Spectral Sequences

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Abstract

This is an introductory/recap talk to the theory of spectral sequences.

A spectral sequence is an algebraic workhorse that takes a filtration on an object

$$\cdots \longrightarrow c_0 \longrightarrow c_1 \longrightarrow c_2 \longrightarrow \cdots \longrightarrow c$$

and tries to piece together the object c from the filtration quotients through certain algebraic information. This algebraic information often comes from a t-structure on a stable ∞-category which we introduce in the beginning of the talk. Then, we explain how to obtain a spectral sequence from a filtered object following Lurie. Having set these up, we discuss several examples from topology and algebra, ranging from the Serre spectral sequences and its relatives, consequences of the Grothendieck spectral sequence to the Adams spectral sequence.

The prison theme somehow came to me while devising the introduction of this talk. I enjoy fun gimmicks and jokes in talks and hope that this is a refreshing one.

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Contents

0 Prison Captive: Inside a Prison Camp **1**
0.1 A History Lesson 1
0.2 Desiderata 1

1 Prison Sentence: Crash Course on t-Structures **2**

2 Prison Work: Spectral Sequence from Filtered Objects **3**
2.1 Construction of Spectral Sequence 3
2.2 Convergence 4

3 Prison Release: Examples of Spectral Sequences **5**
3.1 Spectral Sequences in Topology 5
3.2 Spectral Sequences in Algebra 6
3.3 The Adams Spectral Sequence 7

0 Prison Captive: Inside a Prison Camp

0.1 A History Lesson

We should begin with some history.

> *Myth (Vakil).* Spectral sequences are called 'spectral' because, like spectres, spectral sequences are terrifying, evil and dangerous. I have heard no disagree with this interpretation, which is perhaps not surprising since I just made it up [Vak24, p. 63].

Alright, that was fake history! Onto some real history.

World war 2 broke out. Jean Leray was an officier in the French army and was arrested by the Germans when France was occupied by Germany. He was brought to an officers' prison camp in Edelbach, Austria. Leray's main interests lied in analysis but he feared that his expertise in applied mathematics could lead to him being forced to support the German war effort. So he only mentioned experience in an opaque field instead: topology. In these prison days the concept of spectral sequences was born [McC99, Section 3].

So you might say we are doing prison's work at this EAST but I think the moral of the story is that no matter what hardships you are or will be going through – even if you land in prison, you may still be able to produce groundbreaking mathematics in there.

0.2 Desiderata

Suppose that we are in prison, so we are tasked with the study of spectral sequences.[1]

In homological algebra we learn the omnipresent concept of a long exact sequence: Say we have a short exact sequence of chain complexes

$$0 \longrightarrow C_0 \longrightarrow C \longrightarrow C/C_0 \longrightarrow 0,$$

then, this induces a long exact sequence on homology groups $H_\bullet$. In other words, we want to understand C by piecing together a subobject C_0 and the corresponding quotient C/C_0.

Phrased a bit more generally, let $\mathscr{C}$ be a stable ∞-category and $c \in \mathscr{C}$. Consider a map $c_0 \to c$ together with its cofiber $c_0 \to c \to c/c_0$ and we want to understand c by gluing together certain algebraic information about c_0 and c/c_0. This is the baby case of a spectral sequence: Instead of consider a 1-stage filtration

$$\begin{array}{ccccc} 0 & \longrightarrow & c_0 & \longrightarrow & c \\ & & \downarrow & & \downarrow \\ & & c_0 & & c/c_0 \end{array}$$

we may consider a general filtration

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & c_0 & \longrightarrow & c_1 & \longrightarrow & \cdots & \longrightarrow & c \\ & & \downarrow & & \downarrow & & & & \\ & & c_0/c_{-1} & & c_1/c_0 & & & & \end{array}$$

and wish to study c by virtue of the filtration quotients. In practice, there should be some algebraic information that we can extract – often coming from a *t-structure* – which we piece together through a *spectral sequence*. In other words, there should be some sort of functor

$$\mathbf{Fil}(\mathscr{C}) \to \mathbf{SpSeq}(\mathscr{C}^\heartsuit)$$

from filtered objects on a stable ∞-category $\mathscr{C}$ with a t-structure to spectral sequences valued in the heart $\mathscr{C}^\heartsuit$.

