $i^* j_\sharp \simeq 0$ which can be checked on representables using that U and Z do not intersect. So the same holds for their right adjoint $j^* i_* \simeq 0$. Then, the gluing sequence (3.21) for $i_* E$ gives a cofiber sequence

$$0 \longrightarrow i_* E \xrightarrow{\eta i_*} i_* i^* i_* E,$$

so ηi_* is an equivalence. By the triangle identities also $i_* \varepsilon$ is an equivalence. Hence, we're left to show conservativity of i_*. □

[14] We didn't show this for i_* but let's blackbox this.

3.2.5 Proper Base Change

The idea is

$$\text{PBC} = \text{CBC} + \text{SBC} + \text{ambidexterity}.$$

Let $p : X \to Y$ which we factor as $X \xhookrightarrow{i} P \xrightarrow{q} Y$ where i is a closed immersion and q is smooth and proper. We know base change (3.7) for $q_\sharp$ and ambidexterity is a relation between $q_\sharp$ and q_*. So the strategy will be to discuss ambidexterity and closed base change.

Theorem 3.27 (Ambidexterity)**.** Let $f : X \to Y$ be smooth and proper. Consider the diagram

$$\begin{array}{ccccc} X & & & & \\ & \searrow^{\Delta_f} & & & \\ & & X \times_Y X & \xrightarrow{\pi_1} & X \\ & & \downarrow{\scriptstyle \pi_2} & \lrcorner & \downarrow{\scriptstyle f} \\ & & X & \xrightarrow[f]{} & Y \end{array}$$

then

$$\mathrm{Nm}_f : f_\sharp \simeq f_\sharp(\pi_2)_*(\Delta_f)_* \Rightarrow f_*(\pi_1)_\sharp(\Delta_f)_* \simeq f_*\langle \Omega_f \rangle$$

is an equivalence.

An equivariant analog is the Wirthmüller isomorphism or a sort of Atiyah duality.

Theorem 3.28 (CBC)**.** Consider a pullback square

$$\begin{array}{ccc} Y_Z & \xhookrightarrow{k} & Y \\ {\scriptstyle g}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ Z & \xhookrightarrow[i]{} & S \end{array}$$

where i, k are closed immersions, then $f^* i_* \overset{\simeq}{\Rightarrow} k_* g^*$.

Proof. The functor i_* is fully faithful (3.26), i^* is epic, so we may precompose with i^* and prove this equivalence then. This part is some quick abstract nonsense. □

Theorem 3.29 (CPF)**.** Let i be a closed immersion, then i_* is i^*-linear, i.e. $i_* X \otimes Y \xrightarrow{\simeq} i_*(X \otimes i^* Y)$.

Proof. Let j denote the open complement. Then, i^*, j^* are jointly conservative and we check the abstrat nonsense there. □

Theorem 3.30 (SCBC)**.** Consider

$$\begin{array}{ccc} Z' & \xhookrightarrow{k} & X \\ {\scriptstyle g}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ Z & \xhookrightarrow[i]{} & X \end{array}$$

with closed immersions i, k and smooth f, g, then $f_\sharp k_* \overset{\simeq}{\Rightarrow} i_* g_\sharp$.

Proof. Use that k_* is fully faithful (3.26). □

3.3 Assembling the Six Functors

What do we know? We have a geometric setup $(\mathbf{Sch}^{\text{qcqs}}, \text{lft})$ with a suitable decomposition $I =$ (open immersions) and $P =$ (proper maps) by Nagata compactification. The functor

$$\mathcal{SH}^* : \mathbf{Sch}^{\text{op}} \to \mathbf{CAlg}(\mathbf{Pr}^L)$$

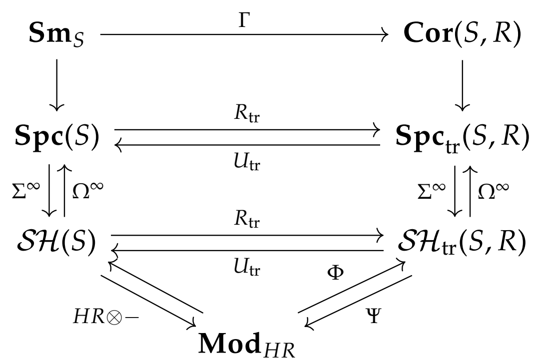
is sheafy, so locally of finite type maps is enough Nisnevich locally.

- I: For $j \in I$ we set $j_! = j_\sharp$ as left adjoint to j^*. Need BC = SBC and PF = SPF.
- P: For $p \in P$ we set $p_* = p_*$ as right adjoint to p^*. Need BC = CBC + (SBC + ambidexterity) and PF = CPF + (SPF + ambidexterity).
- $I \cap P$: Need BC = SPBC $\Longleftarrow$ SCBC + (SSBC + ambidexterity)

Corollary 3.31. The functor $\mathcal{SH}$ extends to a Nisnevich sheafy 6FF on $(\mathbf{Sch}^{\text{qcqs}}, \text{lft})$.

4 Transfers, Motivic Cohomology & EM-Spectra (Thomas Blom)

Let $R \in \mathbf{CRing}$. Consider TALK 4 13.11.2025



Today: Explain the entire part that is not the left column.

(1) $\mathbf{Cor}(S, R)$,

(2) $\mathbf{Spc}_{\text{tr}}(S, R)$,

(3) $\mathcal{SH}_{\text{tr}}(S, R)$,

(4) $HR \circlearrowleft HA$,

(5) Motivic cohomology.

4.1 Correspondences

Idea: Motivic cohomology admits more functoriality than just contravariant functoriality in maps of schemes (2.31(i)) which we encode by $\mathbf{Cor}(S, R)$. This is reminiscent of the wrong-way maps in classical algebraic topology, called *transfers*.

Definition 4.1. The category $\mathbf{Cor}(S, R)$ of **finite correspondences** has the same objects as $\mathbf{Sm}_S$ and mapping R-modules given by

$$\text{Map}_{\mathbf{Cor}(S,R)}(X, Y) = c_0(X \times_S Y/X) \otimes R$$

where roughly $c_0(X \times_S Y/X)$ is the free abelian group on all closed integral subschemes $Z \hookrightarrow X \times_S Y$ such that $Z \to X \times_S Y \to X$ is finite and onto on an irreducible component.[15] Maps can be composed using 'pullback/intersection'.

Remark* 4.2. One should thus think of a finite correspondence as a finite linear combination of (multi-valued) relations.

Remark 4.3. One example is

$$\mathbb{A}^1 \longleftarrow \operatorname{Spec}(k[x,y]/(x^2 - f(y)) \longrightarrow \mathbb{A}^1.$$

giving a finite correspondence $\operatorname{Spec}(k[x,y]/(x^2 - f(y)) \subseteq \mathbb{A}^1 \times \mathbb{A}^1$. More generally, any map $f : X \to Y$ in $\mathbf{Sm}_S$ admits a graph Γ_f, defining a functor $\Gamma : \mathbf{Sm}_S \to \mathbf{Cor}(S, R)$.

Voevodsky defines mixed motives as certain presheaves on $\mathbf{Cor}(S, R)$.

4.2 Spaces with Transfers

The category $\mathbf{Cor}(S, R)$ is additive (with $\amalg$) and symmetric monoidal (with $\times$). Imposing Nisnevich sheaves and $\mathbb{A}^1$-invariance gives

$$\mathbf{Spc}_{\mathrm{tr}}(S, R) \subseteq \mathbf{PSh}^{\Sigma}(\mathbf{Cor}(S, R))$$

where the right side is something like simplicial presheaves (and in Hoyois' paper it actually is). Here, the Nisnevich topology on $\mathbf{Sm}_S$ induces one on $\mathbf{Cor}(S, R)$.

This is modelled by '$\mathbb{A}^1$-invariant non-negative chain complexes of Nisnevich sheaves' with transfers in $\mathbf{Mod}_R$.[16]

Remark 4.4. Alternatively, the functor $\Gamma : \mathbf{Sm}_S \to \mathbf{Cor}(S, R)$ induces by Yoneda extension a functor $\mathbf{PSh}(\mathbf{Sm}_S) \to \mathbf{PSh}^{\Sigma}(\mathbf{Cor}(S, R))$ which is an algebra through Day convolution. Can then define $\mathbf{Spc}_{\mathrm{tr}}(S, R) = \mathbf{PSh}^{\Sigma}(\mathbf{Cor}(S, R)) \otimes_{\mathbf{PSh}(\mathbf{Sm}_S)} \mathbf{Spc}_*(S)$.

I think this perspective also shows that Γ induces $R_{\mathrm{tr}} : \mathbf{Spc}(S) \to \mathbf{Spc}_{\mathrm{tr}}(S, R)$. Note that the usual functoriality in $\mathbf{PSh}_{\mathrm{tr}}(S, R)$ precisely comes from Γ but the correspondences encode a lot more possible maps.

4.3 Spectra with Transfers

Definition 4.5. Let $\mathcal{SH}_{\mathrm{tr}}(S, R) = \mathbf{Spc}_{\mathrm{tr}}(S, R)[(R_{\mathrm{tr}}\mathbb{P}^1)^{-1}] \simeq \mathbf{Spc}_{\mathrm{tr}}(S, R) \otimes_{\mathbf{Spc}_*(S)} \mathcal{SH}(S)$.

4.4 Eilenberg-MacLane Spectra

Definition 4.6. Let $HR = U_{\mathrm{tr}}(\mathbb{1}_{\mathcal{SH}_{\mathrm{tr}}(S,R)})$.

By general module nonsense we get an adjunction

$$\mathbf{Mod}_{HR} \underset{\Psi}{\rightleftarrows} \mathbf{Mod}_{\mathbb{1}}(\mathcal{SH}_{\mathrm{tr}}(S, R)) = \mathcal{SH}_{\mathrm{tr}}(S, R)$$

Theorem 4.7 (Østvær–Röndigs)**.** This restricts to an equivalence between the full subcategories of cellular objects.

Observe:

[15] This is slightly wrong, $Z \to X$ needs to dominate an irreducible component.
[16] This secretly uses hypercompleteness of $\mathbf{Sh}_{\mathrm{Nis}}(\mathbf{Sm}_S)$.

$$\begin{array}{ccc} \mathcal{D}_{\geq 0}(R) & \underset{\longleftarrow}{\overset{\text{const}}{\longrightarrow}} & \mathbf{Spc}_{\text{tr}}(S, R) \\ \downarrow & & \downarrow \\ \mathcal{D}(R) & \underset{\longleftarrow}{\overset{\text{const}}{\longrightarrow}} & \mathcal{SH}_{\text{tr}}(S, R) \end{array}$$

so for $A \in \mathcal{D}(R)$ we get that $HA = U_{\text{tr}} \operatorname{const}(A)$ is canonically an HR-module.

