EAST 2025

Qi Zhu

15 – 19 September 2025

Abstract

These are my (live) TeX'd notes of the European Autumn School in Topology 2025 with invited lecturers Piotr Pstragowski and Joana Cirici.

I only ended up TeX'ing half of Joana's course and since I didn't manage to take satisfying notes (shame on me), I decided to only include Piotr's course here. I also did not type the preparatory talks, gong show talks or contributed talks, but you can find my prison spectral sequence preparatory talk on my https://qizhumath.wixsite.com/math/articles.

Piotr's course was called *Descent in stable homotopy theory* and the following was his abstract: The main calculational tool in the context of stable homotopy theory is given by the Adams spectral sequence. The aim of this lecture series is to give a gentle introduction to this topic, both from the perspective of descent and that of an intermediary between stable and abelian categories. In particular, I plan to cover the following:

- The Adams spectral sequence of a ring spectrum, some classical calculations
- Homology theories, epimorphism classes, injective resolutions
- Encoding spectral sequences using deformations of stable ∞-categories
- Applications: monoidality of the Adams filtration, coherent multiplication on Moore spectra.

Please contact me at qzhu@mpim-bonn.mpg.de (or over social media) for comments or suggestions.

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1 The Adams Spectral Sequence

1.1 The Hurewicz Homomorphism and Adams Resolutions

Often one starts with a problem in geometry or arithmetic and it turns out that this often can be translated into understanding $[X, Y]$ for some $X, Y \in \mathbf{Sp}$. Very often, $\pi_\bullet Y = [\mathbb{S}, Y]_\bullet$.

Example 1.1.

(i) Consider maps $S^{n+k}_{\text{top}} \to S^k_{\text{top}}$ up to homotopy which amounts to $\pi_n \mathbb{S}$ when $k > n$.

(ii) The following example gave rise to a few Fields medals! Let's classify almost complex manifolds up to bordism. This amounts to $\pi_\bullet(\mathrm{MU})$. This was first done by Milnor and is still one of the only ways to understand this problem.

(iii) Consider (higher) algebraic K-theory $K_\bullet(R) \cong \pi_\bullet K(R)$. A lot of study of algebraic K-theory is the study of the spectrum $K(R)$ which is better behaved than its homotopy groups.

Idea: Homology is usually easier to compute, so approximate $\pi_\bullet(-) = [\mathbb{S}, -]$.

Notation 1.2. We write $H_\bullet(X) = H_\bullet(X; \mathbb{F}_p) = \pi_\bullet(\mathbb{F}_p \otimes_{\mathbb{S}} X)$ which is a graded $\mathbb{F}_p$-vector space.

We will mostly work with $p = 2$.

Example 1.3. One has $H_\bullet(\mathbb{S}) \cong \pi_\bullet(\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{S}) \cong \mathbb{F}_p[0]$. This is already quite interesting, for example it tells you $\mathbb{S} \not\simeq 0$.

It is also the starting point of the Adams spectral sequence.

Construction 1.4. Let $X \in \mathbf{Sp}$ and consider the Hurewicz homomorphism

$$\pi_\bullet X \to [\mathbb{S}, X]_\bullet \to \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{F}_p}}(\mathbb{F}_p, H_\bullet X) \cong H_\bullet X \cong \pi_\bullet(\mathbb{F}_p \otimes X).$$

For example, for $X = \mathbb{S}$, the unit in $H_\bullet \mathbb{S} \cong \mathbb{F}_p$ comes from $\mathrm{id}_{\mathbb{S}}$.

Example 1.5. Let $X = \Sigma^\infty_+ \mathbb{R}P^\infty$. Then,

$$H_\bullet(X) \cong H_\bullet(\mathbb{R}P^\infty) \cong \begin{cases} \vdots & \\ \mathbb{F}_2\{\overline{x}^2\} & \bullet = 2, \\ \mathbb{F}_2\{\overline{x}\} & \bullet = 1, \\ \mathbb{F}_2\{\overline{1}\} & \bullet = 0. \end{cases}$$

Homology has better functorial properties but cohomology has a multiplicative structure given by $H^\bullet(X) \cong \mathbb{F}_2[x]$ with $|x| = 1$. Consider

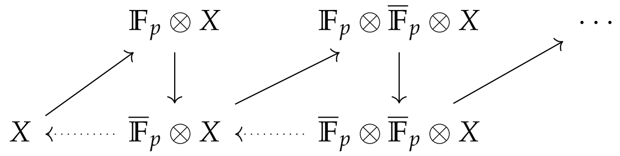
$$\mathrm{Hur} : \pi_\bullet(\Sigma^\infty_+ \mathbb{R}P^\infty) \to H_\bullet(\mathbb{R}P^\infty)$$

and it becomes a non-trivial question now whether $\overline{1}, \overline{x}, \cdots$ come from the Hurewicz homomorphism. As such, the elements in the homology should be viewed as a proof that elements in the homotopy groups are non-trivial.

Construction 1.6 (Adams Resolution). Let $\overline{\mathbb{F}}_p = \mathrm{cofib}(\mathbb{S} \to \mathbb{F}_p)$. So there is a cofiber sequence

$$X \longrightarrow \mathbb{F}_p \otimes X \longrightarrow \overline{\mathbb{F}}_p \otimes X.$$

Consider an element in $\pi_\bullet X$. Applying the Hurewicz leads to $\pi_\bullet(\mathbb{F}_p \otimes X)$ and if this becomes 0, then exactness allows us to lift the element to $\pi_{\bullet+1}(\overline{\mathbb{F}}_p \otimes X)$. So we can try to play the same game again. It leads to



where every triangle is a cofiber sequence, so it gives rise to long exact sequences on $\pi_\bullet$. Then,

$$A_1 = \pi_\bullet\left(\overline{\mathbb{F}}_p^{\otimes s} \otimes X\right) \quad \text{and} \quad E_1 = \pi_\bullet\left(\mathbb{F}_p \otimes \overline{\mathbb{F}}_p^{s} \otimes X\right) \cong H_\bullet\left(\overline{\mathbb{F}}_p^{\otimes s} \otimes X\right).$$

is an exact couple, so we get a spectral sequence with $E_1 = \pi_\bullet\left(\mathbb{F}_p \otimes \overline{\mathbb{F}}_p^{\otimes s} \otimes X\right)$, called the **Adams spectral sequence** which tries to converge to $\pi_\bullet(X)$ up to completion.

1.2 Incorporating the Comodule Structure

There is more structure on $H_\bullet(X)$, and as always in mathematics we should try to use the maximal amount of structure given to us.