[1] Best prison I've ever been in.

1 Prison Sentence: Crash Course on t-Structures

Throughout the entire talk, unless otherwise mentioned, let $\mathscr{C}$ be a stable ∞-category, potentially equipped with a t-structure, defined as follows:

Definition 1.1 (Beilinson-Bernstein-Deligne, 1983)**.** A ***t*-structure** on $\mathscr{C}$ is a pair of full subcategories $(\mathscr{C}_{\geq 0}, \mathscr{C}_{\leq 0})$ on $\mathscr{C}$ satisfying:

(i) If $c \in \mathscr{C}_{\geq 0}, c' \in \mathscr{C}_{\leq 0}$, then $\Sigma c \in \mathscr{C}_{\geq 0}$ and $\Omega c \in \mathscr{C}_{\leq 0}$.

(ii) If $c \in \mathscr{C}_{\geq 0}, c' \in \mathscr{C}_{\leq 0}$, then $\mathrm{Map}_{\mathscr{C}}(c, \Omega c) \simeq *$.

(iii) For every $c \in \mathscr{C}$ there exists a fiber sequence

$$c_{\geq 0} \longrightarrow c \longrightarrow c_{\leq -1}$$

with $c_{\geq 0} \in \mathscr{C}_{\geq 0}$ and $\Sigma c_{\leq -1} \in \mathscr{C}_{\leq 0}$.

We call $\mathscr{C}_{\geq 0}$ resp. $\mathscr{C}_{\leq 0}$ the **connective** resp. **coconnective** parts of $\mathscr{C}$.

Remark 1.2. With a little argument a t-structure on $\mathscr{C}$ is the same thing as a t-structure on $h\mathscr{C}$.

Often, it's easier to describe just the connective or just the coconnective part of a t-structure. However, the other part is already determined.

Lemma 1.3 ([Win24, Lemma 10.4])**.** There are equivalences

$$\begin{aligned} \mathscr{C}_{\geq 0} &\simeq \{c \in \mathscr{C} : \mathrm{Map}_{\mathscr{C}}(c, \Omega c') \simeq * \quad \text{for all } c' \in \mathscr{C}_{\leq 0}\}, \\ \mathscr{C}_{\leq 0} &\simeq \{c' \in \mathscr{C} : \mathrm{Map}_{\mathscr{C}}(c, \Omega c') \simeq * \quad \text{for all } c \in \mathscr{C}_{\geq 0}\}. \end{aligned}$$

Notation 1.4. Let $n \in \mathbb{Z}$. We write $\mathscr{C}_{\geq n} = \Sigma^n \mathscr{C}_{\geq 0}$ and $\mathscr{C}_{\leq -n} = \Omega^n \mathscr{C}_{\leq 0}$.

The following is probably responsible for the etymology of t-structures; namely *tronque*, or truncation [Hum].

Proposition 1.5. Let $m \in \mathbb{Z}$.

(i) There is a Bousfield localization $\tau_{\leq m} : \mathscr{C} \to \mathscr{C}_{\leq m}$ as well as a Bousfield colocalization $\tau_{\geq m} : \mathscr{C} \to \mathscr{C}_{\geq m}$.

(ii) There is a natural equivalence $\tau_{\leq m}\tau_{\geq n} \Rightarrow \tau_{\geq n}\tau_{\leq m}$ of functors $\mathscr{C} \to \mathscr{C}_{\leq m} \cap \mathscr{C}_{\geq n}$.

Proof.

(i) Comparing adjunctions, one can check $\tau_{\geq m} \simeq \Sigma^m \tau_{\geq 0} \Omega^m$, so WLOG $m = 0$.

Adjunctions can be constructed objectwise, so it suffices to show that for $c \in \mathscr{C}_{\geq 0}$ and $c' \in \mathscr{C}$ the natural map

$$\mathrm{Map}_{\mathscr{C}_{\geq 0}}(c, c'_{\geq 0}) \to \mathrm{Map}_{\mathscr{C}}(c, c')$$

is an equivalence. But the fiber sequence $c'_{\geq 0} \to c' \to c'_{\leq -1}$ from 1.1(iii) induces a fiber sequence

$$\mathrm{Map}_{\mathscr{C}}(c, c'_{\geq 0}) \longrightarrow \mathrm{Map}_{\mathscr{C}}(c, c') \longrightarrow \mathrm{Map}_{\mathscr{C}}(c, c'_{\leq -1})$$

and the last term is trivial by 1.1(ii).

