MAX-PLANCK-GESELLSCHAFT

Multiplicative Equivariant Thom Spectra

Zhu, Qi (joint with Ryan Quinn)

Advisor: Prof. Dr. Markus Hausmann

IMPRS Moduli Spaces

Thom Spectra

Classical. In 1954 René Thom [Tho54] gave the following innocent definition: Let $\xi : E \to B$ be a vector bundle, then usually E admits a Riemannian metric, so one can consider the associated disc bundle $D(E) \to B$ and sphere bundle $S(E) \to B$. The **Thom space** of ξ then is $\mathsf{Th}(\xi) = D(E)/S(E)$ which should be viewed as a twisted suspension.

Stabilizing it suitably leads to the notion of **Thom spectra** giving some of the most interesting examples in homotopy theory. Examples include the celebrated bordism spectra MO and MU. Thom [Tho54] showed that MO represents bordism and was awarded a Fields medal for that, while Quillen showed that MU controls chromatic homotopy theory in various ways.

Modern. The modern language of ∞-categories allows for an intrinsically homotopical definition of Thom spectra. Let R be an $\mathbb{E}_\infty$-ring spectrum and let $\mathsf{Pic}(R) \subseteq \mathsf{LMod}_R$ denote the **Picard space** of the ∞-category of left R-modules, i.e. the core of tensor-invertible objects in LMod_R. Beautiful work of Ando-Blumberg-Gepner-Hopkins-Rezk [ABG+14] suggests that the Thom spectrum of a map $f : X \to \mathsf{Pic}(R)$ of spaces is the colimit

$$\mathsf{Th}(f) = \operatorname{colim}\left(X \xrightarrow{f} \mathsf{Pic}(R) \to \mathsf{LMod}_R\right).$$

taken in the ∞-category Cat_∞ of ∞-categories. It is essential here that any space can be viewed as an ∞-category via Grothendieck's homotopy hypothesis $\mathcal{S} \simeq \mathsf{Grpd}_\infty \subseteq \mathsf{Cat}_\infty$.

A map $X \to \mathsf{BO}$ classifies virtual vector bundles of degree 0 and postcomposing by the so-called J-homomorphism $J : \mathsf{BO} \to \mathsf{Pic}(R)$ recovers the classical Thom spectra as shown in [ABG+14].

Multiplicative. Higher categorical tools make formal structure easier to handle. As such, there were two concurrent papers by Ando-Blumberg-Gepner [ABG18] and Antolín-Camarena-Barthel [ACB19] showing that a multiplicative map $f : X \to \mathsf{Pic}(R)$ gives rise to a multiplicative structure on its Thom spectrum $\mathsf{Th}(f)$.

While Ando-Blumberg-Gepner recover more functoriality than Antolín-Camarena-Barthel, the latter authors give a more concrete universal property of multiplicative $\mathsf{Th}(f)$ for all $\mathbb{E}_n$-operads.

Equivariant Thom Spectra

Thom Spectra in Equivariant Mathematics. Let G be a finite group, then equivariant homotopy theory is the study of G-actions. Just as Thom spectra flourish in classical mathematics, they provide an abundant of fascinating examples in the equivariant story. One has to be careful now, as there are multiple bordism Thom spectra like $\mathsf{mO}_G, \mathsf{MO}_G$ and $\mathsf{mU}_G, \mathsf{MU}_G$ enhancing the classical MO and MU but for example, MU_G is still the central G-spectrum in the story of equivariant chromatic homotopy theory.