Observation 1.7. We have *Steenrod operations* $\mathrm{Sq}^n : H_\bullet(X) \to H_{\bullet-n}(X)$ for $n \geq 0$. In other words, $H_\bullet(X)$ is an $\mathcal{A}_*$-comodule., i.e. it has an action of Steenrod operations satisfying the Adem relations and $\mathrm{Sq}^n x = 0$ for every $x \in H_\bullet(X)$ and $n \gg 0$.[1]

There is an action of the Steenrod algebra $\mathcal{A}^\bullet = \mathbb{F}_2[\mathrm{Sq}^n]_n/(\text{Adem relations})$ giving a module structure $\mathcal{A}^\bullet \otimes_{\mathbb{F}_2} H_\bullet(X) \to H_\bullet(X)$ and so a comodule structure $\Delta : H_\bullet(X) \to \mathcal{A}_* \otimes_{\mathbb{F}_2} H_\bullet(X)$.

About the coaction. Consider $\mathcal{A}^\bullet = [\mathbb{F}_p, \mathbb{F}_p]_{-\bullet}$ and so $\mathcal{A}_* = H_\bullet(\mathbb{F}_p) \cong \pi_\bullet(\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p)$. This is a coalgebra via

$$\mathrm{id} \otimes 1 \otimes \mathrm{id} : \mathbb{F}_p \otimes X \to \mathbb{F}_p \otimes \mathbb{F}_p \otimes X \simeq (\mathbb{F}_p \otimes \mathbb{F}_p) \otimes_{\mathbb{F}_p} (\mathbb{F}_p \otimes X).$$

Applying $\pi_\bullet$ and the Künneth isomorphism, we obtain the comultiplication

$$\Delta : H_\bullet(X) \to \mathcal{A}_* \otimes_{\mathbb{F}_p} H_\bullet(X).$$

Proof of Unstability. We can write $X \simeq \operatorname{colim} X_\alpha$ as a filtered colimit of finite spectra, so[2]

$$H_\bullet(X) \cong \operatorname*{colim}_\alpha H_\bullet(X_\alpha)$$

and those terms are all finite-dimensional over $\mathbb{F}_p$, so there are only finite many non-zero Sq^n-actions. □

Remark* 1.8. I'm quite confused about the Steenrod operations and all the duals. It seems like above we construct $\Delta : H_\bullet X \to \mathcal{A}_* \otimes_{\mathbb{F}_2} H_\bullet X$ which corresponds to $\mathcal{A}^\bullet \otimes H^\bullet X \to H^\bullet X$, i.e. the usual Steenrod operations. But there also seems to be a map $H_\bullet X \otimes \mathcal{A} \to H_\bullet X$ coming from

$$\pi_\bullet(X \otimes \mathbb{F}_p) \otimes \pi_{-\bullet}(\mathrm{map}_{\mathbf{Sp}}(\mathbb{F}_p, \mathbb{F}_p)) \to \pi_\bullet(X \otimes \mathbb{F}_p)$$

coming from the evaluation map on $\mathbb{F}_p$. Plugging in elements from $\mathcal{A}$ gives a right action on $H_\bullet X$ which seems to correspond to the usual left action on $H^\bullet X$ under the duality $H^\bullet X \cong \mathrm{Hom}_{\mathbb{F}_p}(H_\bullet X, \mathbb{F}_p)$.

Consider forget : $\mathbf{Comod}_{\mathcal{A}_*} \to \mathbf{Vect}_{\mathbb{F}_p}$ and so we now try to lift things to $\mathbf{Comod}_{\mathcal{A}_*}$.

Observation 1.9. Let $X \in \mathbf{Sp}$ and consider the Hurewicz homomorphism which lifts to a map

$$\pi_\bullet X \to [\mathbb{S}, X]_\bullet \to \mathrm{Hom}_{\mathbf{Comod}_{\mathcal{A}_*}}(\mathbb{F}_p, H_\bullet X) \subseteq H_\bullet X \cong \pi_\bullet(\mathbb{F}_p \otimes X).$$

In particular, the Hurewicz image has those elements without Steenrod operations since $\mathbb{F}_p$ has none.

[1] This is not always true for the cohomology of a spectrum.
[2] This is one reason why homology is better than cohomology.

This is a first step into detecting whether an element lies in the image of the Hurewicz homomorphism.

Example 1.10. Recall 1.5. Since $\mathrm{Sq}^1 x = x^2$ for $\mathbb{R}P^\infty$, we have $\mathrm{Sq}^1 \overline{x}^2 = \overline{x}$ and this shows that $\overline{x}^2$ is not in the image of the Hurewicz homomorphism.

The functor forget : $\mathbf{Comod}_{\mathcal{A}_*} \to \mathbf{Vect}_{\mathbb{F}_p}$ has a right adjoint $R(-) = \mathcal{A}_* \otimes_{\mathbb{F}_p} -$ giving rise to the cofree comodule.

Example 1.11. Let $X \in \mathbf{Mod}_{\mathbb{F}_p}(\mathbf{Sp})$, then $\pi_\bullet(X) \in \mathbf{Vect}_{\mathbb{F}_p}$ and

$$H_\bullet X \cong \pi_\bullet(\mathbb{F}_p \otimes X) \cong \pi_\bullet((\mathbb{F}_p \otimes \mathbb{F}_p) \otimes_{\mathbb{F}_p} X) \cong \mathcal{A}_* \otimes_{\mathbb{F}_p} \pi_\bullet X \cong R(\pi_\bullet X).$$

Thus, the Hurewicz homomorphism is a map

$$\pi_\bullet X \to \mathrm{Hom}_{\mathbf{Comod}_{\mathcal{A}_*}}(\mathbb{F}_p, R(\pi_\bullet(X))) \cong \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{F}_p}}(\mathbb{F}_p, \pi_\bullet X) \cong \pi_\bullet X$$

which can be checked to be an isomorphism. This shows that the map

$$\pi_\bullet X \to \mathrm{Hom}_{\mathbf{Comod}_{\mathcal{A}_*}}(\mathbb{F}_p, H_\bullet X)$$

is an isomorphism for $X \in \mathbf{Mod}_{\mathbb{F}_p}(\mathbf{Sp})$.

Example* 1.12. Consider $X = \mathbb{F}_p$. Then, the Hurewciz homomorphism is

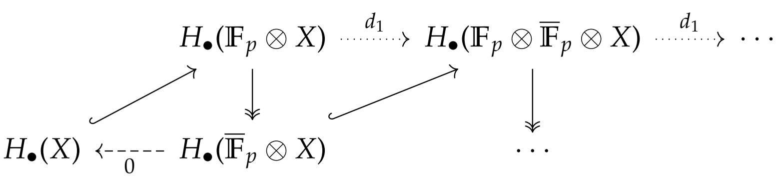
$$\mathbb{F}_p \to \mathrm{Hom}_{\mathbf{Comod}_{\mathcal{A}_*}}(\mathbb{F}_p, \mathcal{A}_*) \subseteq \mathcal{A}_*$$

where the first map is the map picking out elements in $\mathcal{A}_*$ with trivial Steenrod action. Those are precisely those concentrated in degree 0, so we are picking out degree 0 elements of $\mathcal{A}_*$.