(ii) See [Lur17, Proposition 1.2.1.10].

□

Definition 1.6.

(i) The **heart** of the t-structure is $\mathscr{C}^{\heartsuit} = \mathscr{C}_{\geq 0} \cap \mathscr{C}_{\leq 0}$.

(ii) We define $\pi_0 = \tau_{\geq 0}\tau_{\leq 0} : \mathscr{C} \to \mathscr{C}^{\heartsuit}$ and $\pi_n = \pi_0 \Omega^n : \mathscr{C} \to \mathscr{C}^{\heartsuit}$ for $n \in Z$.

Proposition 1.7 ([Lur17, Remark 1.2.1.12])**.** The heart $\mathscr{C}^{\heartsuit}$ is an abelian (1-)category.

Proposition 1.8 ([Win24, Proposition 10.8])**.** Let $c \to c' \to c''$ be a fiber sequence in $\mathscr{C}$. Then, there is a LES

$$\cdots \longrightarrow \pi_n c \longrightarrow \pi_n c' \longrightarrow \pi_n c'' \longrightarrow \pi_{n-1} c \longrightarrow \cdots$$

in $\mathscr{C}^{\heartsuit}$.

Example 1.9.

(i) There is a t-structure $(\mathbf{Sp}, \mathbf{Sp}_{\geq 0}, \mathbf{Sp}_{\leq 0})$ consisting of (co-)connective spectra.

(ii) Let $R \in \mathbf{CRing}$, then the derived ∞-category $\mathcal{D}(R)$ has a t-structure $(\mathcal{D}(R)_{\geq 0}, \mathcal{D}(R)_{\leq 0})$ controlled by homology.

One could combine the previous two points into the same definition by taking the derived ∞-category of a connective $\mathbb{E}_1$-ring spectrum R [Ant24, Example 2.13(b)].

(iii) Let E be a nice spectrum. Piotr defined the so-called ∞-category of *synthetic spectra* $\mathbf{Syn}_E$ with a natural t-structure whose heart is $\mathbf{Syn}_E^{\heartsuit} \simeq \mathbf{Comod}_{E_\bullet E}$ [Pst23, Proposition 4.16]. We will see more about this category in this autumn school.

There is still a lot of interesting current research on t-structures but that would stray us too far for mere prison inmates.

2 Prison Work: Spectral Sequence from Filtered Objects

2.1 Construction of Spectral Sequence

Classical approaches e.g. via exact couples are arguably more intuitive but we will sketch Lurie's construction [Lur17, Section 1.2.2]. There is an equivalent construction via décalage due to Antieau [Ant24].

Definition 2.1. Let $\mathscr{C} \in \mathbf{Cat}_{\infty}^{\mathrm{st}}$. A **filtered object** of $\mathscr{C}$ is a functor $X : \mathbb{Z} \to \mathscr{C}$. The ∞-category of filtered objects in $\mathscr{C}$ is denoted by $\mathbf{Fil}(\mathscr{C})$.

We visualize this as a sequence of objects and maps

$$\cdots \longrightarrow X_{-1} \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \cdots$$

together with homotopies of composites. Lurie constructs a bunch of auxiliary objects through Kan extensions [Lur17, 1.2.2.2 – 1.2.2.4] but it seems like the main essence lies in the following:

Construction 2.2. Let $\mathscr{C}$ be a stable ∞-category with a t-structure and $X \in \mathbf{Fil}(\mathscr{C})$.

(i) Consider the objects

$$E_r^{p,q} = \mathrm{im}\left(\pi_{p+q}(X_p/X_{p-r}) \to \pi_{p+q}(X_{p+r-1}/X_{p-1})\right)$$

where the map is induced by the filtration structure maps of X.

(ii) Restricting the connecting homomorphism coming from the cofiber sequence

$$X_{p-1}/X_{p-r-1} \longrightarrow X_{p+r-1}/X_{p-r-1} \longrightarrow X_{p+r-1}/X_{p-1}$$

yields a differential $d_r : E_r^{p,q} \to E_r^{p-r,q+r-1}$.