Definition 4.8. We write $K(A(q), p) = U_{\text{tr}}(R_{\text{tr}} S^{p,q} \otimes_R \operatorname{const} A)$.

4.5 Motivic Cohomology

Theorem 4.9. Let A be an R-module and $X \in \mathbf{Sm}_S$ be essentially smooth over a field k.[17] Then,

$$H^{p,q}(X, A) \cong [\Sigma^\infty_+ X, \Sigma^{p,q} HA].$$

Proof Idea. 'Essentially by definition' one obtains $H^{p,q}(X, A) \cong [X_+, K(A(q), p)]$ and deduce the stable version from this. Hoyois uses the definition

$$H^{p,q}(X; \mathbb{Z}) = H^{p-q}_{\text{Zar}}\left(X, \mathbb{Z}_{\text{tr}}(\mathbb{G}_m^{\otimes q})[-q]\right)$$

or perhaps some slightly different indexing. □

5 Motivic Steenrod Algebra (David Wiedemann)

Let k be a perfect field and $\ell \neq \operatorname{char} k$ with $c = c(k)$ the characteristic exponent. Talk 5 20.11.2025

Let $\mathcal{M}^{\bullet,\bullet}$ be the bi-graded algebra of stable motivic cohomology operations with $\mathbb{Z}/\ell$-coefficients. These contain 'reduced power operations' $P^i \in \mathcal{M}^{2i(\ell-1),i(\ell-1)}$ and Bockstein operation $\beta \in \mathcal{M}^{1,0}$ as well as $H^{\bullet,\bullet}(k, \mathbb{Z}/\ell)$. Let $\mathcal{A}^{\bullet,\bullet} \subseteq \mathcal{M}^{\bullet,\bullet}$ be the algebra spanned by these.

We are in the situation $\mathcal{A}^{\bullet,\bullet} \hookrightarrow \mathcal{M}^{\bullet,\bullet} \twoheadleftarrow H\mathbb{Z}/\ell^{\bullet,\bullet} H\mathbb{Z}/\ell$ [HKOsr17, p. 2].

Theorem 5.1. Let S/k be a Noetherian scheme of finite Krull dimension which is essentially smooth over k.

(i) The map $\mathcal{A}^{\bullet,\bullet} \to \mathcal{M}^{\bullet,\bullet}$ is an isomorphism with explicit basis as an $H^{\bullet,\bullet}(S, \mathbb{Z}/\ell)$-module given by $\{\beta^{\varepsilon_r} P^{i_r} \cdots \beta^{\varepsilon_1} P^{i_1} \beta^{\varepsilon_0} : r \geq 0, i_j \geq 0, \varepsilon_j \in \{0,1\}, i_{j+1} \geq \ell i_j + \varepsilon_j\}$.

(ii) The map $(H\mathbb{Z}/p)^{\bullet,\bullet}(H\mathbb{Z}/p) \to \mathcal{M}^{\bullet,\bullet}$ is an isomorphism.

(iii) There is an equivalence of $H\mathbb{Z}/\ell$-modules $H\mathbb{Z}/\ell \otimes H\mathbb{Z}/\ell \simeq \bigoplus_\alpha \Sigma^{p_\alpha,q_\alpha} H\mathbb{Z}/\ell$.

This result was essentially proven by Voevodsky in characteristic zero and the point of [HKOsr17] is to push this to positive characteristics through étale cohomology. As such, we will often reduce to Voevodsky's works throughout this talk.

Definition 5.2. Let S be a scheme and $\mathbf{Sch}_S$ be the category of separated finite-type schemes. We call a full subcategory $\mathscr{C} \subseteq \mathbf{Sch}_S$ **admissible** if:

(i) $S, \mathbb{A}^1_S \in \mathscr{C}$,

(ii) If $X \in \mathscr{C}$ and $U \to X$ is a finite étale map, then $U \in \mathscr{C}$,

(iii) $\mathscr{C}$ is closed under finite (co-)products.

[17] This probably means that it's a filtered colimit of smooth things.

Example 5.3. The most important will be $\mathbf{Sm}_S$ but also normal schemes will be relevant.[18]

Theorem 5.4 (Fundamental square)**.** Let R be a $\mathbb{Z}_{(l)}$-algebra and $i : \mathscr{C} \to \mathscr{D}$ is an inclusion admissible subcategories with $\mathscr{C} \subseteq \mathbf{Sm}_k$. Then, the square

$$\begin{array}{ccc} \mathcal{H}^*_{\mathrm{Nis},\mathbb{A}^1}(\mathscr{D}) & \xrightarrow{i^*} & \mathcal{H}^*_{\mathrm{Nis},\mathbb{A}^1}(\mathscr{C}) \\ {\scriptstyle R^{\mathrm{tr}}}\downarrow & & \downarrow{\scriptstyle R^{\mathrm{tr}}} \\ \mathcal{H}^{\mathrm{tr}}_{\mathrm{Nis},\mathbb{A}^1}(\mathscr{D},R) & \xrightarrow[i^*]{} & \mathcal{H}^{\mathrm{tr}}_{\mathrm{Nis},\mathbb{A}^1}(\mathscr{C},R) \end{array}$$

commutes.

We will prove this later.

Theorem 5.5. Let S be essentially smooth over k and A be a finitely generated $\mathbb{Z}[1/c]$-module and F be a field of characteristic $\neq c$. Let $p \geq 2$ and $q \geq 0$. Then, $F^{\mathrm{tr}}K(A(q),p)_{\mathbf{Sm}_S}$ is a direct sum of $F^{\mathrm{tr}}S^{a,b}$ with $a \geq 2b$ and $b \geq q$. If $L \in \mathcal{H}^{\mathrm{tr}}_{\mathbb{A}^1,\mathrm{Nis}}(\mathbf{Sm}_S, F)$ is a direct sum of $F^{\mathrm{tr}}S^{a,b}$ for $a \geq 2b, b \geq \dim S$, it is called **split proper Tate of weight** $\geq \dim S$.

Proof Sketch. For admissible $\mathscr{C}$ let $K_{\mathscr{C}} = K(A(q),p)_{\mathscr{C}}$. If $\mathscr{C}$ is the subcategory of normal schemes over S, then Voevodsky shows the result. Let $i : \mathbf{Sm}_S \hookrightarrow \mathscr{C}$, then

$$i^* F^{\mathrm{tr}} K_{\mathscr{C}} \simeq F^{\mathrm{tr}} i^* K_{\mathscr{D}} \simeq F^{\mathrm{tr}} K_{\mathbf{Sm}_S}$$

using the fundamental square. But $i_! \dashv i^*$ and $c_!$ is fully faithful. □

Fix $H = H\mathbb{Z}/\ell$ and $K_n = K(\mathbb{Z}/\ell(n), 2n)$.

Corollary 5.6. The map $H^\bullet H \to \mathcal{M}^{\bullet,\bullet}$ is an isomorphism.

Proof. There is a $\lim^1$-sequence

$$0 \longrightarrow \lim^1_n \widetilde{H}^{p-1+2n,q+n}(K_n,\mathbb{Z}/\ell) \longrightarrow H^{p,q}H \longrightarrow \lim_n \widetilde{H}^{p+2n,q+n}(K_n,\mathbb{Z}/\ell) \longrightarrow 0$$

By the theorem above $(\mathbb{Z}/\ell)^{\mathrm{tr}}K_n \simeq \Sigma^{2n,n}H_n$ for a split proper Tate H_n of weight ≥ 0. One computes that

$$\widetilde{H}^{p-1+2n,q-1+n}(K_n,\mathbb{Z}/\ell) \cong [\Sigma^\infty H_n, \Sigma^{p-1,q}(\mathbb{Z}/\ell)^{\mathrm{tr}}\mathbb{1}].$$

It is a general fact that

$$\bigoplus_n H_n \longrightarrow \bigoplus_n H_n \longrightarrow \mathrm{colim}_n H_n$$

is split by Voevodsky. So the $\lim^1$-term vanishes. □

Corollary 5.7. There is a splitting $H \otimes H \simeq \bigwedge_\alpha \Sigma^{p_\alpha,q_\alpha} H$ as H-modules.

Proof. The object $\mathrm{colim}_n H_n$ is split proper Tate by work of Voevodsky of weight ≥ 0 and $\mathrm{colim}_n H_n \simeq \bigoplus_\alpha (\mathbb{Z}/\ell)^{\mathrm{tr}} S^{p_\alpha,q_\alpha}$. Thus,

$$(\mathbb{Z}/\ell)^{\mathrm{tr}}(H) \simeq \Sigma^\infty \operatorname*{colim}_n H_n \simeq (\mathbb{Z}/\ell)^{\mathrm{tr}} \bigoplus_\alpha \Sigma^\infty S^{p_\alpha,q_\alpha}.$$

So $(\mathbb{Z}/\ell)^{\mathrm{tr}}(H)$ is cellular which is $\Phi(H \otimes H)$ for $\Phi : \mathbf{Mod}_H \to \mathcal{SH}^{\mathrm{tr}}(\mathscr{C}, \mathbb{Z}/\ell)$. Therefore, we get $H \otimes H \simeq H \otimes \bigoplus_\alpha \Sigma^\infty S^{p_\alpha,q_\alpha}$. □

[18] Throughout the talk the issue came up about what is actually meant by this, since normal schemes are not be closed under finite products. I'm not sure what an answer to this is.

5.1 Comparison with Étale Steenrod Algebra

Let k be algebraically closed. Let $a_{\text{ét}} : \mathcal{H}^*(\mathbf{Sch}_k) \to \mathcal{H}^*_{\text{ét}}(\mathbf{Sch}_k)$ and $u_{\text{ét}} : D(\mathbf{Sm}_k) \to D(\mathbf{Sm}_k)$. There is an isomorphism

$$\mathbb{Z}(1)[1] \simeq よ_{\mathbb{G}_m} \in D_{\text{Nis}}(\mathbf{Sm}_k).$$

By Kummer, if $m \in k^\times$, there is an isomorphism $a_{\text{ét}}\mathbb{Z}/m(1) \simeq \mu_m$ which implies

$$a_{\text{ét}}(\mathbb{Z}/m(q)[p]) \simeq \mu_m^{\otimes q}[p].$$

So $\widetilde{H}^{p,q}(X, \mathbb{Z}/m) \cong \widetilde{H}^p_{\text{Nis}}(X, \mathbb{Z}/m(q)) \to \widetilde{H}^p_{\text{ét}}(X, \mu_m^{\otimes q})$ for all pointed presheaves.

Theorem 5.8. This is an isomorphism for $p \le q$.

Corollary 5.9. Let $X \in \mathcal{H}^*(\mathbf{Sm}_k)$. Étale sheafification induces an isomorphism

$$\widetilde{H}^{\bullet,\bullet}(X, \mathbb{Z}/\ell)[\tau^{-1}] \cong H^\bullet_{\text{ét}}(X, \mu_\ell^{\otimes \bullet})$$

where $\tau \in \mu_\ell(k)$ is a primitive root of unity with $\tau \in H^{0,1}(\operatorname{Spec} k, \mathbb{Z}/\ell)$.