Observation 1.13. Back to the Adams spectral sequence. Consider

$$E_1 \cong \pi_\bullet\left(\mathbb{F}_p \otimes \overline{\mathbb{F}}_p^{\otimes s} \otimes X\right) \cong \mathrm{Hom}_{\mathbf{Comod}_{\mathcal{A}_*}}\left(\mathbb{F}_p, H_\bullet(\mathbb{F}_p \otimes \overline{\mathbb{F}}_p^{\otimes s} \otimes X)\right)$$

by Hurewicz. On homology we have



The top row gives an injective[3] resolution of $H_\bullet(X)$ in comodules. It is a so-called **Adams resolution**.

Theorem 1.14 (Adams)**.**

(i) There exists an isomorphism $E_2 \cong \mathrm{Ext}^{s,t}_{\mathbf{Comod}_{\mathcal{A}_*}}(\mathbb{F}_p, H_\bullet X)$.

(ii) If X is finite-type and bounded below, then this converges to $\pi_\bullet(X_p^\wedge) \cong \pi_\bullet(X) \otimes \mathbb{Z}_p$.

Why did this work? Essentially three steps.

(i) The Hurewicz map is an isomorphism for Eilenberg-MacLane spectra giving rise to this identification of the E_1-page.

(ii) $H_\bullet(-)$ of Eilenberg-MacLane spectra is injective as comodules for the injective resolution.

(iii) $H_\bullet(-)$ takes cofiber sequences to LESs for the injective resolution. Note here that we have those injective maps by virtue of the Hurewicz homomorphism for $\mathbb{F}_p$-modules (1.11). By exactness, the bottom arrows must be 0 and thus the vertical arrows must be surjective. This can be used as a diagram chase to see that the top row is exact.

□

[3] This uses that the right adjoint of an injective object is injective and everything in $\mathbf{Vect}_{\mathbb{F}_p}$ is injective. It also uses that these triangles are short exact sequences.

1.3 A Short Example

It's time to draw an Adams chart.

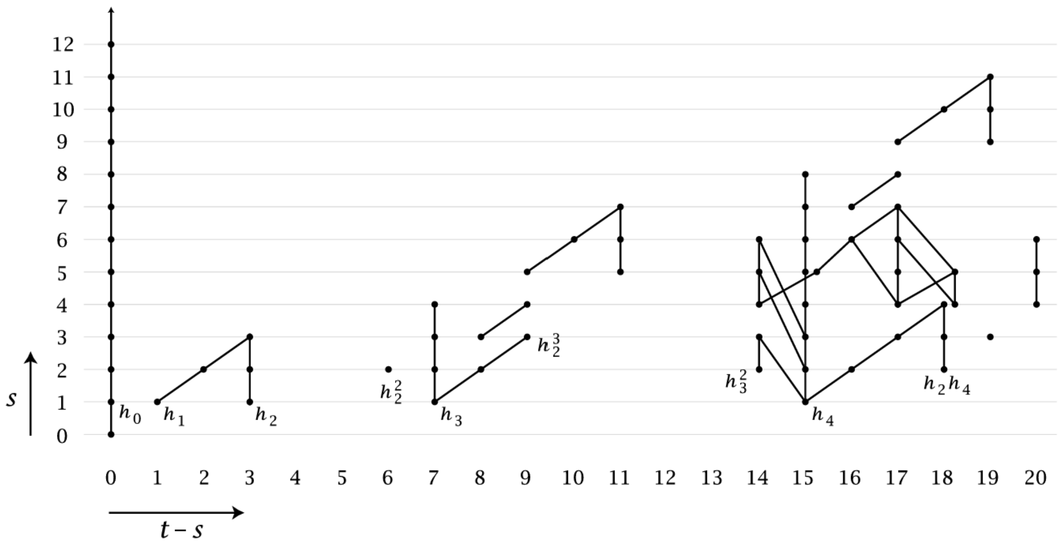


Figure 1: This is from [Hat04, p. 599].

Example 1.15. Let $X = \mathbb{S}$. Then, $E_2 = \mathrm{Ext}^{s,t}_{\mathbf{Comod}_{\mathcal{A}_*}}(\mathbb{F}_2, \mathbb{F}_2)$.

(i) Let $s = 0$. Then, $\mathrm{Ext}^0 = \mathrm{Hom}_{\mathbf{Comod}_{\mathcal{A}_*}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[0]$. This is the dot detecting the identity.

(ii) Let $s = 1$. Then,

$$\mathrm{Ext}^{1,t} = \mathrm{Ext}^1(\mathbb{F}_2[t], \mathbb{F}_2) = \{\text{extensions in comodules } \mathbb{F}_2 \to M \to \mathbb{F}_2[t]\}.$$

So additively, we must have $M \cong \mathbb{F}_2 \oplus \mathbb{F}_2[t]$. There is only a single possible Steenrod operation $\mathrm{Sq}^t : \mathbb{F}_2[t] \to \mathbb{F}_2$. It can either be 0 or an isomorphism. When do we get a comodule? There is e.g. the Adem relation $\mathrm{Sq}^3 = \mathrm{Sq}^1\,\mathrm{Sq}^2$, so this already shows that Sq^3 must act via 0. It's a fact that exactly those Sq^{2^n} can be taken as isomorphisms, and so we get an element $h_n \in \mathrm{Ext}^{1,2^n}$.

There is a Yoneda product on Ext giving rise to a multiplicative structure on the spectral sequence. Plus, on the first horizontal part of the picture there are no differentials. There might be one from h_1 to 8, because 8 is non-trivial. From this picture we get

$$\pi_k(\mathbb{S})^\wedge_2 = (\mathbb{Z}_2, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/8, 0, 0, \cdots)$$

allowing us to read off the first few stable homotopy groups of spheres.

2 Abstractification of Adams Spectral Sequence

It turns out you can abstractify some of the tools used for the Adams spectral sequence to generalize this. Let $\mathscr{C}$ be a stable ∞-category and $\mathscr{A}$ be an abelian category. TALK 2 17.09.2025

2.1 Homological Functors

Definition 2.1. A functor $H : \mathscr{C} \to \mathscr{A}$ is **homological** if it is additive and if for cofiber sequences $x \to y \to z$ in $\mathscr{C}$ the sequence $H(x) \to H(y) \to H(z)$ is exact.

Remark 2.2. Since $\mathscr{A}$ is a 1-category, this factors uniquely through $h\mathscr{C}$, as h is a left adjoint. So these things can be phrased for triangulated categories.

Remark 2.3. Let $x \to y \to z$ be a cofiber sequence. So we get

$$\cdots \longrightarrow \Sigma^{-1}y \longrightarrow \Sigma^{-1}z \longrightarrow x \longrightarrow y \longrightarrow z \longrightarrow \Sigma x \longrightarrow \cdots$$

which gives rise to a LES

$$\cdots \longrightarrow H(\Sigma^{-1}z) \longrightarrow H(x) \longrightarrow H(y) \longrightarrow H(z) \longrightarrow H(\Sigma x) \longrightarrow \cdots$$

This is how you should think in general if you don't keep track of gradings (as opposed to classical homology) which would be encoded by the (de-)suspensions.