The **spectral sequence** associated to X is $\{E_r^{p,q}, d_r\}_{r,p,q}$.

So the r-th page of the spectral sequence takes into account r steps of a filtration and their filtration quotients. The magic is that these fit together conveniently to extract a lot of interesting information. To me it's still mysterious why Lurie's construction actually works and would be happy about any further intuition.

Example 2.3. The E_1-page is given by $E_1^{p,q} = \pi_{p+q}(X_p/X_{p-1})$ with differential

$$d_1 : E_1^{p,q} = \pi_{p+q}(X_p/X_{p-1}) \to \pi_{p-1+q}(X_{p-1}/X_{p-2}) = E_1^{p-1,q}$$

given by the connecting homomorphism.

Theorem 2.4 ([Lur17, Proposition 1.2.2.7])**.** The spectral sequence $\{E_r^{p,q}, d_r\}_{r,p,q}$ associated to X is a **spectral sequence** in $\mathscr{C}^\heartsuit$, i.e. it consists of:

(i) objects $E_r^{p,q} \in \mathscr{C}^\heartsuit$ for $r \geq 1$ and $p, q \in \mathbb{Z}$,

(ii) differentials $d_r^{p,q} : E_r^{p,q} \to E_r^{p-r,q+r-1}$ for $r \geq 1$ and $p, q \in \mathbb{Z}$,

(iii) isomorphisms $H^\bullet(E_r^{\bullet,\bullet}) \cong E_{r+1}^{\bullet,\bullet}$.

Lurie states (without proof) that this recovers the usual spectral sequence associated to a filtered complex [Lur17, Example 1.2.2.11].

2.2 Convergence

A spectral sequence on its own doesn't yet need to make a statement about the object we are studying. For this we are need to understand the concept of *convergence*.

We will not go in depth and only treat the simplest case as discussed in [Lur17]. For more precise treatments we refer the listener to [Boa99].

Proposition 2.5 ([Lur17, Proposition 1.2.2.14])**.** Suppose that $\mathscr{C}$ admits sequential colimits and that the t-structure on $\mathscr{C}$ is compatible with sequential colimits, i.e. that $\mathscr{C}_{\geq 0}$ is closed under sequential colimits in $\mathscr{C}$. Let $X \in \mathbf{Fil}(\mathscr{C})$ with $X_n \simeq 0$ for $n \ll 0$. Then, the associated spectral sequences **converges**:

$$E_r^{p,q} \Rightarrow \pi_{p+q}\left(\operatorname*{colim}_n X_n\right),$$

where $\pi_{p+q}(\operatorname{colim}_n X_n)$ is called **abutment** of the spectral sequence, i.e.:

(i) For fixed p, q the differentials $d_r : E_r^{p,q} \to E_r^{p-r,q+r-1}$ vanishes for $r \gg 0$.

So for $r \gg 0$ there is a sequence of epimorphisms $E_r^{p,q} \twoheadrightarrow E_{r+1}^{p,q} \twoheadrightarrow E_{r+2}^{p,q} \twoheadrightarrow \cdots$ and we set $E_\infty^{p,q} = \operatorname{colim}_r E_r^{p,q}$ in $\mathscr{C}^\heartsuit$.

(ii) Let $n \in \mathbb{Z}$ and $H_n = \pi_n \operatorname{colim}_r X_r$. Then, there exists a filtration thereon:

$$\cdots \hookrightarrow F^{-1}H_n \hookrightarrow F^0 H_n \hookrightarrow F^1 H_n \hookrightarrow \cdots$$

of H_n with $F^m H_n \cong 0$ for $m \ll 0$ and $\operatorname{colim}_m F^m H_n \cong H_n$.

(iii) For every p, q there are isomorphisms $E_\infty^{p,q} \cong F^p H_{p+q}/F^{-1}H_{p+q}$ in $\mathscr{C}^\heartsuit$.

Typically,[2] the filtrations in (ii) and the isomorphisms in (iii) are part of the structure of convergence.

3 Prison Release: Examples of Spectral Sequences

It's time to release you from the prison.

3.1 Spectral Sequences in Topology

Example 3.1. Let $p : E \to B$ be a Serre fibration of spaces with (homotopy) fiber F and simply connected B. Suppose that B is a CW complex[3] with skeleton $\emptyset = B_{-1} \subseteq B_0 \subseteq \cdots \subseteq B$ which induces a filtration $\{E_k = p^{-1}(B_k)\}_k$ on E.