Let $\mathcal{H}^{\bullet,\bullet}_{\text{ét}} = \lim_n H^{\bullet+2n}(K_n^{\text{ét}}, \mu_\ell^{\otimes \bullet+n})$ with connecting maps coming from $\mathbb{P}^1 \otimes K_n^{\text{ét}} \to K_{n+1}^{\text{ét}}$. Since k is algebraically closed, μ_ℓ is constant, so $K_n^{\text{ét}}$ is constant. Since $\operatorname{Spec} k$ has no non-trivial covers, $R\Gamma : \mathcal{H}_{\text{ét}}(\mathbf{Sm}_k) \to \mathcal{S}$ is given by evaluation at $\operatorname{Spec} k$. Thus, $\mathrm{id} \to R\Gamma \circ c$ is an isomorphism. Thus,

$$\widetilde{H}^\bullet(K_n^{\text{ét}}, A) \cong H^\bullet_{\text{ét}}\left(K(\mu_\ell(k)^{\otimes 2n}, n), A\right) \cong \widetilde{H}^\bullet_{\text{Top}}\left(K(\mu_\ell(k)^{\otimes 2n}, n), A\right).$$

Take $A = \mathbb{Z}/\ell$. This gives an isomorphism of algebras $\chi : \mathcal{A}^\bullet \to \mathcal{M}^{\bullet,0}_{\text{ét}}$ where $\mathcal{A}^\bullet$ is the topological Steenrod algebra.

This allows us to define $P^i_{\text{ét}} = \tau^{i(\ell-1)}\chi(P^i)$. Note that $\mathcal{H}^{\bullet,\bullet}_{\text{ét}}$ is generated by $P^i_{\text{ét}}, \beta^i_{\text{ét}}$ and $\tau^{\pm 1}$. Use that $\tau : \mathcal{M}^{\bullet,n}_{\text{ét}} \to \mathcal{H}^{\bullet,n+1}_{\text{ét}}$. This defines a map $\psi : \mathcal{A}^{\bullet,\bullet} \to \mathcal{H}^{\bullet,\bullet}_{\text{ét}}$.

Lemma 5.10. The diagram

$$\begin{array}{ccc} \mathcal{A}^{\bullet,\bullet} & \longrightarrow & \mathcal{H}^{\bullet,\bullet} \\ & \searrow^{\psi} & \downarrow \\ & & \mathcal{H}^{\bullet,\bullet}_{\text{ét}} \end{array}$$

commutes.

Proof of 5.1.

(i) The theorem holds over any perfect field by Voevodsky, but we deal with algebraically closed k. It suffices to prove:

(a) The map $\mathcal{A}^{\bullet,\bullet}/\tau \to \mathcal{H}^{\bullet,\bullet}/\tau$ is injective.

(b) The map $\mathcal{A}^{\bullet,\bullet}[\tau^{-1}] \to \mathcal{H}^{\bullet,\bullet}[\tau^{-1}]$ is surjective.

We omit how to see that this is sufficient.

(a) This was shown by Voevodsky. For every $P = \sum_I a_I P^I$ a space X and $w \in H^{\bullet,\bullet}(X, \mathbb{Z}/\ell)$ such that $P(w) \neq 0$. The example $X = (B\mu_\ell)^N$ for $N \gg 0$ works.

(b) Consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{A}^{\bullet,\bullet}[\tau^{-1}] & \longrightarrow & \mathcal{H}^{\bullet,\bullet}[\tau^{-1}] & \longrightarrow & \widetilde{H}^{\bullet+2n,\bullet+n}(K_n, \mathbb{Z}/\ell)[\tau^{-1}] \\ & \searrow & \downarrow{\scriptstyle \varphi} & & \downarrow \\ & & \mathcal{H}^{\bullet,\bullet}_{\text{ét}} & \longrightarrow & \widetilde{H}^{\bullet+2n}_{\text{ét}}(K_n^{\text{ét}}, \mathbb{Z}/\ell) \end{array}$$

It suffices to see that φ is injective. Let $x = (x_0, x_1, \cdots) \in \mathcal{H}^{\bullet,\bullet}[\tau^{-1}]$ and $\operatorname{supp} \varphi(x) = 0$. Thus, $x_n \mapsto 0$ in $\widetilde{H}^{\bullet+2n,n}(K_n, \mathbb{Z}/\ell)[\tau^{-1}]$. But $(\mathbb{Z}/\ell)^{\mathrm{tr}} K_n$ is split proper Tate, so there is no τ-torsion, whence $x_n = 0$.

(iii) We compute
$$\mathcal{H}^{\bullet,\bullet} \cong [H, \Sigma^{\bullet,\bullet} H] \cong [H \otimes H, \Sigma^{\bullet,\bullet} H]_H \cong \prod_\alpha H^{\bullet - p_\alpha, \bullet, q_\alpha}.$$

Omit the result for general base schemes.

□

5.2 Back to Fundamental Square

Definition 5.11.

(i) An **fpsl cover** is a faithfully flat cover $f : U \to X$ such that $f_* \mathcal{O}_U$ is free of rank prime to ℓ.

(ii) An **ldh-cover** is a cover which cdh-locally is fpsl.

Theorem 5.12. Let $X \in \mathbf{Sch}_k$. There exists an ldh-cover by smooth & quasi-projective schemes.

Proposition 5.13. The map $i^* : \mathcal{H}^{\mathrm{tr}}(\mathbf{Sch}_k, R) \to \mathcal{H}^{\mathrm{tr}}(\mathbf{Sm}_k, R)$ preserves $R^{\mathrm{tr}} W_{\mathrm{ldh}}$-local equivalences.

Finally, the hard part of the theorem:

Corollary 5.14. Let R be a $\mathbb{Z}_{(\ell)}$-algebra. Then,

$$\begin{array}{ccc} \mathcal{H}^{\mathrm{tr}}(\mathbf{Sm}_k, R) & \longrightarrow & \mathcal{H}^{\mathrm{tr}}_{\mathrm{Nis},\mathbb{A}^1}(\mathbf{Sm}_k, R) \\ & \searrow \qquad \nearrow_{\exists} & \\ & \mathcal{H}^{\mathrm{tr}}_{\mathrm{ldh}}(\mathbf{Sm}_k, R) & \end{array}$$

Proof of 5.4. By Voevodsky we can assume that $\mathscr{C} = \mathbf{Sm}_k$ and $\mathscr{D} = \mathbf{Sch}_k$. It suffices to show that $R^{\mathrm{tr}} i^* F \to i^* R^{\mathrm{tr}} F$ is an isomorphism for $F \in \mathcal{H}^*_{\mathrm{Nis}}(\mathscr{C})$. Consider

$$\begin{array}{ccc} R^{\mathrm{tr}} i^* i_! i^* & \longrightarrow & R^{\mathrm{tr}} i^* \\ \downarrow & & \downarrow \\ i'^* R^{\mathrm{tr}} i_! i^* & \longrightarrow & i^* R^{\mathrm{tr}} \end{array}$$

The top and left maps are equivalences. It's left to show that the bottom map is an equivalence. When restricted to $F \in \mathbf{Sm}_k$ the map $i_! i^* F \to F$ is an equivalence. So $i_! i^* F \to F$ is a ldh-local equivalence. So $i^* R^{\mathrm{tr}} i_! i^* F \to i^* R^{\mathrm{tr}} F$ is an $R^{\mathrm{tr}} W_{\mathrm{ldh}}$-local equivalence, but these become equivalences in $\mathcal{H}^{\mathrm{tr}}_{\mathrm{Nis},\mathbb{A}^1}(\mathscr{D}, R)$. □

6 Motivic Dual Steenrod Algebra (Lucas Piessevaux)

Let k be a perfect field, c be the exponential characteristic of k and $\ell \neq \operatorname{char} k$. TALK 6 27.11.2025

Recollection 6.1. Recall $\mathcal{A}^{\bullet,\bullet} \cong \mathcal{M}^{\bullet,\bullet} \cong H^{\bullet,\bullet} H$ where

$$\begin{aligned} \mathcal{M}^{\bullet,\bullet} &= \text{algebra of bigraded } \mathbb{P}^1\text{-stable operations in } H \\ &= \lim_n \widetilde{H}^{\bullet+2n,\bullet+n}(K(\mathbb{F}_\ell(n); 2n); \mathbb{F}_\ell) \end{aligned}$$

as well as $\mathcal{A}^{\bullet,\bullet} \subseteq \mathcal{M}^{\bullet,\bullet}$ is the subalgebra on βP^i and $H^{\bullet,\bullet} H = \pi_{-\bullet,-\bullet} \operatorname{map}(H, H)$. Moreover, $H \otimes H \simeq \bigoplus_\alpha \Sigma^{p_\alpha, q_\alpha} H$ with $p_\alpha \geq 2q_\alpha$ and $q_\alpha \geq 0$, called split proper Tate.

Let us indicate how to get these βP^i. In the case $\ell = 2$ one gets

$$H^{\bullet,\bullet}(X;\mathbb{F}_2) \to H^{\bullet,\bullet}(X \times B_{\text{ét}}\mu_2;\mathbb{F}_2),\ x \mapsto \{\mathrm{Sq}^i\, x\}.$$

One can compute $H^{\bullet,\bullet}(B_{\text{ét}}\mu_2;\mathbb{F}_2) \cong H^{\bullet,\bullet}[\![t]\!]/[2]$.

Warning 6.2.

(i) The ∞-category $\mathbf{Mod}(\mathcal{SH}(k);H)$ is not a derived category of a field. So $\lim^1$ terms may potentially show up.

(ii) The algebra $H^{\bullet,\bullet} = H^{\bullet,\bullet}(k;\mathbb{F}_\ell)$ is not concentrated in one degree.

6.1 Panorama/Spoilers

We want to study the Hoyois–Hopkins–Morel equivalence. There is a map

$$f : \mathrm{MGL}[1/c]/(a_1, a_2, \cdots) \to H\mathbb{Z}[1/c]$$

which Hoyois shows to be an equivalence. The proof strategy is the following.

- $H\mathbb{Q} \otimes f$ is an equivalence by a computation of $H\mathbb{Q} \otimes \mathrm{MGL}$ which will be done by Sven.
- $H\mathbb{F}_\ell \otimes f$ is an equivalence by a computation of $H\mathbb{F}_\ell \otimes H\mathbb{Z}$ and coming from $H\mathbb{F}_\ell \otimes H\mathbb{F}_\ell$. This will be done today. We will also need something for $H\mathbb{F}_\ell \otimes \mathrm{MGL}$ which will be Fabio's talk.

So $H\mathbb{Z} \otimes f$ is an equivalence via fracture square and connectivity arguments.