Example 2.4.

(i) Consider $\pi_0 : \mathbf{Sp} \to \mathbf{Ab}$ and $R \in \mathbf{Sp}$. Then, $R_\bullet(-) = \pi_\bullet(R \otimes -) : \mathbf{Sp} \to \mathbf{grAb}$ is homological.

(ii) Let $(\mathscr{C}_{\geq 0}, \mathscr{C}_{\leq 0})$ be a t-structure, then $\pi_0^\heartsuit(-) = \tau_{\geq 0}\tau_{\leq 0}(-) : \mathscr{C} \to \mathscr{C}^\heartsuit$ is homological.

(iii) Let $H : \mathscr{C} \to \mathscr{A}$ be a homological functor and $F : \mathscr{D} \to \mathscr{C}$ be an exact functor of stable ∞-categories, the $HF : \mathscr{D} \to \mathscr{A}$ is homological. Combining (ii) and (iii) recovers (i) as a special case.

(iv) Let $c \in \mathscr{C}$, then $[c, -] = \pi_0 \operatorname{map}_{\mathscr{C}}(c, -) : \mathscr{C} \to \mathbf{Ab}$ is a homological functor and similarly $\text{よ}_c(-) = [-, c] : \mathscr{C}^{\mathrm{op}} \to \mathbf{Ab}$.

(v) The Yoneda embedding $\text{よ} : \mathscr{C} \to \mathbf{PSh}(\mathscr{C}, \mathbf{Ab})$ is homological. It factors through a fully faithful functor

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\text{よ}} & \mathbf{PSh}(\mathscr{C}, \mathbf{Ab}) \\ \downarrow & \nearrow & \\ h\mathscr{C} & & \end{array}$$

which arguably says that homological functors are a powerful tool to study stable categories. This fully faithful functor remembers all information of the homotopy category through this homological functor.

2.2 The Freyd Envelope

Definition 2.5. A presheaf $X : \mathscr{C}^{\mathrm{op}} \to \mathbf{Ab}$ is **finitely presented** if there exists some $f : c \to d$ such that $X \simeq \operatorname{coker}(\text{よ}(c) \to \text{よ}(d))$.

Definition 2.6. We write $\mathcal{A}(\mathscr{C}) \subseteq \mathbf{PSh}(\mathscr{C}, \mathbf{Ab})$ for the category of finitely presented presheaves, called the **Freyd envelope**.

Note that this person is not pronounced 'Freud' – as Ieke remarks.

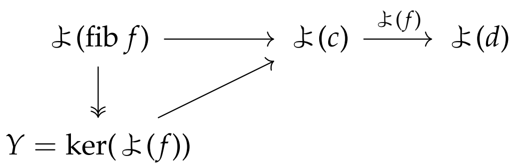
Remark* 2.7. Since $\mathbf{Ab}$ is a 1-category, we can use adjunction to obtain $\mathbf{PSh}(\mathscr{C}, \mathbf{Ab}) \simeq \mathbf{PSh}(h\mathscr{C}, \mathbf{Ab})$ and $\mathcal{A}(\mathscr{C}) \simeq \mathcal{A}(h\mathscr{C})$. In particular, we are dealing with 1-categories here.

Theorem 2.8 (Freyd)**.** The subcategory $\mathcal{A}(\mathscr{C}) \subseteq \mathbf{PSh}(\mathscr{C}, \mathbf{Ab})$ is an abelian subcategory, i.e. it is closed under cokernels, kernels and extensions.

Remark 2.9. I asked about the relationship to $\mathcal{A}(\mathscr{C})$ to $\mathbf{PSh}(\mathscr{C}, \mathbf{Ab})^{\omega}$ and Marius answered that they agree.

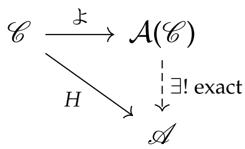
Remark 2.10. Consider $\mathbf{Mod}_R(\mathbf{Ab}) \supseteq \mathbf{Mod}_R^{\mathrm{fp}}(\mathbf{Ab})$. Then, the subcategory is abelian if and only if R is coherent.

Proof Sketch of 2.8. A priori there is no reason we're allowed to take kernels. Let $f : c \to d$ and consider $Y = \ker(よc \to よd) \in \mathbf{PSh}(\mathscr{C}, \mathbf{Ab})$. Consider $\operatorname{fib} f \to c \to d$ and the preferred comparison map



and it's a quick diagram chase to show that this left vertical map is surjective. This only shows finite generation but it turns out that you get finite presentation by modding out by the kernel of that surjection. □

Theorem 2.11 (Neeman)**.** The functor $よ : \mathscr{C} \to \mathcal{A}(\mathscr{C})$ is the initial homological functor out of $\mathscr{C}$, i.e.



for a homological functor $H : \mathscr{C} \to \mathscr{A}$.

Corollary 2.12. The notion of homological functors out of $\mathscr{C}$ depends only on $h\mathscr{C}$ as a category (not the triangulated structure).

Proof. It only depends on exact functors out of $\mathcal{A}(\mathscr{C})$ which ignores triangles. □

2.3 Adapted Homology Theories

Definition 2.13.

(i) We say that $H : \mathscr{C} \to \mathscr{A}$ has **lifts of injectives** if $\mathscr{A}$ has enough injectives and for every injective $i \in \mathscr{A}$ there exists $c_i \in \mathscr{C}$ and a map $H(c_i) \to i$ such that the composite

$$[-, c_i] \longrightarrow \operatorname{Hom}_{\mathscr{A}}(H(-), H(c_i)) \longrightarrow \operatorname{Hom}_{\mathscr{A}}(H(-), i)$$

is an isomorphism.[4]

(ii) We say that H is **adapted** if it has lifts of injectives for for every injective $i \in \mathscr{A}$ the map $H(c_i) \to i$ is an isomorphism.

Remark 2.14.

[4] In other words, $\operatorname{Hom}_{\mathscr{A}}(H(-), i)$ is represented by c_i.

(i) Note that $\mathrm{Hom}_{\mathscr{A}}(H(-), i)$ is a homological functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{A}$ since i is injective. The essence is the representability which essentially says that *Eilenberg-MacLane objects exists*.

Indeed, let $\mathscr{C}$ be presentable and H preserve (infinite) direct sums, then H has lifts of injectives by Brown representability [PP23, Corollary 2.17].

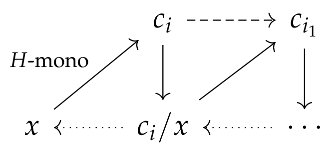
(ii) Adaptedness says that the Hurewicz map $[-, c_i] \to \mathrm{Hom}_{\mathscr{A}}(H(-), H(c_i))$ is an isomorphism.