Parametrized Higher Category Theory. Initiated by Barwick-Dotto-Glasman-Nardin-Shah [BDG+16] equivariant mathematics now have their own category theory. There is the so-called orbit category Orb_G consisting of the orbits G/H for subgroups $H \le G$ as well as the G-equivariant maps between them. Then,

$$\mathsf{Cat}_{G,\infty} = \mathsf{Fun}(\mathsf{Orb}_G^{\mathsf{op}}, \mathsf{Cat}_\infty) \simeq \mathsf{coCart}(\mathsf{Orb}_G^{\mathsf{op}})$$

is the ∞-category of **G-∞-categories** inspired by the classical Elmendorf theorem $\mathcal{S}_G \simeq \mathsf{Fun}(\mathsf{Orb}_G^{\mathsf{op}}, \mathcal{S})$.

This can also be stated for more general base ∞-category $\mathcal{T}^{\mathsf{op}}$ instead of $\mathsf{Orb}_G^{\mathsf{op}}$. Essentially all categorical notions now have a G-counterpart.

Modern Equivariant Thom Spectra. The insight of Ando-Blumberg-Gepner-Hopkins-Rezk [ABG+14] in defining Thom spectra is that any space can be viewed as an ∞-category. With parametrized higher category theory we can now view every G-space as a G-∞-category, so nothing stops us from enhancing the ∞-categorical Thom spectrum construction. Let R be a genuinly commutative G-ring spectrum and $f : X \to \mathsf{Pic}_G(R)$ be a map of G-spaces, then

$$\mathsf{Th}_G(f) = \operatorname{colim}_G\left(X \xrightarrow{f} \mathsf{Pic}_G(R) \to \mathsf{LMod}_R^G\right)$$

is the **G-Thom spectrum** of f where we replace every notion by their G-counterparts. In that regard, this is a G-colimit taken in the G-∞-category of G-∞-categories.

What happens multiplicatively?

Multiplicative Equivariant Thom Spectra

Goal. The main aim of this project is to equivariantize Antolín-Camarena-Barthel's work [ACB19].

Setup. Let $\mathcal{O}$ be a nice enough G-∞-operad and $R \in \mathsf{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathsf{Sp}_G)$. Then, we show that there exists a G-∞-category of left modules LMod_R^G which obtains an $\mathcal{O}$-monoidal structure. There also exists a suitable G-Picard space notion and for any $A \in \mathsf{Alg}_{\mathcal{O}}(\mathsf{LMod}_R^G)$ we define the pullback square

$$\begin{array}{ccc} \mathsf{Pic}_G^{\otimes}(R)_{\downarrow A} & \longrightarrow & (\mathsf{LMod}_R^{G,\otimes})_{/A} \\ \downarrow & \lrcorner & \downarrow \\ \mathsf{Pic}_G^{\otimes}(R) & \longrightarrow & \mathsf{LMod}_R^{G,\otimes} \end{array}$$

in the ∞-category of $\mathcal{O}$-monoidal G-∞-categories.

Main Theorem. Let $f : X \to \mathsf{Pic}_G^{\otimes}(R)$ be an $\mathcal{O}$-monoidal map of $\mathcal{O}$-monoidal G-spaces. Then, its G-Thom spectrum is an $\mathcal{O}$-algebra $\mathsf{Th}_G^{\otimes}(f)$ and a map $\mathsf{Th}_G^{\otimes}(f) \to A$ corresponds to an $\mathcal{O}$-monoidal lift

$$\begin{array}{ccc} & & \mathsf{Pic}_G^{\otimes}(R)_{\downarrow A} \\ & \nearrow & \downarrow \\ X & \xrightarrow[f]{} & \mathsf{Pic}_G^{\otimes}(R) \end{array}$$

For the trivial group $G = e$ this specializes to Antolín-Camarena-Barthel's theorem [ACB19].

Technical Ingredients. This project builds on some parametrized higher algebra that either needed to be put together from the literature or required a new proof.

- We provide a monoidal structure on parametrized slice categories in great generality and prove a universal property about it.
- We prove a 'microcosmic' version of parametrized monoidal straightening unstraightening which allows for a suitable functorial description of parametrized Thom spectra.
- The hardest work was to build an $\mathcal{O}$-monoidal parametrized left module category and showing that it is distributive with respect to equivariant tensor products after [NS22]. It is required to run operadic left Kan extensions which is an essential tool in this story.