6.2 Duality & Künneth

Goal: Set up conditions that guarantee a Künneth formula.

Definition 6.3. An HR-module M is **split** if there is a decomposition $M \simeq \bigoplus_\alpha \Sigma^{p_\alpha,q_\alpha} HR$.

Remark 6.4. These indices are unique, unless $S = \emptyset$.

Proof. We want to see $\pi_{p,q} s_q \Sigma^{p,q} HR \simeq R$. Recall that $s_q = \mathrm{cofib}(f_{q+1} \to f_q)$ where f_0 is the coreflection into $\mathcal{SH}^{\mathrm{eff}}(S) = \langle \Sigma^{n,0}\Sigma^\infty_+ X : n \in \mathbb{Z}\rangle$ and $f_q = \Sigma^{q,q} f_0 \Sigma^{-q,-q}$. We then apply a theorem from Levine: $s_0 \mathbb{1}_k \simeq H\mathbb{Z}_k$ for a perfect field k. □

Remark 6.5. Split implies cellular, but not the other way.

Counterexample. Let $R = \mathbb{F}_\ell$. Consider

$$\tau_\ell \in H^{0,\ell-1}(k;\mathbb{F}_\ell) \cong H^{0,\ell-1}_{\text{ét}}(k;\mathbb{F}_\ell) \cong H^0_{\text{ét}}(k;\mu_\ell^{\otimes(\ell-1)})$$

via $\tau_\ell \leftrightarrow (\zeta_\ell, \zeta_\ell^2, \cdots, \zeta_\ell^{\ell-1})$.

- Claim: The spectrum $\tau_{\ell-1} H\mathbb{F}_\ell$ is cellular but not split. (The spectrum $\tau_\ell^{-1} H\mathbb{F}_\ell \neq 0$ represents étale motivic cohomology.) It follows from the next claim.
- Claim′: For every q we have

$$\begin{aligned} \forall q : s_q \tau_\ell^{-1} H\mathbb{F}_\ell = 0 &\impliedby \tau_\ell^{-1} H\mathbb{F}_\ell \in \bigcap_q \mathcal{SH}^{\mathrm{eff}}(k)(q) \\ &\impliedby \exists q : \tau_\ell^{-1} H\mathbb{F}_\ell \in \mathcal{SH}^{\mathrm{eff}}(k)(q) \text{ which is furthermore } (0,\ell-1)\text{-periodic} \\ &\impliedby \tau_\ell^{-1} H\mathbb{F}_\ell \in \mathcal{SH}^{\mathrm{eff}}(k) \end{aligned}$$

The last step is by Levine's result.

□

Lemma 6.6. Let M, N be HR-modules. The Künneth map

$$\pi_{\bullet,\bullet}M \otimes_{HR_{\bullet,\bullet}} \pi_{\bullet,\bullet}N \to \pi_{\bullet,\bullet}(M \otimes_{HR} N)$$

is an isomorphism if one of the following are satisfied:

(i) M is split.

(ii) M is cellular and $\pi_{\bullet,\bullet}N$ is flat over $HR_{\bullet,\bullet}$.

Proof.

(i) Clear.

(ii) By flatness the map

$$\pi_{\bullet,\bullet}(-) \otimes_{HR_{\bullet,\bullet}} \pi_{\bullet,\bullet}N \to \pi_{\bullet,\bullet}(- \otimes_{HR} N)$$

is a transformation of homological functors. They agree on $S^{p,q}$ by (i), so they agree on cellular spectra by commuting with colimits.

□

For the following $\underline{\pi}$ lands in $\mathbf{Sh}_{\mathrm{Nis}}(\mathbf{Sm}_S; \mathbf{Ab})$ which with $\Gamma(k, -)$ yields π which lands in $\mathbf{Ab}$.

Lemma 6.7. The following are equivalent for an HR-module M.

(i) M is split.

(ii) $\underline{\pi}_{\bullet,\bullet}M$ is free over $\underline{\pi}_{\bullet,\bullet}HR$.

(iii) $\pi_{\bullet,\bullet}M$ is free over $\pi_{\bullet,\bullet}HR$ and M is cellular.

Proof.

(i) $\Longrightarrow$ (ii), (iii) Okay.

(ii) $\Longrightarrow$ (i) Pick a basis over $\underline{\pi}_{\bullet,\bullet}R$, so we get an $\underline{\pi}_{\bullet,\bullet}$-equivalence, and hence an equivalence.

(iii) $\Longrightarrow$ (i) Pick a basis over $\pi_{\bullet,\bullet}R$, so we get an $\pi_{\bullet,\bullet}$-equivalence and thus a cellular equivalence (recalling that $\pi_{p,q} = [S^{p,q}, -]$).

□

Definition 6.8. A split HR-module is **psf** (= proper, slicewise finite) if for all p_α, q_α such that $p \geq 2q, q \geq 0$ and for all q finitely many copies of q appear.

Remark 6.9. The spectrum $HR \otimes X$ is psf for X a Grassmannian, Thom spectrum, MGL, $H\mathbb{F}_\ell, \cdots$.

Goal: $H\mathbb{F}_\ell \otimes H\mathbb{F}_\ell$ is psf.

Lemma 6.10. Let X be smooth of dimension d over k and A be any HR-module. Then, $H^{p,q}(X; A) = 0$ if either of the following hold:

(i) $q < 0$,

(ii) $p > q + d$,

(iii) $p > 2q$.

Proof. We do the proof for $R = A = \mathbb{Z}$.

(i), (ii) These are immediate from

$$H^{p,q}(X;\mathbb{Z}) \cong \mathbb{H}^p_{\mathrm{Zar}}(X;\mathbb{Z}(q)).$$

Indeed, $\mathbb{Z}(q) = 0$ for $q < 0$ and $\mathbb{Z}(q)$ is concentrated in degrees $\leq q$.

(iii) Recall $H^{p,q}(X;\mathbb{Z}) \cong \mathrm{CH}^q(X; 2q-p)$.

□

Consider $H^{\bullet,\bullet}(\mathbb{C};\mathbb{F}_\ell)$. We have $H^{i,i}(\mathbb{C};\mathbb{F}_\ell) \cong K_i^M(\mathbb{C})/\ell$. Recall $K_\bullet^M(k) \cong T_{\mathbb{Z}}[k^\times]/([a][1-a]=0)$. For $a \in \mathbb{C}^\times$ we have $[a] = \ell[\sqrt[\ell]{a}] = [\sqrt[\ell]{a}^\ell]$. In particular, we get $H^{i,i}(\mathbb{C},\mathbb{F}_\ell) \cong \mathbb{F}_\ell$ for $i \geq 0$. So $H^{\bullet,\bullet}(\mathbb{C},\mathbb{F}_\ell) \cong \mathbb{F}_\ell[\tau]$.

Lemma 6.11. Let M, N be psf over HR. Let $M^\vee = \mathrm{map}_{HR}(M, HR)$.

(i) The module $M \otimes_{HR} N$ is psf.

(ii) The module $M^\vee$ is split.

(iii) The map $M^\vee \otimes_{HR} M \to HR$ is a perfect pairing.

(iv) There is an equivalence $M^\vee \otimes_{HR} N^\vee \simeq (M \otimes_{HR} N)^\vee$.

(v) The duality on homotopy is perfect.

(vi) Künneth holds and it is compatible with duality.

Proof. Most parts are not so hard. It remains to show all those statements involving duality. This follows from the observation

$$\coprod_\alpha \Sigma^{\pm p_\alpha, \pm q_\alpha} HR \xrightarrow{\simeq} \prod_\alpha \Sigma^{\pm p_\alpha, \pm q_\alpha} HR.$$

This is some finite type argument. □

Proposition 6.12. The spectrum $H\mathbb{F}_\ell \otimes H\mathbb{F}_\ell$ is psf.

Proof. In the last talk we saw that it was SPT of weight ≥ 0, on generators corresponding to βP^I's. □

Corollary 6.13. The algebra $\mathcal{A}^{\bullet,\bullet}$ admits a coproduct Δ over $H^{\bullet,\bullet}$.

Proof. There is an isomorphism

$$H^{\bullet,\bullet}(H \otimes H) \cong H^\bullet H \otimes_{H^{\bullet,\bullet}} H^{\bullet,\bullet} H \cong \mathcal{A}^{\bullet,\bullet} \otimes_{H^{\bullet,\bullet}} \mathcal{A}^{\bullet,\bullet}.$$

Take $\mu^* : \mathcal{A}^{\bullet,\bullet} = H^{\bullet,\bullet}H \to H^{\bullet,\bullet}(H \otimes H)$. □

6.3 Cooperations

Definition 6.14. We write $\mathcal{A}_{\bullet,\bullet}$ for the $H^{\bullet,\bullet}$-linear dual of $\mathcal{A}^{\bullet,\bullet}$.

Remark 6.15. There is an isomorphism $\mathcal{A}_{\bullet,\bullet} \cong \pi_{\bullet,\bullet}(H \otimes H)$ by the hard work we did. In fact, $H_{\bullet,\bullet}M \cong \mathcal{A}_{\bullet,\bullet} \otimes_{H_{\bullet,\bullet}} \pi_{\bullet,\bullet}M$ for every $M \in \mathbf{Mod}_H$. Setting $M = H^{\otimes i}$ we see that $(H_{\bullet,\bullet}, \mathcal{A}_{\bullet,\bullet})$ forms a bigraded Hopf algebroid and $H_{\bullet,\bullet} : \mathcal{SH}(S) \to \mathbf{Mod}_{H_{\bullet,\bullet}}$ lifts to $\mathbf{Comod}_{\mathcal{A}_{\bullet,\bullet}}$.

Recall that a Hopf algebroid (A, Γ) is a cogroupoid in rings together with structure maps $\eta_L, \eta_R : A \to \Gamma$ and $\Delta : \Gamma \to \Gamma \otimes_A \Gamma$ and $\varepsilon : \Gamma \to A$.

Definition 6.16. The Hopf algebroid (A, Γ) is

- $A = \mathbb{F}_\ell[\rho, \tau]$,
- $\Gamma \cong A[\tau_0, \tau_1, \cdots, \xi_1, \xi_2, \cdots]/(\tau_i^2 - \tau\xi_{i+1} - \rho\tau_{i+1} - \rho\tau_0\xi_{i+1} = 0)$,
- η_L is the preferred inclusion,
- $\eta_R(\tau) = \tau + \rho\tau_0$,
- $\Delta(\rho) = \rho \otimes 1$,
- $\Delta(\tau) = \tau \otimes 1$,
- $\Delta(\tau_r) = \tau_r \otimes 1 + 1 \otimes \tau_r + \sum_{i=0}^{r-1} \xi_{r-i}^{\ell^i} \otimes \tau_i$,
- $\Delta(\xi_r) = \xi_r \otimes 1 + 1 \otimes \xi_r + \sum_{i=0}^{r-1} \xi_{r-i}^{\ell^i} \otimes \xi_i$.