(iii) The functor $H_\bullet(-) : \mathbf{Sp} \to \mathbf{Vect}_{\mathbb{F}_p}$ has lifts of injectives but is not adapted. Indeed,

$$\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{F}_p}}(H_\bullet(-), \mathbb{F}_p) \cong H^\bullet(-)$$

by duality, so it is represented by $H\mathbb{F}_p$. On the other hand, $H_\bullet(\mathbb{F}_p) \cong \mathcal{A}_* \not\cong \mathbb{F}_p$ which shows that it is not adapted. It turns out that $H_\bullet(-) : \mathbf{Sp} \to \mathbf{Comod}_{\mathcal{A}_*}$ is adapted.

Construction 2.15. Let $H : \mathscr{C} \to \mathscr{A}$ be adapted with $x \in \mathscr{C}$ and $f : H(x) \hookrightarrow i$ using that $\mathscr{A}$ has enough injectives. We lift this to a map $x \to c_i$ in $\mathscr{C}$ where we can take the cofiber c_i/x. We proceed inductively by now lifting $H(c_i/x) \hookrightarrow i_1$ and so on, so we obtain an Adams resolution



with $H(c_i/x) \cong i/H(x)$. For $d \in \mathscr{C}$ we get

$$E_1 = [d, c_{i_\bullet}] \cong \mathrm{Hom}_{\mathscr{A}}(H(d), c_{i_\bullet})$$

where $i_\bullet$ is an injective resolution of $H(x)$. So this gives rise to

$$E_2 \cong \mathrm{Ext}^{\bullet,\bullet}_{\mathscr{A}}(H(d), H(x)) \Rightarrow [d, x]_\bullet,$$

the ***H*-based Adams spectral sequence**.

*Proof**. Injectivity is by construction. Moreover, these maps $H(\mathrm{cofib}) \to i_k$ are injective by construction. Exactness implies that the dotted arrows are 0 on homology, so the vertical arrows are surjective on homology. This can be used to diagram chase that $c_{i_\bullet}$ is a resolution. □

2.4 Classification of Adams Spectral Sequences

Question 2.16. Can we classify adapted homological functors? In other words: Can we classify Adams spectral sequences?

Example 2.17. Consider $\mathrm{MU}_\bullet : \mathbf{Sp} \to \mathbf{Comod}_{\mathrm{MU}_\bullet \mathrm{MU}}$. It leads to the Adams-Novikov spectral sequence.

Let $\mathscr{C}$ be idempotent complete.

Theorem 2.18 ([PP23, Lemma 2.55, Theorem 2.56])**.** Let $H : \mathscr{C} \to \mathscr{A}$ be homological such that $\mathscr{A}$ has enough injectives. This leads to $L : \mathcal{A}(\mathscr{C}) \to \mathscr{A}$. Then,

(i) H has lifts of injectives if and only if L has a right adjoint $R : \mathscr{A} \to \mathcal{A}(\mathscr{C})$.

(ii) H is moreover adapted if and only if R is fully faithful, i.e. $L : \mathcal{A}(\mathscr{C}) \to \mathscr{A}$ is a Bousfield localization.

Corollary 2.19. If H is adapted, then $\mathscr{A} \simeq \mathcal{A}(\mathscr{C}) / \ker L$ is the *Gabriel/Serre quotient*.

Remark* 2.20. In fact, this is an 'if and only if' [PP23, Theorem 2.56].

So understanding $\ker L$ is equivalent to understanding H. Let

$$X \simeq \operatorname{coker}(\text{よ}(f) : \text{よ}(c) \to \text{よ}(d)) \in \mathcal{A}(\mathscr{C}).$$

Then,

$$\begin{aligned} X \in \ker L &\iff LX \simeq 0 \\ &\iff \operatorname{coker}(L\text{よ}(f) : L\text{よ}(c) \to L\text{よ}(d)) \simeq 0 \\ &\iff \operatorname{coker}(H(f) : H(c) \to H(d)) \simeq 0 \\ &\iff f \text{ is an } H\text{-epimorphism.} \end{aligned}$$

The slogan is that adapted homology theories (i.e. Adams spectral sequences) are determined by homology epis.

Corollary 2.21. The category $\mathbf{Comod}_{\mathcal{A}_*}$, as an abelian category, is completely determined by the class of maps of spectra which are $H_\bullet(-;\mathbb{F}_p)$-epis.

Proof. Indeed, $\mathbf{Comod}_{\mathcal{A}_*} \simeq \mathcal{A}(\mathbf{Sp})/\ker L$ by 2.19 where L comes from $H_\bullet(-)$. □

It's quite magical that this is an invariant of $\mathbf{Sp}$ (and this choice of epimorphism class). Indeed, $\mathbf{Comod}_{\mathcal{A}_*}$ is a presentation of some category of quasicoherent sheaves on some stack but 2.21 has as consequence that it doesn't depend on this presentation.

3 The Derived ∞-Category of a Homology Theory

Last lecture we constructed Adams resolutions associated to adapted homology theories (2.15). TALK 3 18.09.2025

3.1 Problem: Categorification of Adams Resolutions

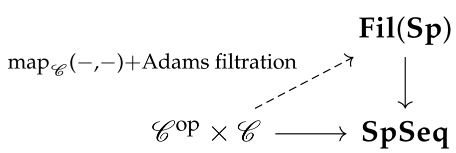
There are some problems:

(i) There is in general no canonoical choice of an Adams resolution.

(ii) You can write down an Adams spectral sequence functor

$$\mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{SpSeq}, \ (d, c) \mapsto E_2(d, c).$$

On the other hand, $\mathbf{SpSeq}$ is almost always a shadow of a more homotopical construction, namely a filtered spectrum. So we hope for:



Mapping into the sequence of maps on the bottom row of 2.15 will give this filtration.

Question 3.1. Can we construct an ∞-category $\mathcal{D}(\mathscr{C}, H)$ where Adams resolutions live and for which we have an 'Adams resolution' functor $\nu : \mathscr{C} \to \mathcal{D}(\mathscr{C}, H)$?

3.2 The Derived Category of an Abelian Category

About the abelian case: There is an assignment $\mathscr{A} \rightsquigarrow \mathbf{Ch}(\mathbf{Inj})$ to the dg-category of chain complexes of injectives. A clever combinatorial argument gives:

Definition 3.2. The **derived ∞-category** of $\mathscr{A}$ is $\mathcal{D}(\mathscr{A}) = N_{\mathrm{dg}}(\mathbf{Ch}(\mathbf{Inj}))$.

Fact 3.3. This is a stable ∞-category and there are homology functors $H_k(-) : \mathcal{D}(\mathscr{A}) \to \mathscr{A}$ giving rise to the standard t-structure $(\mathcal{D}_{\geq 0}(\mathscr{A}), \mathcal{D}_{\leq 0}(\mathscr{A}))$ with $H_0 : \mathcal{D}(\mathscr{A})^{\heartsuit} \xrightarrow{\simeq} \mathscr{A}$.

In particular, it has a unique inverse $i : \mathscr{A} \to \mathcal{D}(\mathscr{A})^{\heartsuit}$. This essentially sends $A \in \mathscr{A}$ to an injective resolution in a coherent way.