(i) This induces a filtration on the associated singular cochain complex $C^\bullet(E)$ via $C^\bullet(E/E_k)$ which leads to *Serre's spectral sequence*

$$E_2^{p,q} = H^p(B, H^q(F)) \Rightarrow H^{p+q}(E)$$

from his PhD thesis. It takes an argument to see that this is the E_2-page.

(ii) If A is a spectrum, we can instead consider a filtration on $\operatorname{map}_{\mathbf{Sp}}(E, A)$ which leads to the *Atiyah-Hirzebruch spectral sequence*

$$E^2_{p,q} = H^p(B, A^q(F)) \Rightarrow A^{p+q}(E).$$

Convergence is less clear here.[4] There is also an alternative construction by filtering A instead of E, first shown by Maunders [Ant24, Corollary 9.3]. Namely, one takes the Whitehead filtration of A. An upshot is that the E^1-page is not 'canonical' but it becomes so on the E_2-page. On the other hand, when filtering A, its E^1-page is immediately this E^2 page as described above – related by a décalage.

(iii) Let $N \trianglelefteq G$ be a normal subgroup of a discrete group G. The Serre spectral sequence applied to the fiber sequence $BN \to BG \to B(G/N)$ recovers the *Lyndon-Hochschild-Serre spectral sequence*

$$E_2^{p,q} = H^p(G/N, H^q(N)) \Rightarrow H^{p+q}(G)$$

on group cohomology.

The following example solely made it into this talk for the purpose of a joke.[5] As prisoners we are *behind bars*. We are part of the *bar spectral sequence*.

Example 3.2. Let $X \in \mathbf{Alg}_{\mathbb{E}_1}(\mathcal{S})$. Then, for a field k the *(topological) bar spectral sequence* has the signature

$$E^2_{p,q} \cong \operatorname{Tor}_{p,q}^{H_\bullet(X;k)}(k,k) \Rightarrow H_{p+q}(BX;k)$$

which comes from filtering the bar construction.

End of joke.

[2] This doesn't seem clear in Lurie's formulations.
[3] For the sake of the prison, this is a *cell* complex. Credits to Julius for the joke.
[4] In general, it only converges conditionally.
[5] Sorry.

3.2 Spectral Sequences in Algebra

Many spectral sequences in algebra come from *double complexes*.

Example 3.3. Let $C^{\bullet,\bullet}$ be a double complex, i.e. there are horizontal and vertical differentials d_h, d_v such that the horizontal and vertical sequences are chain complexes and all squares anticommute. We form the *total complex*

$$\operatorname{Tot} C^{\bullet,\bullet} = \left(\prod_{p+q=n} C^{p,q},\ d = d_h + d_v \right)_n$$

which we can filter in two ways: filter the double complex horizontally and vertically. This leads to two spectral sequences

$$E_2^{p,q} = H_v^p(H_h^q(C^{\bullet,\bullet})) \Rightarrow H^{p+q}(\operatorname{Tot} C^{\bullet,\bullet}) \quad \text{and} \quad E_2^{p,q} = H_h^p(H_v^q(C^{\bullet,\bullet})) \Rightarrow H^{p+q}(\operatorname{Tot} C^{\bullet,\bullet}),$$

the *double complex spectral sequences*.

Remark 3.4. We warn that the convergence does not mean that they necessarily lead to the same filtration on the abutment, rather that we get two (possibly different) E_∞-pages which give rise to filtrations on $H^{p+q}(\operatorname{Tot} C^{\bullet,\bullet})$.

Comparing these two spectral sequences give many cool results. For example, one can give brief proofs of the Five Lemma or the Snake Lemma by virtue of these [Vak24, 1.7.B, 1.7.6].

Theorem 3.5 (Grothendieck Spectral Sequence)**.** Let $G : \mathscr{A} \to \mathscr{B}$ and $F : \mathscr{B} \to \mathscr{C}$ be left-exact functors of abelian categories and suppose that G maps injective objects to F-acyclic objects. Let $A \in \mathscr{A}$, then there is a spectral sequence

$$E_2^{p,q} = (R^q F)(R^p G(A)) \Rightarrow R^{p+q}(F \circ G)(A),$$

the so-called *Grothendieck spectral sequence*

Idea Sketch. Let $A \to I^\bullet$ be an injective resolution. One needs to choose a well-behaved injective resolution $(J^{\bullet,\bullet})$ of $F(I^\bullet)$ via the Horseshoe Lemma. So we obtain a double complex $(G(J^{\bullet,\bullet}))$ and need to compare the double complex spectral sequences associated to this.