Remark 6.17.

(i) For $p = 2$ we have $\mathcal{A}_\bullet \cong \mathbb{F}_2[\xi_1, \xi_2, \cdots]$.

(ii) For $p = 3$ we have $\mathcal{A}_\bullet \cong \mathbb{F}_p[\xi_1, \xi_2, \cdots] \otimes \Lambda(\tau_0, \tau_1, \cdots)$.

Construction 6.18. Define a map $A \to H_{\bullet,\bullet}$ such that:

- Case 1: ℓ is odd. Kill ρ, τ.
- Case 2: $\ell = 2$. We put

$$\begin{aligned} \rho &\mapsto [-1] \in H_{-1,-1} \cong H^1_{\text{ét}}(S; \mu_2) \leftarrow H^0_{\text{ét}}(S; \mathbb{G}_m) \\ \tau &\mapsto \text{generator of } H_{0,-1} \cong \operatorname{Hom}(\pi_0 S, C_2) \end{aligned}$$

Theorem 6.19. Using this map, and setting $|\tau_r| = (2\ell^r - 1, \ell^r - 1)$ and $|\xi_r| = (2\ell^r - 2, \ell^r - 1)$ we get $\mathcal{A}_{\bullet,\bullet} \cong \Gamma \otimes_A H_{\bullet,\bullet}$.

Construction 6.20. Let $E = (\varepsilon_i)$ and $R = (r_i)$ where $\varepsilon_i \in \{0, 1\}$ and $r_i \in \mathbb{N}$. We put

$$\tau^E = \tau_0^{\varepsilon_0}\tau_1^{\varepsilon_1}\cdots \quad \text{and} \quad \xi^R = \xi_1^{r_1}\xi_2^{r_2}\cdots.$$

Set $\rho(E, R) \in \mathcal{A}^{\bullet,\bullet}$ for the dual to $\tau^E\xi^R$. Set $\rho(E, R) = Q(E)P^R$, so $Q_0 = \beta =$ dual to τ_0.

Recall $H^\bullet(\mathbb{Z}, \mathbb{F}_2) \cong \mathcal{A}^\bullet / \operatorname{Sq}^1 \cong \mathcal{A}^\bullet/\mathcal{E}(0)$ where $\operatorname{Sq}^1 = \beta = Q_1$. So $\mathbb{Z} = \mathrm{BP}\langle 0\rangle$. In that regard,

$$H_\bullet(\mathrm{BP}, \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p[\xi_1, \xi_2, \cdots] & p \geq 3, \\ \mathbb{F}_2[\xi_1^2, \xi_2^2, \cdots] & p = 2. \end{cases}$$

Lemma 6.21. The left ideal of $\mathcal{A}^{\bullet,\bullet}$ generated by the Q_i's is equivalent as a left module to the left module generated by $\rho(E, R)$ with $E \neq \emptyset$.

Definition 6.22. We write $\mathcal{P}_{\bullet,\bullet} \subseteq \mathcal{A}_{\bullet,\bullet}$ for the left submodule generated by ξ^R.

Corollary 6.23. The submodule $\mathcal{P}_{\bullet,\bullet} \subseteq \mathcal{A}_{\bullet,\bullet}$ is dual to $\mathcal{A}^{\bullet,\bullet} \to \mathcal{A}^{\bullet,\bullet}/(Q_0, Q_1, \cdots)$.

6.4 Homology of $H\mathbb{Z}$

Here is some notation.

(i) Write $\mathcal{B}$ for the basis $\{\tau^E \xi^R\}$ of $\mathcal{A}_{\bullet,\bullet}$ over $H_{\bullet,\bullet}$. Then, the map $\bigoplus_{\zeta\in\mathcal{B}} \Sigma^{|\zeta|}H \to H \otimes H$ is an equivalence.

(ii) Write ∂ for the Bockstein associated to

$$H\mathbb{Z} \xrightarrow{\ell} H\mathbb{Z} \longrightarrow H\mathbb{F}_\ell \xrightarrow{\partial} \Sigma^{1,0}H\mathbb{Z}.$$

It induces a SES of comodules

$$0 \longrightarrow H_{\bullet,\bullet}H\mathbb{Z} \longrightarrow H_{\bullet,\bullet}H\mathbb{F}_\ell \xrightarrow{\partial} H_{\bullet-1,\bullet}H\mathbb{Z} \longrightarrow 0.$$

(iii) Recall that β is the composite

$$H\mathbb{F}_\ell \xrightarrow{\partial} \Sigma^{1,0}H\mathbb{Z} \xrightarrow{\pi} H\mathbb{F}_\ell.$$

So $H_{\bullet,\bullet}H\mathbb{Z} \cong \ker \beta_*$.

(iv) Write $\mathcal{B}_\mathbb{Z} = \{\tau^E\xi^R : \varepsilon_0 = 0\}$.

Theorem 6.24. The map $\bigoplus_{\zeta\in\mathcal{B}_\mathbb{Z}} \Sigma^{|\zeta|}H \to H \otimes H\mathbb{Z}$ is an equivalence.

Proof. Consider

$$\begin{array}{ccccc}
\bigoplus_{\zeta\in\mathcal{B}_\mathbb{Z}} \Sigma^{|\zeta|}H & \longrightarrow & \bigoplus_{\zeta\in\mathcal{B}} \Sigma^{|\zeta|}H & \longrightarrow & \bigoplus_{\zeta\in\mathcal{B}\setminus\mathcal{B}_\mathbb{Z}} \Sigma^{|\zeta|}H \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\
H\otimes H\mathbb{Z} & \longrightarrow & H \otimes H & \xrightarrow[\partial]{} & H \otimes \Sigma^{1,0}H\mathbb{Z}
\end{array}$$

For γ note that $\mathcal{B}\setminus\mathcal{B}_\mathbb{Z} \cong \mathcal{B}_\mathbb{Z}, (\varepsilon_0 = 1) \leftrightarrow (\varepsilon_0 = 0)$. So γ 'is' just α. Here, α is a mono on $\underline{\pi}_{\bullet,\bullet}$ by the Five Lemma. But γ 'is' α, so $\underline{\pi}_{\bullet,\bullet}\alpha$ is surjective as well. □

7 Algebraic Cobordism & Landweber Exactness

The main goal of this talk is to compute $\pi_{\bullet,\bullet}(\mathrm{MGL} \otimes H\mathbb{Q})$ which is one of the ingredients needed to prove the Hopkins–Morel theorem. It turns out that we might as well replace $H\mathbb{Q}$ be an oriented motivic ring spectrum, and arguments remain the same. TALK 7 04.12.2025

We will relate MGL to formal groups.

7.1 (Grading) Conventions

Fix a regular Noetherian finite scheme of finite Krull dimension. If $M_{\bullet,\bullet}$ is bigraded, we write $M_\bullet = M_{(2,1)\bullet}$. This means that given $M_{\bullet,\bullet}$ and $N_\bullet$ we can talk about maps $N_\bullet \to M_{\bullet,\bullet}$ meaning $N_\bullet \to M_\bullet = M_{(2,1)\bullet}$.

7.2 Grassmannians

Definition 7.1.

(i) We write $\mathrm{Gr}_n(\mathbb{A}^k)$ for the scheme parametrizing subvector bundles of rank n of the trivial bundle of rank k such that the inclusion is locally split.

(ii) We write $G(n,d)$ for the scheme parametrizing locally free quotients of rank d of the trivial rank n bundle.

Remark 7.2.

(i) There is an isomorphism $G(n,d) \cong \mathrm{Gr}_{n-d}(\mathbb{A}^n)$.

(ii) There is a SES of VBs on $G(n,d)$:

$$0 \longrightarrow \mathcal{K}_{n,d} \longrightarrow \mathcal{O}^n \longrightarrow \mathcal{Q}_{n,d} \longrightarrow 0.$$

Definition 7.3. Let $\mathrm{Gr}_n = \mathrm{colim}_k \mathrm{Gr}_n(\mathbb{A}^{n+k})$. Consider the maps

$$\Sigma^{\mathbb{A}^1} \mathrm{Th}(\gamma_n) \longrightarrow \mathrm{Th}(\gamma_{n+1}|_{\mathrm{Gr}_n}) \longrightarrow \mathrm{Th}(\gamma_{n+1})$$

along which we take a colimit to define $\mathrm{MGL} = \mathrm{colim}_n \Sigma^{-\mathbb{A}^n}\Sigma^\infty \mathrm{Th}(\gamma_n)$.

More 'canonically': Bachmann–Hoyois define a functor $\mathbf{Sm}_S \to \mathcal{SH}(S)$, $(f : X \to S) \mapsto f_\sharp \mathrm{Th}_X(j)$. Here, $f_\sharp$ was the left adjoint to f^*. We define $\mathrm{MGL} = \mathrm{colim}_{f\in \mathbf{Sm}_S} f_\sharp \mathrm{Th}_X(j)$. This naturally gives gives MGL a normed structure, in particular an $\mathbb{E}_\infty$-algebra in $\mathcal{SH}(S)$.

7.3 Cohomology of Grassmannians

Definition 7.4. An **oriented motivic ring spectrum** E is a motivic ring together with a class $c_\infty \in E^{2,1}(\mathbb{P}^\infty)$ which restricts to the unit $c_1 \in \widetilde{E}^{2,1}(\mathbb{P}^1) \cong E^{0,0}(*)$.

Example 7.5. The spectra MGL and KGL are canonically oriented.

Definition 7.6. Let ξ be a line bundle over X, then this is classified by $f : X \to \mathbb{P}^\infty$. We define $c_1(\xi) = f^*(c_\infty) \in \widetilde{E}^{2,1}(X)$.

Proposition 7.7 (Projective Bundle Theorem)**.** Let ξ be a vector bundle over X, and E be oriented. Then,

$$E^{\bullet,\bullet}(\mathbb{P}(\xi)) \cong E^{\bullet,\bullet}(X)[x]/\left(x^d + c_1(\xi)x^{d-1} + \cdots + c_d(\xi)\right)$$

where $x = c_1(\gamma_{\mathbb{P}(\xi)})$.

Theorem 7.8. Let E be oriented. Then, $E^{\bullet,\bullet}(G(n,d)) \cong E^{\bullet,\bullet}[x_1, \cdots, x_{n-d}]/(s_{d+1}, \cdots, s_n) = R_{n,d}$. Here:

- the s_i are determined by $1 + \sum_{i=1}^\infty s_i t^i = (1 + x_1 t + \cdots + x_{n-d}t^{n-d})^{-1}$ in $\mathbb{Z}[x_1, \cdots, x_{n-d}][\![t]\!]$,
- $x_i = c_i(\mathcal{K}_{n,d})$.