Question 3.4. Can we characterize $\mathcal{D}(\mathscr{A})$?

Observation 3.5. The functor $i : \mathscr{A} \to \mathcal{D}^{\flat}_{\geq 0}(\mathscr{A})$ send SES in $\mathscr{A}$ to cofiber sequences.

Theorem 3.6 (BCKW, 2019)**.** The functor i is the universal additive functor out of $\mathscr{A}$ such that the target is additive and has finite colimits and the functor sends SES to cofiber sequences.

Remark 3.7. You can also consider $\mathscr{A} \to \mathcal{D}^{\flat}(\mathscr{A})$ and then replace the word additive by stable.

We want to start mimicking the above behaviour with the following universal construction.

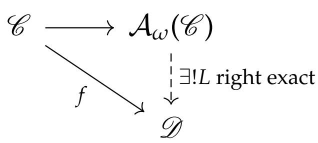
Definition 3.8. Let $\mathscr{C}$ be an additive ∞-category. Its **prestable Freyd envelope**

$$\mathcal{A}_{\omega}(\mathscr{C}) \subseteq \mathbf{PSh}_{\Sigma}(\mathscr{C}, \mathbf{Sp}_{\geq 0})$$

is the full subcategory generated under finite colimits by the image of よ.[5] We obtain a functor $\nu : \mathscr{C} \to \mathcal{A}_{\omega}(\mathscr{C})$.

Remark 3.9.

(i) For every additive ∞-category with finite colimits $\mathscr{D}$ and additive functor $f : \mathscr{C} \to \mathscr{D}$ there is a unique factorization



essentially by construction of $\mathcal{A}_{\omega}(\mathscr{C})$ (by freely adjoining finite colimits).

(ii) The heart is the classical Freyd envelope $\mathcal{A}_{\omega}(\mathscr{C})^{\heartsuit} \simeq \mathcal{A}(\mathscr{C})$.

(iii*) There are two sensible ways of infinitizing the Freyd envelope and this is the one which Patchkoria-Pstragowski call *perfect prestable Freyd envelope* [PP23, Definition 4.20].

Theorem 3.10 ([PP23, Theorem 4.26])**.** Let $\mathscr{C}$ be an additive ∞-category with finite limits. Then, $\mathcal{A}_{\omega}(\mathscr{C})$ also has finite limits.

Comment.* This is quite a bit of work in the paper. See also [PP23, Remark 4.23] for a remark on the crux of difficulty. □

The classical theorem of Freyd only requires weak limits, here you actually need limits.

Corollary 3.11. Let $\mathscr{C}$ have finite limits. Then, $\mathcal{A}_{\omega}(\mathscr{C})$ is the connective part of a t-structure on its Spanier-Whitehead stabilization: $\mathcal{A}_{\omega}(\mathscr{C}) \simeq (\mathcal{A}_{\omega}(\mathscr{C})^{\mathrm{st}})_{\geq 0}$.

[5] Note that $\mathbf{Sp}_{\geq 0}$ is the free additive ∞-category on a single object, and hence an ∞-categorical analogue of **Ab**.

Observation 3.12. Let $\mathscr{A}$ be abelian. Consider

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\nu} & \mathcal{A}_\omega(\mathscr{C}) \\ & \searrow & \downarrow{\scriptstyle L} \\ & & \mathcal{D}^\flat_{\geq 0}(\mathscr{A}) \end{array}$$

then L forces the image of SESs in $\mathscr{A}$ to be cofiber sequences.

In that regard, we are pretty close to $\mathcal{D}^\flat_{\geq 0}(\mathscr{A})$.

3.3 Relations via Sheaf Theory

Definition 3.13. An **additive ∞-site** is an additive ∞-category with an Grothendieck pretopology such that covering families $\{f_i : c_i \to c\}_i$ are singletons.

Example 3.14.

(i) If $\mathscr{A}$ is abelian, then it can be made into an additive site with coverings being epimorphisms [PP23, Definition 5.5].

(ii) Let $H : \mathscr{C} \to \mathscr{A}$ be a homology theory. Then, $\mathscr{C}$ becomes an additive ∞-site where coverings are H-epimorphisms.

Theorem 3.15 ([Pst23, Theorem 2.8])**.** Let $\mathscr{C}$ be an additive ∞-site and $X : \mathscr{C}^{\mathrm{op}} \to \mathbf{Sp}_{\geq 0}$ be an additive functor. TFAE:

(i) X is a sheaf.

(ii) For every covering $p : d \to c$ the sequence

$$X(c) \longrightarrow X(d) \longrightarrow X(\mathrm{fib}\, p)$$

is a fiber sequence.

Remark 3.16. I.e. $\nu(\mathrm{fib}\, p) \to \nu d \to \nu c$ is a cofiber sequence of sheaves.

Fact 3.17. Let $\mathscr{C}$ be an additive ∞-site and consider the free cocompletion under colimits inside additive ∞-categories $\mathbf{PSh}_\Sigma(\mathscr{C}, \mathbf{Sp}_{\geq 0})$. Then,

$$\mathbf{Sh}_\Sigma(\mathscr{C}, \mathbf{Sp}_{\geq 0}) \hookrightarrow \mathbf{PSh}_\Sigma(\mathscr{C}, \mathbf{Sp}_{\geq 0})$$

is a free completion subject to the relation $\mathrm{fib}\, p \to d \to c$ is a cofiber sequence for every covering p. There exists a sheafification functor $L : \mathbf{PSh}_\Sigma(\mathscr{C}, \mathbf{Sp}_{\geq 0}) \to \mathbf{Sh}_\Sigma(\mathscr{C}, \mathbf{Sp}_{\geq 0})$.

Corollary 3.18. The functor L is left-exact.

Theorem 3.19. Let $\mathscr{A}$ be abelian. Then, $\mathcal{D}^\flat_{\geq 0}(\mathscr{A}) \simeq \mathcal{A}_\omega(\mathscr{C}) \cap \mathbf{Sh}_\Sigma$ with the epimorphism topology on $\mathscr{A}$.[6]

Definition 3.20. Let $H : \mathscr{C} \to \mathscr{A}$ be an adapted homology theory. Then,

$$\begin{aligned} \mathcal{D}^\flat_{\geq 0}(\mathscr{C}; H) &= \mathcal{A}_\omega(\mathscr{C}) \cap \mathbf{Sh}_\Sigma \\ &= \{X : \mathscr{C}^{\mathrm{op}} \to \mathbf{Sp}_{\geq 0} : a \to b \xrightarrow{p} c \text{ with } (*)\} \end{aligned}$$

where $(*)$ is that if this is a cofiber sequence with p an H-epi, then $X(c) \to X(b) \to X(a)$ is a fiber sequence and $X \in \mathcal{A}_\omega(\mathscr{C})$.

We denote by $\nu : \mathscr{C} \hookrightarrow \mathcal{D}^\flat_{\geq 0}(\mathscr{C})$ the Yoneda embedding.

[6] These are also called *perfect*.