A comment from Marius was that this can also be immediately obtained from a filtration which is left as an exercise. □

Corollary 3.6.

(i) Let $f : X \to Y$ be a map of spaces and $\mathcal{F} \in \mathbf{Sh}(X)$. The Grothendieck spectral sequence yields the *Leray spectral sequence*

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

In the case $f : E \to B$ is a Serre fibration of spaces with simply-connected B and connected fiber F and $\mathcal{F}$ is the constant sheaf at $A \in \mathbf{Ab}$, then one can check that the Leray spectral sequence takes the form of the Serre spectral sequence.

(ii) Let $f : A \to B$ be a map of commutative rings, $M \in \mathbf{Mod}_A$ and $N \in \mathbf{Mod}_B$. The Grothendieck spectral sequence yields the Ext *base change spectral sequence*

$$E_2^{p,q} = \operatorname{Ext}_B^p(M, \operatorname{Ext}_A^q(B, N)) \Rightarrow \operatorname{Ext}_A^{p+q}(M, N).$$

There is also a Tor base change spectral sequence.

(iii) Let $\mathcal{F}, \mathcal{G} \in \mathbf{Coh}_X$. Then, the Grothendieck spectral sequence yields the *local-to-global spectral sequence*

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(F,G)) \Rightarrow \mathrm{Ext}^{p+q}(F,G).$$

We refer the reader to Belmans' notes [Bel14] for the *Čech-to-derived functor spectral sequence* but it also contains other algebraic examples not coming from the Grothendieck spectral sequence. A more geometric example coming from double complex spectral sequences would be the *Frölicher spectral sequence* interpolating between the Dolbeault cohomology and complex de Rham cohomology. This could be more in flavour of Joana's work.

3.3 The Adams Spectral Sequence

We construct the Adams spectral sequence through the Bousfield-Kan spectral sequence.

Example 3.7. Let $X^\bullet : \Delta \to \mathscr{C}$ be a cosimplicial object. Then, it yields[6] a filtered object

$$\cdots \longrightarrow \mathrm{Tot}_{\leq 2}(X^\bullet) \longrightarrow \mathrm{Tot}_{\leq 1}(X^\bullet) \longrightarrow X^0 = X^0 = \cdots$$

where $\mathrm{Tot}_{\leq n}(X^\bullet) = \lim_{\Delta_{\leq n}} X^\bullet|_{\Delta_{\leq n}}$. Its associated spectral sequence is the *Bousfield-Kan spectral sequence* of $X^\bullet$ [BK72, Chapter X.6, X.7].

Example 3.8. Let $A \in \mathbf{Alg}_{\mathbb{E}_1}(\mathbf{Sp})$ and consider the cobar[7] complex $\mathrm{CB}^\bullet(A) : \Delta \to \mathbf{Sp}$ given by

$$A \rightrightarrows A \otimes A \ \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} \cdots$$

Let $X \in \mathbf{Sp}$. The following spectral sequences will be phrased in the Adams grading convention as opposed to the Serre grading from above. This is merely a convention and results from a certain shearing map. Since we are not performing any calculation, we will refrain from saying more.

(i) The Bousfield-Kan spectral sequence to $\mathrm{CB}^\bullet(A) \otimes X$ is the *A-Adams spectral sequence* which for nice enough A has E_2-page

$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{A_\bullet A}(A_\bullet, A_\bullet X) \Rightarrow \pi_{q-p}(\widehat{X}_A)$$

where $\widehat{X}_A$ is the so-called A-nilpotent completion.

(ii) Setting $A = \mathrm{MU}$ (and X connective) yields

$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{\mathrm{MU}_\bullet \mathrm{MU}}(\mathrm{MU}_\bullet, \mathrm{MU}_\bullet X) \Rightarrow \pi_{q-p}X,$$

the so-called *Adams-Novikov spectral sequence*. People also like to set $A = \mathrm{BP}$.