Proof Sketch. Recall the SES

$$0 \longrightarrow \mathcal{K}_{n,d} \longrightarrow \mathcal{O}^n \longrightarrow \mathcal{Q}_{n,d} \longrightarrow 0.$$

This implies $c(\mathcal{K}_{n,d})c(\mathcal{Q}_{n,d}) = 1$. This tells us that $s_{d+1} = \cdots = s_n = 0$ in $E^{\bullet,\bullet}(G(n,d))$. So we have a map $\varphi_{n,d} : R_{n,d} \to E^{\bullet,\bullet}(G(n,d))$. Now, argue by induction.

- Base case: For $d = 0$ or $d = n$ we have $G(n, d) = S$.
- Inductive step: Assume that $\varphi_{n,d}$ and $\varphi_{n,d+1}$ are isomorphisms. Consider the composite

  $$\alpha : \mathcal{O}^n_{G(n+1,d+1)} \longrightarrow \mathcal{O}^{n+1}_{G(n+1,d+1)} \longrightarrow \mathcal{Q}_{n+1,d+1}.$$

  Let $U = \{\text{Complement of support of } \alpha\} \subseteq G(n+1, d+1)$. It has a map $U \to G(n, d+1)$ which is an affine bundle. So this is a motivic equivalence.

Consider $i : G(n, d) \to G(n+1, d+1)$, then $N(i) \cong \mathcal{K}_{n,d}$ which follows from $TG(n, d) \cong \underline{\mathrm{Hom}}(\mathcal{K}_{n,d}, \mathcal{Q}_{n,d})$. Write $r = \mathrm{codim}(i) = n - d$. Then,

$$E^{\bullet,\bullet}(\mathrm{Th}(N(i)) \cong E^{\bullet-2r,\bullet-r}(G(n, d)).$$

All this combines to a diagram

$$\begin{array}{ccccc} E^{\bullet-2r,\bullet-r}(G(n,d)) & \xrightarrow{\alpha} & E^{\bullet,\bullet}(\mathrm{Gr}(n+1,d+1)) & \xrightarrow{\beta} & E^{\bullet,\bullet}(G(n,d+1)) \\ \uparrow{\scriptstyle \varphi_{n,d}} & & \uparrow{\scriptstyle \varphi_{n+1,d+1}} & & \uparrow{\scriptstyle \varphi_{n,d+1}} \\ R_{n,d}[-2r,-2] & \longrightarrow & R_{n+1,d+1} & \longrightarrow & R_{n,d+1} \end{array}$$

The bottom sequence is a SES. The diagram can be checked to commute, e.g.

$$\begin{aligned} \beta(\varphi_{n+1,d+1}(x_i)) &= \beta(c_i(\mathcal{K}_{n+1,d+1})) \\ &= c_i(\mathcal{K}_{n+1,d+1}|_U) \\ &= c_i(\mathcal{K}_{n,d+1} \oplus \mathcal{O}_{G(n,d+1)}) \\ &= \pi^*(c_i(\mathcal{K}_{n,d+1})) \\ &= \varphi_{n,d+1}(\pi(x_i)). \end{aligned}$$

□

Theorem 7.9. Let E be a oriented ring spectrum.

(i) There is an isomorphsim $E^{\bullet,\bullet}(\mathrm{BGL}_d) \cong E^{\bullet,\bullet}[\![c_1, \cdots, c_d]\!]$.

(ii) There are isomorphisms

$$E^{\bullet,\bullet}(\mathrm{BGL}) \cong E^{\bullet,\bullet}[\![c_1, c_2, \cdots]\!] \quad \text{and} \quad E_{\bullet,\bullet}(\mathrm{BGL}) \cong E_{\bullet,\bullet}[\beta_0, \beta_1, \cdots]/(\beta_0 = 1)$$

with $\beta_i \in E_{2i,i}(\mathrm{BGL})$ dual to $c_1^i \in E^{2i,i}(\mathrm{BGL})$.

(iii) There are isomorphisms $E^{\bullet,\bullet}(\mathrm{BGL}) \cong E^{\bullet,\bullet}(\mathrm{MGL})$ and $E_{\bullet,\bullet}(\mathrm{BGL}) \cong E_{\bullet,\bullet}(\mathrm{MGL})$.

Corollary 7.10. The pair $(\mathrm{MGL}_{\bullet,\bullet}, \mathrm{MGL}_{\bullet,\bullet}\,\mathrm{MGL})$ is a flat Hopf algebroid and $\mathrm{MGL}_{\bullet,\bullet}\,X$ is a comodule over it.

Proof. Let E be such that $E_{\bullet,\bullet}E$ is flat over $E_\bullet$. Then,

$$E_{\bullet,\bullet}E \otimes_{E_{\bullet,\bullet}} E_{\bullet,\bullet}X \xrightarrow{\sim} (E \otimes E \otimes X)_{\bullet,\bullet}.$$

Now use $E \otimes X \to E \otimes E \otimes X$. □

Remark 7.11. Also get: $\mathrm{MGL}_{\bullet,\bullet}\,\mathrm{MGL} \cong \mathrm{MGL}_\bullet\,\mathrm{MGL} \otimes_{\mathrm{MGL}_\bullet} \mathrm{MGL}_{\bullet,\bullet}$.

7.4 Formal Groups

Definition 7.12. We will ad hoc'ly write $\mathcal{M}_{\mathrm{fg}}^{\mathrm{mot}} = [\mathrm{MGL}_\bullet / \mathrm{MGL}_\bullet \mathrm{MGL}]/\mathbb{G}_m$.

Definition 7.13. Write $\mathcal{F} : \mathcal{SH}(S) \to \mathbf{QCoh}(\mathcal{M}_{\mathrm{fg}}^{\mathrm{mot}})$ lifting $\mathrm{MGL}_\bullet(-)$.

We have a map $\mathrm{MGL}^{\bullet,\bullet}(\mathbb{P}^\infty) \to \mathrm{MGL}^{\bullet,\bullet}(\mathbb{P}^\infty \times \mathbb{P}^\infty)$ which gives a formal group law over $\mathrm{MGL}_{\bullet,\bullet}$. In particular, we get a ring map $\mathrm{MU}_\bullet \to \mathrm{MGL}_{\bullet,\bullet}$ and this factors over $\mathrm{MGL}_\bullet \hookrightarrow \mathrm{MGL}_{\bullet,\bullet}$.

Proposition 7.14. There is an isomorphism $\mathrm{MGL}_{\bullet,\bullet}\mathrm{MGL} \cong \mathrm{MGL}_{\bullet,\bullet} \otimes_{\mathrm{MU}_\bullet} \mathrm{MU}_\bullet\mathrm{MU}$.

Corollary 7.15. We have a pullback

$$\begin{array}{ccc} \operatorname{Spec}\mathrm{MGL}_\bullet/\mathbb{G}_m & \longrightarrow & \operatorname{Spec}\mathrm{MU}_\bullet/\mathbb{G}_m \\ \downarrow & & \downarrow \\ \mathcal{M}_{\mathrm{fg}}^{\mathrm{mot}} & \longrightarrow & \mathcal{M}_{\mathrm{fg}} \end{array}$$

7.5 Landweber Exactness

Recall: A graded $\mathrm{MU}_\bullet$-module $M_\bullet$ is *Landweber exact* if for all p the sequence $p, v_1, v_2, \cdots$ is a regular sequence on $M_\bullet$. It turns out that it is Landweber is and only if the functor

$$q^*(-) \otimes_{\mathrm{MU}_\bullet} M_\bullet : \mathbf{QCoh}(\mathcal{M}_{\mathrm{fg}}) \to \mathbf{grAb}$$

is exact, where $q : \operatorname{Spec}\mathrm{MU}_\bullet/\mathbb{G}_m \to \mathcal{M}_{\mathrm{fg}}$.

Proposition 7.16. Let $A_\bullet$ be a Landweber $\mathrm{MU}_\bullet$-algebra. Then,

$$\mathrm{MGL}_{\bullet,\bullet}(-) \otimes_{\mathrm{MU}_\bullet} A_\bullet : \mathcal{SH}(S) \to \mathbf{bigrAb}$$

is a multiplicative homology theory.

Proof. Consider $\mathrm{MGL}_{\bullet,\bullet}(-) \otimes_{\mathrm{MU}_\bullet} A_\bullet \cong \mathrm{MGL}_{\bullet,\bullet}(-) \otimes_{\mathrm{MGL}_\bullet} \mathrm{MGL}_\bullet \otimes_{\mathrm{MU}_\bullet} A_\bullet$. Consider the pullback squares

$$\begin{array}{ccccc} \operatorname{Spec}B_\bullet/\mathbb{G}_m & \longrightarrow & \operatorname{Spec}\mathrm{MGL}_\bullet/\mathbb{G}_m & \longrightarrow & \mathcal{M}_{\mathrm{fg}}^{\mathrm{mot}} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}A_\bullet/\mathbb{G}_m & \longrightarrow & \operatorname{Spec}\mathrm{MU}_\bullet/\mathbb{G}_m & \longrightarrow & \mathcal{M}_{\mathrm{fg}} \end{array}$$

Have $\mathrm{MGL}_\bullet(X) \otimes_{\mathrm{MU}_\bullet} A_\bullet \cong \Gamma(\operatorname{Spec}B/\mathbb{G}_m, p^*\mathcal{F}(X))$. □

Remark 7.17. Can extend this to:

(i) $\mathrm{MU}_\bullet$-module $M_\bullet$: For this, we show that $\mathcal{M}_{\mathrm{fg}}^{\mathrm{mot}} \to \mathcal{M}_{\mathrm{fg}}$ is affine. Then,

$$\Gamma(\operatorname{Spec}\mathrm{MU}_\bullet/\mathbb{G}_m, M_\bullet \otimes_{\mathrm{MU}_\bullet} q^*i_*\mathcal{F}(-))$$

is exact.

(ii) Start with an ungraded M. Then, form $M_\bullet = M[u^\pm]$ with $|u| = (2,1)$.

Proposition 7.18. Let E be a Landweber exact motivic ring. Then, write E^{top} for the Landweber exact spectrum associated to $E_\bullet$.

(i) There is an isomorphism $E_{\bullet,\bullet}E \cong E_\bullet^{\mathrm{top}}E^{\mathrm{top}} \otimes_{E_\bullet^{\mathrm{top}}} E_{\bullet,\bullet}$.

(ii) The Hopf algebroid $(E_{\bullet,\bullet}, E_{\bullet,\bullet}E)$ is induced from $(\mathrm{MGL}_{\bullet,\bullet}, \mathrm{MGL}_{\bullet,\bullet}\,\mathrm{MGL})$ along the map $\mathrm{MGL}_{\bullet,\bullet} \to \mathrm{MGL}_{\bullet,\bullet} \otimes_{\mathrm{MU}_\bullet} E_\bullet \cong E_{\bullet,\bullet}$.