Equivalently, $\mathcal{D}_{\geq 0}(\mathscr{C}; H)$ is built from the representables $\nu(c) = (\tau_{\geq 0}\,\mathrm{map}(-, c))^{\#}$ for $c \in \mathscr{C}$ by closing under finite colimits and desuspensions.

Theorem 3.21. The standard t-structure on $\mathbf{Sh}(\mathscr{C}, \mathbf{Sp})$ restricts.

(i) The category $\mathcal{D}^{\flat}_{\geq 0}(\mathscr{C})$ is a prestable ∞-category with finite colimits, so $\mathcal{D}^{\flat}_{\geq 0}(\mathscr{C}) \simeq (\mathcal{D}^{\flat}_{\geq 0}(\mathscr{C})^{\mathrm{st}})_{\geq 0}$ and $\mathcal{D}^{\flat}(\mathscr{C}) = \mathcal{D}^{\flat}_{\geq 0}(\mathscr{C})^{\mathrm{st}}$.

(ii) There is an equivalence $\mathcal{D}^{\flat}(\mathscr{C})^{\heartsuit} \simeq \mathscr{A}$.

(iii) The composite

$$\mathscr{C} \xrightarrow{\nu} \mathcal{D}^{\flat}_{\geq 0}(\mathscr{C}) \xrightarrow{\tau_{\leq 0}} \mathcal{D}^{\flat}(\mathscr{C})^{\heartsuit} \simeq \mathscr{A}$$

is equivalent to H.

The functor $\nu : \mathscr{C} \to \mathcal{D}^{\flat}(\mathscr{C})$ is about lifting H from being valued in an abelian category to a functor valued in a stable ∞-category with a t-structure. Moreover, ν is fully faithful, so it really stores all of the information from $\mathscr{C}$.

4 Towards Deformation Theory

4.1 Deformation Theory

Let's start with some examples of the derived ∞-categories that we constructed last lecture. TALK 4 19.09.2025

Example 4.1.

(i) Let $\mathscr{A} = \mathcal{A}(\mathscr{C})$ and consider the universal homology theory. Then,

$$\mathcal{D}^{\flat}_{\geq 0}(\mathscr{C}) \simeq \mathcal{A}_{\omega}(\mathscr{C}) \simeq \mathbf{PSh}_{\Sigma}(\mathscr{C}).$$

This is the largest derived category.

(ii) Let $\mathscr{A} \simeq 0$. Then, $\nu : \mathscr{C} \xrightarrow{\simeq} \mathcal{D}^{\flat}(\mathscr{C})$. This is the smallest derived category.

(iii) Let $E \in \mathbf{Sp}$ be nice enough, say Adams-type. Consider $E_{\bullet} : \mathbf{Sp} \to \mathbf{Comod}_{E_{\bullet}E}$. Then, $\mathcal{D}^{\flat}(\mathbf{Sp}, E_{\bullet}) \hookrightarrow \mathbf{Syn}_E$ into the E-based synthetic spectra where the left side is the thick subcategory generated by νX for $X \in \mathbf{Sp}$. This is the minimal thing in $\mathbf{Syn}_E$ containing the representables, so it is enough to talk about Adams spectral sequences.

Idea: The category $\mathcal{D}^{\flat}(\mathscr{C}, H)$ is a categorical deformation interpolating between $\mathscr{C}$ and $\mathcal{D}^{\flat}(\mathscr{A})$. The mapping spaces of $\mathcal{D}^{\flat}(\mathscr{A})$ are Ext-groups which should be the E_2-page while the mapping spaces in $\mathscr{C}$ are just mapping spaces, i.e. the abutment.

Observation 4.2. Assume: If $x \to y$ is H-epi, then $\Sigma^k x \to \Sigma^k y$ is H-epi for all k. Then,

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{H} & \mathscr{A} \\ {\scriptstyle \Sigma^k}\downarrow & & \downarrow{\scriptstyle [k]} \\ \mathscr{C} & \xrightarrow[H]{} & \mathscr{A} \end{array}$$

from the quotient universal property via the Freyd envelope, so we get a canonical autoequivalence, i.e. $H(\Sigma^k x) \simeq H(x)[k]$.

Example 4.3. Let $H = H_{\bullet}(-; \mathbb{F}_p)$, then $[k]$ is the grading shift. We get

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\nu} & \mathcal{D}^\flat(\mathscr{C}) \\ \Sigma^k \downarrow & & \downarrow [k] \\ \mathscr{C} & \xrightarrow[\nu]{} & \mathcal{D}^\flat(\mathscr{C}) \end{array}$$

Observation 4.4. Let $\nu : \mathscr{C} \to \mathcal{D}^\flat_{\geq 0}(\mathscr{C})$ leading to a canonical comparison map

$$\Sigma\nu(c) \longrightarrow \nu(\Sigma c) \simeq \nu(c)[1]$$

which gives rise to a canonical natural transformation $\Sigma X \to X[1]$ for all $X \in \mathcal{D}^\flat(\mathscr{C})$ or equivalently $\tau : \Sigma X[-1] \to X$. This gives rise to the deformation picture

$$\mathscr{C} \xleftarrow{\tau^{-1}} \mathcal{D}^\flat(\mathscr{C}) \xrightarrow{-/\tau = C\tau\otimes -} \mathcal{D}^\flat(\mathscr{A})$$

where on the right side $\otimes$ doesn't quite make sense in this generality since there is in general not a monoidal structure. Nonetheless we use this notation for $-/\tau$ since $C\tau \otimes -$ has become the classical notation.

Theorem 4.5.

(i) There is an equivalence $\mathscr{C} \simeq \mathcal{D}^\flat(\mathscr{C})[\tau^{-1}]$.

(ii) The endofunctor $C\tau \otimes X \simeq \operatorname{cofib}(\tau : \Sigma X[-1] \to X)$ in X has a canonical monad structure and $\mathbf{Mod}_{C\tau}(\mathcal{D}^\flat(\mathscr{C})) \simeq \mathcal{D}^\flat(\mathscr{A})$.[7]

Proof Idea. Concretely, τ is given by

$$\Sigma\nu(c)[1] \simeq \Sigma\nu(\Sigma^{-1}c) \simeq \Sigma\tau_{\geq 0}\operatorname{map}(-, \Sigma^{-1}c)^\# \simeq \tau_{\geq 1}\operatorname{map}(-, c)^\# \to \tau_{\geq 0}\operatorname{map}(-, c)^\# = \nu(c),$$

the 1-connective cover of $\nu(c)$. Thus, we deduce $C\tau \otimes \nu(c) = \nu(c)/\tau \in \mathcal{D}^\flat(\mathscr{C})^\heartsuit$.