(iii) Setting $A = H\mathbb{F}_p$ yields

$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{\mathcal{A}_p^*}(\mathbb{F}_p, H_\bullet(X;\mathbb{F}_p)) \Rightarrow \pi_{q-p}(X)^\wedge_p,$$

the *classical (mod p) Adams spectral sequence*.

Completely understanding the Adams spectral sequence is essentially impossible, as it amounts to understanding the stable homotopy of spheres. In the words of Mahowald:

[6] Cosimplicial objects actually correspond to towers by the ∞-categorical Dold-Kan correspondence [Lur17, Theorem 1.2.4.1].

[7] As noticed by Bhavna and Laurent, this was another opportunity for a pun – to be outside of the bars. I completely missed this one.

> **The Mahowald Uncertainty Principle.** Any spectral sequence converging to the homotopy groups of spheres with an E_2-term that can be named using homological algebra will be infinitely far from the actual answer.

In that regard results about the Adams spectral sequence usually have extremely interesting consequences.

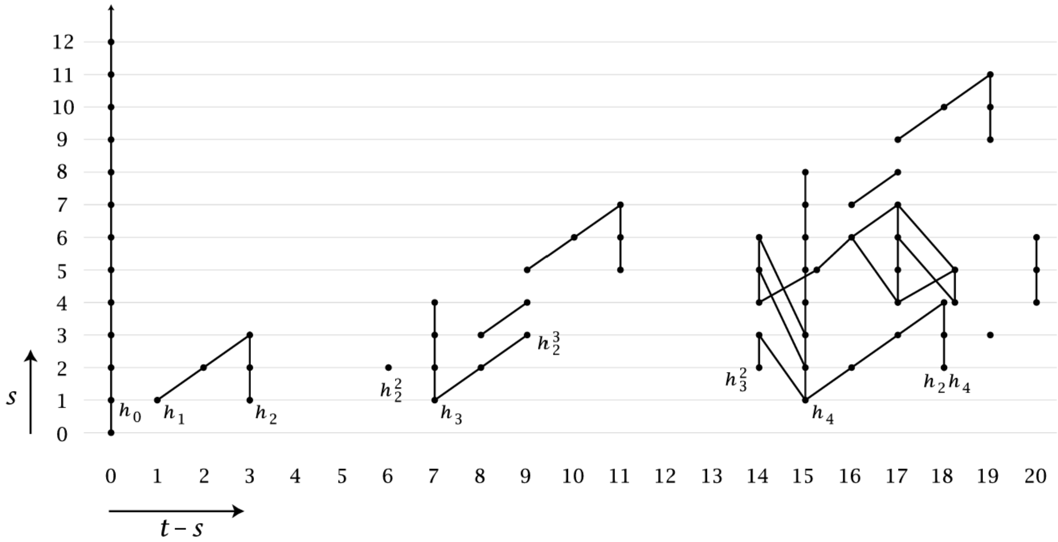

Figure 1: Let $X = \mathbb{S}$. This is from [Hat04, p. 599].

For example, consider the classical Adams spectral sequence for $X = \mathbb{S}$. It's an algebraic exercise to compute the horizontal 1-line which consists of the elements h_i at position $(1, 2^i)$.

- It turns out that the non-trivial elements on the E_∞-page detected by h_i are precisely the Hopf invariant one elements of the famous *Hopf invariant one problem*. In fact, Adams invented his spectral sequence to study this problem [AA66]! If one unravels further, then it turns out that this is equivalent to the (non-)vanishing of a certain d_2-differential [Wan67]. I recently gave a talk on this topic where you can find a bit more information on this [Zhu25].
- Studying h_i^2 corresponds to the famous resolution of the Kervaire invariant one problem from geometry topology via equivariant homotopy theory by Hill-Hopkins-Ravenel [HHR09] and recently Lin-Wang-Xu [LWX25].
- The elements h_i^3 are related to characteristic numbers of framed manifolds with corners as studied by Burklund-Xu [BX25].
- Higher powers seem very much like open problems.

This shows a glimpse of how geometric problems can be translated into the algebra of differentials on a spectral sequence.

This talk of course only scratches the beginnings of spectral sequences but you are now released from prison. Enjoy your freedom![8]

[8] Is freedom another word for synthetic spectra?...

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