Proof. We have

$$\begin{aligned} E_{\bullet,\bullet}E &\cong \mathrm{MGL}_{\bullet,\bullet}\, E \otimes_{\mathrm{MU}_\bullet} E_\bullet^{\mathrm{top}} \\ &\cong \mathrm{MGL}_{\bullet,\bullet}(E) \otimes_{\mathrm{MGL}_\bullet} \mathrm{MGL}_\bullet \otimes_{\mathrm{MU}_\bullet} E_\bullet^{\mathrm{top}} \\ &\cong \mathrm{MGL}_{\bullet,\bullet}(E) \otimes_{\mathrm{MGL}_\bullet} E_\bullet^{\mathrm{top}}. \end{aligned}$$

Likewise: $E_{\bullet,\bullet}E \cong \mathrm{MGL}_{\bullet,\bullet}(E) \otimes \mathrm{MGL}_{\bullet,\bullet}\, E_{\bullet,\bullet}$. Thus,

$$E_{\bullet,\bullet}E \cong E_{\bullet,\bullet} \otimes_{\mathrm{MGL}_{\bullet,\bullet}} \mathrm{MGL}_{\bullet,\bullet}\,\mathrm{MGL} \otimes_{\mathrm{MGL}_{\bullet,\bullet}} E_{\bullet,\bullet}.$$

Using $\mathrm{MGL}_{\bullet,\bullet}\,\mathrm{MGL} \cong \mathrm{MGL}_{\bullet,\bullet} \otimes_{\mathrm{MU}_\bullet} \mathrm{MU}_\bullet\,\mathrm{MU}$ we get:

$$\begin{aligned} E_\bullet^{\mathrm{top}} \otimes_{\mathrm{MU}_\bullet} \mathrm{MGL}_{\bullet,\bullet}\,\mathrm{MGL} \otimes_{\mathrm{MU}_\bullet} E_\bullet^{\mathrm{top}} &\cong \mathrm{MGL}_{\bullet,\bullet} \otimes_{\mathrm{MU}_\bullet} (E_\bullet^{\mathrm{top}} \otimes_{\mathrm{MU}_\bullet} \mathrm{MU}_\bullet\,\mathrm{MU} \otimes_{\mathrm{MU}_\bullet} E_\bullet^{\mathrm{top}}) \\ &\cong \mathrm{MGL}_{\bullet,\bullet} \otimes_{\mathrm{MU}_\bullet} E_\bullet^{\mathrm{top}} E^{\mathrm{top}} \\ &\cong \mathrm{MGL}_{\bullet,\bullet} \otimes_{\mathrm{MU}_\bullet} E_\bullet^{\mathrm{top}} \otimes_{E_\bullet^{\mathrm{top}}} E_\bullet^{\mathrm{top}} E^{\mathrm{top}} \\ &\cong E_{\bullet,\bullet} \otimes_{E_\bullet^{\mathrm{top}}} E_\bullet^{\mathrm{top}} E^{\mathrm{top}}. \end{aligned}$$

□

8 From Algebraic Cobordism to Motivic Cohomology (Fabio Neugebauer)

Let S be smooth over a field k with characteristic exponent c. We are still working towards: TALK 8 11.12.2025

Theorem 8.1 (Hopkins–Morel)**.** There is an equivalence $\mathrm{MGL}/(a_1, a_2, \cdots)[1/c] \simeq H\mathbb{Z}[1/c]$.

There will be lots of applications next week.

Proof. This is the proof strategy.

- Check that this is true on $HR^{\bullet,\bullet}(-)$ for $R = \mathbb{Q}, \mathbb{F}_\ell$. The rational case can be deduced from last week's work and we will see this next week again. Today, our goal is $\mathbb{F}_\ell$.
- You want to apply Hurewicz now but this is only applicable on connective theories.
- Check that this is true after $(-)_{\leq 0}$, which we will do today.
- Next week, we will conclude with the motivic Hurewicz theorem that the theorem holds.

□

8.1 $\mathrm{MGL}_{\leq 0}$

Recall $\mathcal{SH}(S)_{\geq d} = \langle \Sigma^{p,q}\Sigma^\infty_+ X \mid X \in \mathbf{Sm}_S, p-q \geq d\rangle_{\mathrm{loc}}$.

Lemma 8.2. Let E be a rank d vector bundle over $X \in \mathbf{Sm}_S$. Then, $\Sigma^\infty\,\mathrm{Th}(E) \in \mathcal{SH}(S)_{\geq d}$.

Proof. Consider a cover $\{U_\alpha\}$ on X trivializing E. Then, $\mathrm{Th}(E)$ is the colimit of

$$\cdots \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} \;\bigvee_{\alpha,\beta} \mathrm{Th}(E|_{U_{\alpha\beta}}) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \bigvee_\alpha \mathrm{Th}(E|_{U_\alpha})$$

Since E is trivial on $U_{\alpha_1,\cdots,\alpha_n}$ we have

$$\mathrm{Th}(E|_{U_{\alpha_1,\cdots,\alpha_n}}) \simeq \mathbb{A}^d/(\mathbb{A}^d - \{0\}) \wedge U_{\alpha_1,\cdots,\alpha_n+}$$

where the first term is $S^{2d,d}$ and $2d - d = d$. □

Notation 8.3. Let V be a vector bundle over S. We write

$$\begin{aligned} \mathrm{Gr}(r,V) &= \text{scheme of } r\text{-planes in } V, \\ E(r,V) &= \text{tautological bundle over } \mathrm{Gr}(r,V). \end{aligned}$$

Let $W \hookrightarrow V$ be an inclusion, then we get a closed immersion $i_W : \mathrm{Gr}(r,W) \hookrightarrow \mathrm{Gr}(r,V)$. Let $V_1,\cdots,V_t$ be VBs, then we get a closed immersion

$$j_{V_1,\cdots,V_t} : \mathrm{Gr}(r_1,V_1) \times \cdots \times \mathrm{Gr}(r_t,V_t) \to \mathrm{Gr}(r_1+\cdots+r_t, V_1 \oplus \cdots \oplus V_t).$$

Example 8.4. Let $W \subseteq V = \mathbb{A}^n$ with $W \oplus L = V$ for $\dim L = 1$. Put

$$j = j_{L,W} : \mathrm{Gr}(r-1,W) \cong \mathrm{Gr}(1,L) \times \mathrm{Gr}(r-1,W) \hookrightarrow \mathrm{Gr}(r,V)$$

We get $i_W : \mathrm{Gr}(r,W) \hookrightarrow \mathrm{Gr}(r,V) \setminus \operatorname{im} j$.

Fact 8.5. The map r_W is the zero section of an r-dimensional vector bundle

$$p : \mathrm{Gr}(r,V) \setminus \operatorname{im} j \to \mathrm{Gr}(r,W)$$

where $p(E)$ is morally the orthogonal projection to W.

Lemma 8.6. The map i_W induces an equivalence $\mathrm{Th}(E(r,W)) \to \mathrm{Th}(E(r,V)|_{\mathrm{Gr}(r,V)\setminus \operatorname{im} j}$.

Proof. We have $p \circ i_W = \mathrm{id}$ and an $\mathbb{A}^1$-equivalence $i_W \circ p \simeq \mathrm{id}$. These maps extend to vector bundle maps. □

Lemma 8.7. The cofiber of

(i) $i_W : \mathrm{Gr}(r,W) \to \mathrm{Gr}(r,V)$,

(ii) $i_W : \mathrm{Th}(E(r,W)) \to \mathrm{Th}(E(r,V))$,

(iii) $j : \mathrm{Gr}(r-1,W) \to \mathrm{Gr}(r,V)$,

(iv) $j : \mathrm{Th}(E(r-1,W)) \to \mathrm{Th}(E(r,V))$

is stably

(i) $n-r$,

(ii) n,

(iii) r,

(iv) $2r$

connective.

Proof.

(i) Consider

$$\begin{array}{ccc} \mathrm{Gr}(r,W) & \xrightarrow{i_W} & \mathrm{Gr}(r,V) \\ & \nwarrow^{p} \quad \nearrow & \\ & \mathrm{Gr}(r,V)\setminus \operatorname{im} j & \end{array}$$

The left map is a homotopy equivalence, so i_W factors. Thus,

$$\mathrm{cofib}(i_W) = \mathrm{Gr}(r,V)/(\mathrm{Gr}(r,V)\setminus \operatorname{im} j) \cong \mathrm{Th}(N_j)$$

by the purity theorem where N_j is the normal bundle of j and has rank $n-r$. We conclude with the connectivity of Thom spaces result.

(ii) By Lemma we have

$$\begin{aligned} \mathrm{cofib}(2) &\simeq \mathrm{Th}(E(r,V))/\,\mathrm{Th}(E(r,V)|_{\mathrm{Gr}(r,V)\setminus \operatorname{im} j} \\ &\simeq E(r,V)/(E(r,V)\setminus \operatorname{im} j) \\ &\simeq \mathrm{Th}(N_{s\circ j}) \end{aligned}$$

The normal bundle has rank $n-r+r=n$.

(iii), (iv) Similar.

□

Definition 8.8. The **Hopf map** is $h : \mathbb{A}^2 - \{0\} \to \mathbb{P}^1$, $(x,y) \mapsto [(x,y)]$. Let $C(h) = \mathrm{cofib}\, h$.

Note:

$$\begin{array}{ccccc} \mathbb{A}^2-\{0\} & \xrightarrow{h} & \mathbb{P}^1 & \longrightarrow & C(h) \\ \downarrow & & \uparrow\cong & & \uparrow\cong \\ E(1,2)\setminus s(\mathbb{P}) & \hookrightarrow & E(1,2) & \longrightarrow & \mathrm{Th}(E(1,2)) \end{array}$$

Construction 8.9. Consider the map

$$\begin{array}{ccccc} \Sigma^{-2,-1}\Sigma^\infty C(h) & \longrightarrow & \Sigma^{-2,-1}\,\mathrm{Th}(E(1,\infty)) & \longrightarrow & \mathrm{colim}_r\, \Sigma^{-2r,-r}\,\mathrm{Th}(E(r,\infty)) \\ \uparrow & & & & \| \\ \mathbb{1} = \Sigma^{-2,-1}\Sigma^\infty\mathbb{P}^1 & & \xrightarrow{\quad\text{unit}\quad} & & \mathrm{MGL} \end{array}$$

where the left vertical map is $(x,y) \mapsto ((x,y) \in [(x,y)])$. Write $\eta = \Sigma^{-2,-1}\Sigma^\infty h : \mathbb{S}^{1,1} \to \mathbb{1}$. So the top left is $\mathbb{1}/\eta$.

Theorem 8.10. The unit induces an equivalence $(\mathbb{1}/\eta)_{\leq 0} \to \mathrm{MGL}_{\leq 0}$.