This implies that $\mathbf{Mod}_{C\tau}(\mathcal{D}^\flat(\mathscr{C}))$ is generated by objects in the heart. In particular, it is a derived category of an abelian category

$$\mathcal{D}^\flat(\mathscr{C}) \underset{H_*}{\overset{H^*}{\rightleftarrows}} \mathcal{D}^\flat(\mathscr{A})$$

induced by $H : \mathscr{C} \to \mathscr{A}$. One checks that the adjunction $H^* \dashv H_*$ is monadic and that the associated monad can be identified with $C\tau \otimes -$. □

4.2 Adams Resolutions

Let's go back to

$$\begin{array}{ccc} & & \mathbf{Fil}(\mathbf{Sp}) = \operatorname{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathbf{Sp}) \\ & \overset{F_\bullet \operatorname{map}_{\mathscr{C}}(-,-)}{\nearrow} & \downarrow \\ \mathscr{C}^{\mathrm{op}} \times \mathscr{C} & \longrightarrow & \mathbf{SpSeq} \end{array}$$

that appeared in 3.1.

Construction 4.6. Let $c \in \mathscr{C}$. We get

[7] So $C\tau \otimes - \in \operatorname{End}(\mathcal{D}^\flat(\mathscr{C}))$ is an algebra therein.

$$\begin{array}{ccccccc} \tau_{\geq 1} \operatorname{map}_{\mathscr{C}}(-,c)^{\#} & \longrightarrow & \tau_{\geq 0} \operatorname{map}_{\mathscr{C}}(-,c)^{\#} & \longrightarrow & \tau_{\geq -1} \operatorname{map}_{\mathscr{C}}(-,c)^{\#} & \longrightarrow & \cdots \\ \| & & \| & & \| & & \\ \Sigma\nu(c)[-1] & \xrightarrow[\tau]{} & \nu(c) & \xrightarrow[\tau]{} & \Sigma^{-1}\nu(c)[1] & \longrightarrow & \cdots \end{array}$$

giving rise to $V_\bullet(c) \in \mathbf{Fil}(\mathcal{D}^\flat(\mathscr{C}))$ with $V_n(c) = \Sigma^n \nu(c)[-n]$. For $d \in \mathscr{C}$ we then put

$$F_\bullet \operatorname{map}_{\mathscr{C}}(d,c) = \operatorname{map}_{\mathcal{D}^\flat(\mathscr{C})}(d, V_\bullet(c)) \in \mathbf{Fil}(\mathbf{Sp}).$$

Then,

(i) $\operatorname{colim} F_\bullet \operatorname{map}_{\mathscr{C}}(d,c) \simeq \operatorname{colim}_k \tau_{\geq k} \operatorname{map}(d,c)^{\#} \simeq \operatorname{colim} \tau_{\geq k} \operatorname{map}(d,c) \simeq \operatorname{map}(d,c)$.

(ii) There is an equivalence

$$\begin{aligned} \operatorname{gr} F_\bullet \operatorname{map}_{\mathscr{C}}(d,c) &\simeq \operatorname{map}_{\mathcal{D}^\flat(\mathscr{C})}(\nu d, \Sigma^n \nu(c)/\tau[-n]) \\ &\simeq \operatorname{map}_{C\tau}(\nu d/\tau, \Sigma^n \nu(c)/\tau[-n]) \\ &\simeq \operatorname{map}_{\mathcal{D}^\flat(\mathscr{A})}(H(d), \Sigma^n H(c)[-n]), \end{aligned}$$

so its homotopy groups are given by Ext_A. In fact, the spectral sequence associated to $F_\bullet \operatorname{map}_{\mathscr{C}}(d,c)$ is the H-ASS via some functoriality arguments.

Let $R \in \mathbf{Alg}(\mathbf{Sp})$ and $X, Y \in \mathbf{Sp}$. Then, one can write down a cosimplicial diagram

$$R \otimes X \rightrightarrows R \otimes R \otimes X \rightrightarrows \cdots,$$

the *Amitsur resolution* which corresponds to a filtered spectrum by the Dold-Kan correspondence. Then,

$$\widetilde{F} \operatorname{map}_{\mathbf{Sp}}(Y,X) \simeq \operatorname{Tot}(\tau_{\geq \bullet} \operatorname{map}(Y, R^{\otimes \bullet+1} \otimes X))$$

where $\operatorname{Tot} \circ \tau_{\geq \bullet}$ is the décalage. This is the R-Adams filtered mapping spectrum. That's one reason classically this lift problem (3.1) is usually not discussed. That one is a non-canonical resolution while we can just write down a canonical resolution as above.

Let $R \in \mathbf{CAlg}(\mathbf{Sp})$, then

$$\widetilde{F} \operatorname{map}_{\mathbf{Sp}}(\mathbb{S}, X) = \operatorname{Tot}(\tau_{\geq \bullet} \mathrm{AR})$$

gives a lax symmetric monoidal functor $\mathbf{Sp} \to \mathbf{Fil}(\mathbf{Sp})$, the R-Adams filtration functor.

Our final goal is to describe such a functor without the assumption $R \in \mathbf{CAlg}(\mathbf{Sp})$.

Observation 4.7. Assume that R has a right unital multiplication. Then, there is a homology theory $H : \mathbf{Sp} \to \mathbf{Ab}$ such that $f : X \to Y$ is H-epi if and only if $R \otimes X \to R \otimes Y$ has a section. The H-epis are closed under $- \otimes_{\mathbf{Sp}} -$ giving rise to a unique symmetric monoidal structure on $\mathcal{D}^\flat(\mathbf{Sp})$ such that ν is symmetric monoidal.

Theorem 4.8. Let R be a spectrum with a right-unital multiplication. Then, the R-Adams filtration functor

$$\mathbf{Sp} \xrightarrow{V_\bullet(-)} \mathbf{Fil}(\mathcal{D}^\flat(\mathbf{Sp})) \xrightarrow{\operatorname{map}(\nu\mathbb{S},-)} \mathbf{Fil}(\mathbf{Sp})$$

is lax symmetric monoidal.

4.3 Multiplicative Structure on Moore Spectra

Let p be odd. It is known that $\mathbb{S}/p$ is a homotopy associative unital ring (but not $\mathbb{E}_1$).

Theorem 4.9 (Burklund)**.** The Moore spectrum $\mathbb{S}/p^{n+1}$ can be made $\mathbb{E}_n$.

Idea. Since $\mathbb{S}/p$ has a right unital multiplication, we obtain a symmetric monoidal structure on $\mathcal{D}^\flat(\mathbf{Sp}, \mathbb{S}/p)$ where he sets up an obstruction theory to make something into a ring. He shows that the obstructions vanish for $\nu\mathbb{S}/(p/\tau)^{n+1}$. So $\nu\mathbb{S}/(p/\tau)^{n+1} \in \mathbf{Alg}_{\mathbb{E}_n}(\mathcal{D}^\flat(\mathbf{Sp}, \mathbb{S}/p))$ and applying τ^{-1} lets us land in $\mathbf{Sp}$. □

This makes much use of the lax symmetric monoidal $\mathbb{S}/p$-Adams resolution allowing us to preserve algebras! It can now be done despite $\mathbb{S}/p$ not having much structure – we only need it to have a right unital multiplication (**4.8**).

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