Consequences. The multiplicative Thom spectrum business allows us to harvest a number of formal consequences.

- Thom spectra behave well with respect to equivariant norms.
- Thom spectra behave well with respect to inducing up algebra structures $\mathsf{Ind}_{\mathcal{O}}^{\mathcal{P}}$.
- There are multiplicative Thom isomorphisms.
- If X is grouplike in a suitable sense, then maps $\mathsf{Th}_G^{\otimes}(f) \to A$ correspond to so-called $\mathcal{O}$-A-orientations.
- There are Chadwick-Mandell type results [CM15] allowing us to untwist $\mathsf{Map}_{\mathsf{Alg}_{\mathcal{O}}(\mathsf{LMod}_R)}(\mathsf{Th}_G^{\otimes}(f), A)$ by a Thom isomorphism.

Additional Philosophy. We spent much effort making this work for general G-∞-operads than restricting to the fully commutative or the $\mathbb{N}_\infty$-case. Much of equivariant higher algebra nowadays seems to be focused on commutative G-ring spectra but we intend to make the statement that lower commutativities rightfully have their place in equivariant homotopy theory. For example, there is the entire family of interesting $\mathbb{E}_V$ operads for G-representations V that deserve to be studied. Already non-equivariantly not every spectrum is $\mathbb{E}_\infty$, so equivariantly there is also no chance for every spectrum to be $\mathbb{N}_\infty$.

Multiplicative Structure on Real Brown-Peterson Spectra

Brown-Peterson Spectrum. For every prime p, the p-local complex bordism spectrum $\mathsf{MU}_{(p)}$ splits into a infinite direct sum of spectra BP, the so-called **Brown-Peterson spectra**. As such, this is another of those designer spectra in chromatic homotopy theory. It contains essentially the same information as $\mathsf{MU}_{(p)}$ but is much smaller leading to more computational power. This is one of the spectra that is known not to be $\mathbb{E}_\infty$ [Law18, Sen24]. It is known to have an $\mathbb{E}_4$-structure due to Basterra-Mandell [BM13] and an $\mathbb{E}_3$-MU-algebra structure due to Hahn-Wilson [HW22]. These had groundbreaking applications like being a way to attack the redshift conjecture [HW22]. Earlier, Chadwick-Mandell [CM15] put an $\mathbb{E}_2$-structure on BP.

Real Brown-Peterson Spectra. The spectrum MU is intimately connected to the infinite unitary group U which has an intrinsic C_2-action given by complex conjugation. Taking this C_2-action into account gives a C_2-spectrum $\mathsf{MU}_{\mathbb{R}}$, the **Real complex bordism spectrum**. Similarly, this leads to $\mathsf{BP}_{\mathbb{R}}$. As of now, it is not even known that $\mathsf{BP}_{\mathbb{R}}$ is a homotopy commutative ring spectrum!

Multiplicative Structure on Real Brown-Peterson Spectra. Ryan Quinn and I prove that $\mathsf{BP}_{\mathbb{R}}$ admits an $\mathbb{E}_\rho$-algebra structure where ρ is the regular C_2-representation.

Super Rough Proof Strategy. We use the Chadwick-Mandell type result obtained from the multiplicative equivariant Thom spectrum formalism to lift the $\mathbb{E}_2$-Quillen idempotent $\mathsf{MU} \to \mathsf{MU}$ to an $\mathbb{E}_\rho$-Quillen idempotent $\mathsf{MU}_{\mathbb{R}} \to \mathsf{MU}_{\mathbb{R}}$. This splits off an $\mathbb{E}_\rho$-spectrum $\mathsf{BP}_{\mathbb{R}}$.

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2025 Max-Planck Institute for Mathematics, Bonn
qzhu@mpim-bonn.mpg.de