Proof. Apply connectivity lemma to the transition maps in the colimit of MGL. □

8.2 Motivic Quotients of MGL

8.2.1 Chromatic Recollection

Recall: Fix a homotopy ring map $\Theta : \mathrm{MGL} \to E$. Then, you can compute

$$E^{\bullet,\bullet}\,\mathrm{Gr}(r,\infty) \cong E^{\bullet,\bullet}[\![c_1,\cdots,c_r]\!]$$

where the c_i are called the **Chern classes**. Let $\mathrm{BGL} = \mathrm{colim}_r\, \mathrm{Gr}(r,\infty)$ which is a group object using the direct sum. Then,

$$E^{\bullet,\bullet}\,\mathrm{BGL} = E^{\bullet,\bullet}[\![c_1,c_2,\cdots]\!]$$

is a cogroup object. So it has a comultiplication $\Delta(c_n) = \sum_{i+j=n} c_i \otimes c_j$.

Definition 8.11. Let $\beta^n \in E_{\bullet,\bullet}\,\mathrm{BGL}$ be the dual of c_1^n. One computes

$$E_{\bullet,\bullet}(\mathrm{BGL}) \cong E_{\bullet,\bullet}[\beta_1, \beta_2, \cdots]$$

The linear span of the β_n is $E_{\bullet,\bullet}\,\mathrm{Gr}(1,\infty)$. This is essentially because the β_n all come from c_1. By the Thom isomorphism

$$E_{\bullet,\bullet}(\mathrm{MGL}) \cong E_{\bullet,\bullet}[b_1, b_2, \cdots].$$

Moreover, $\mathrm{Gr}(1,\infty)$ is a group object. So $E^{\bullet,\bullet}(\mathrm{Gr}(1,\infty)) \cong E^{\bullet,\bullet}[\![c_1]\!]$ is a cogroup and the FGL $\Delta(c_1)$ gives rise to a map $\mathrm{MU}_\bullet \to E_{\bullet,\bullet}$. In particular, we get a map $\mathrm{MU}_\bullet \to \mathrm{MGL}_{\bullet,\bullet}$.

8.2.2 Hurewicz Map

Let R be a ring.

Definition 8.12. The **Hurewicz map** $h_R : (-)_{\bullet,\bullet} \to HR_{\bullet,\bullet}(-)$ is defined by the unit $\mathbb{1} \to HR$.

We get a diagram

$$\begin{array}{ccc}
\mathrm{MU}_\bullet & \xrightarrow{h_R} & R_\bullet\,\mathrm{MU}) = R[b_1, b_2, \cdots] \\
\downarrow & & \downarrow \\
\mathrm{MGL}_{\bullet,\bullet} & \xrightarrow[h_R]{} & HR_{\bullet,\bullet}\,\mathrm{MGL} = HR_{\bullet,\bullet}[b_1, b_2, \cdots]
\end{array}$$

This commutes as both h_R's classify the coordinate change of the additive formal group law with new coordinate $\sum_{n\geq 0} b_n x^{n+1}$.

Theorem 8.13 (Lazard). Let $I \subseteq \mathrm{MU}_\bullet$ be the ideal generated by positive degree elements. Then, $h_{\mathbb{Z}} : \mathrm{MU}_\bullet \to \mathbb{Z}[b_1, b_2, \cdots]$ restricts to $h_{\mathbb{Z}} : I/I^2 \to (b_i)/(b_i)^2$ and the induced map

$$h_{\mathbb{Z}} : (I/I^2)_{2n} \to \left((b_i)/(b_i)^2\right)_{2n} = b_n\mathbb{Z}$$

is injective with

$$\operatorname{im} h_{\mathbb{Z}} = \begin{cases} \mathbb{Z}\ell b_n & n = \ell^r - 1,\ \ell \text{ prime}, \\ \mathbb{Z} & \text{else}. \end{cases}$$

Choose lifts generators $a_n \in \mathrm{MU}_{2n}$ of $(I/I^2)_{2n}$. Then, $\mathrm{MU}_\bullet \cong \mathbb{Z}[a_1, a_2, \cdots]$.

These generators are not canonical but you can do a bit better.

Definition 8.14. Let ℓ be a prime. An element $v \in \mathrm{MU}_{2(\ell^r-1)}$ is called **ℓ-typical** if

(i) $h_{\mathbb{Z}/\ell}(v) = 0$,

(ii) $h_{\mathbb{Z}/\ell^2}(v) \not\equiv 0 \pmod{(b_i)^2}$.

Definition 8.15. A set of generators $a_n \in \mathrm{MU}_{2n}$ for $n \geq 1$ as in Lazard's theorem is **adequate** if $a_{2(\ell^r-1)}$ is ℓ-typical for all primes ℓ and all r.

These exist: Take logarithms for the universal formal group law.

8.3 Regular Quotients of MGL

Let $H = H\mathbb{F}_\ell$ and $h = h_{\mathbb{F}_\ell}$ with $\ell \neq \operatorname{char} k$. Recall

$$\mathcal{P}_{\bullet,\bullet} = H_{\bullet,\bullet}[\xi_1, \cdots] \subseteq \mathcal{A}_{\bullet,\bullet} = H_{\bullet,\bullet}H$$

which is the polynomial part of the dual Steenrod algebra.

Observation 8.16. The image $h(\mathrm{MU}_\bullet) \subseteq \mathbb{F}_\ell[b_1, b_2, \cdots]$ is a polynomial ring $\mathbb{F}_\ell[b'_n : n \neq \ell^r - 1]$ with $b_n \cong b'_n \pmod{(b_n)^2}$.

Theorem 8.17. The coaction $\Delta : H_{\bullet,\bullet}\mathrm{MGL} \to \mathcal{A}_{\bullet,\bullet} \otimes_{H_{\bullet,\bullet}} H_{\bullet,\bullet}\mathrm{MGL}$ induces an isomorphism

$$H_{\bullet,\bullet}[b_1, b_2, \cdots] \cong H_{\bullet,\bullet}\mathrm{MGL} \cong \mathcal{P}_{\bullet,\bullet} \otimes h(\mathrm{MU}_\bullet).$$

Proof Idea. Can compute Δ on b_i by dually computing $\mathcal{A}^{\bullet,\bullet}$ acting on $c_1^n \in H^{\bullet,\bullet}(\mathrm{Gr}(1, \infty))$. Match polynomial generators. □

Definition 8.18. Let x be a sequence of elements of $\mathrm{MU}_\bullet$.

(i) It is **h-regular** if $h(x)$ is a regular sequence.

(ii) It is **maximal** if $h(\mathrm{MU}_\bullet)/h(x) = \mathbb{F}_\ell$.

Example 8.19. If $a_1, a_2, \cdots$ are adequate, then $x(a) = \{a_i : i \neq \ell^r - 1\}$ is maximal h-regular and $v(a) = \{a_i : i = \ell^r - 1\}$ is ℓ-typical.

Corollary 8.20. Let x be maximal h-regular. Then,

$$H_{\bullet,\bullet}(\mathrm{MGL}/x) \cong \mathcal{P}_{\bullet,\bullet} \otimes h(\mathrm{MU}_\bullet)/h(x) \cong \mathcal{P}_{\bullet,\bullet}.$$

Proof. The first isomorphism uses regularity, the second isomorphism uses maximality. □

Dualizing gives:

Corollary 8.21. There is an isomorphism

$$\mathcal{A}^{\bullet,\bullet}/(Q_0, Q_1, \cdots) \to H^{\bullet,\bullet}(\mathrm{MGL}/x), \ [\varphi] \mapsto \varphi(\Theta)$$

where $\Theta : \mathrm{MGL} \to H$ is the orientation.

Let $M = \mathrm{MGL}/x$ where x is maximal h-regular.

Theorem 8.22. Let $J \subseteq \mathbb{N}$. For each $j \in J$ let $v_j \in \mathrm{MU}_{2(\ell^j - 1)}$ be ℓ-typical. Then,

$$\mathcal{A}^{\bullet,\bullet}/(Q_i : i \notin J) \to H^{\bullet,\bullet}(M/(v_i : i \in J)), \ [\varphi] \mapsto \varphi(\Theta)$$

is an isomorphism.

Proof. Induction of $\#J$. If $J = \emptyset$, then this is by the Corollary. Suppose that this is true for J and $r \notin J$. Because $h_{\mathbb{F}_\ell}(v_r) = 0$, then sequence induced by $- \cdot v_r$ is short exact.

$$H^{\bullet,\bullet}(\mathrm{MGL}) \xrightarrow{\ \delta\ } H^{\bullet,\bullet}(\mathrm{MGL}/v_r) \longrightarrow H^{\bullet,\bullet}(\mathrm{MGL}).$$

Cotensor this sequence over $H^{\bullet,\bullet}(\mathrm{MGL})$ with $H^{\bullet,\bullet}(M/(v_i : i \in J))$. We obtain:

$$\begin{array}{ccccc}
\mathcal{A}^{\bullet,\bullet}/(Q_i : i \notin J) & \xrightarrow{-\cdot Q_r} & \mathcal{A}^{\bullet,\bullet}/(Q_i : i \notin J \cup \{r\}) & \xrightarrow{-/Q_r} & \mathcal{A}^{\bullet,\bullet}/(Q_i : i \notin J) \\
\cong \downarrow & & \downarrow & & \downarrow \cong \\
H^{\bullet,\bullet}(M/(v_i : i \in J)) & \longrightarrow & H^{\bullet,\bullet}(M/(v_i : i \in J)) \square H^{\bullet,\bullet}(\mathrm{MGL}/v_r) & \longrightarrow & H^{\bullet,\bullet}(M/v_i) \\
 & & \cong \downarrow & & \\
 & & H^{\bullet,\bullet}(M/(v_i : i \in J \cup \{r\})) & &
\end{array}$$

So the middle map is an isomorphism. □

So $H^{\bullet,\bullet}(H\mathbb{Z}) \cong \mathcal{A}^{\bullet,\bullet}/Q_0 \to H^{\bullet,\bullet}(\mathrm{MGL}/a_1, a_2, \cdots)$.

9 Rest of the Talks

Unfortunately, I don't have notes for the remaining talks; sorry about that. The talks were the following:

- The Hopkins–Morel Equivalence (Emma Brink): The previous work was assembled here to finish the proof of the Hopkins–Morel equivalence and to give some direct applications.
- Motivic Spectra (Christian Kremer): We started a new section and developed foundations to define non-$\mathbb{A}^1$-invariant motivic spectra.
- Moduli of Vector Bundles (Sayan Kundu): Described the Grassmannian model for the moduli stack of vector bundles and proceeded to defined oriented motivic spectra.
- Projective Bundle Formula (Qingyuan Bai): Projective bundle formula for oriented motivic spectra and oriented cohomology of Grassmannians.
- Universality of K-theory (TBD): Universal property of algebraic K-theory and Selmer K-theory.

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