Trace Methods & Localization Sequences

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Abstract

We rapidly introduce the trace maps $K \to \mathrm{THH}$ and $K \to \mathrm{TC}$ as the master tool to understand algebraic K-theory. We indicate an application by sketching Antieau–Barthel–Gepner's argument showing that the localization sequence

$$K(\mathrm{BP}\langle n-1\rangle) \longrightarrow K(\mathrm{BP}\langle n\rangle) \longrightarrow K(E(n))$$

is not a fiber sequence for $n \geq 2$. This answers a question of Rognes negatively.

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1 Trace Methods

1.1 Die Hexenküche

Mathematische Arbeitstagungen are regularly organized at the MPIM; these are special conferences where the schedule is decided 'in real time' and not in advance. The last one was about condensed mathematics in 2023![1] These were founded by Friedrich Hirzebruch to bring mathematics in West Germany back into its place after the war. A central theme in early iterations of these workshops was K-theory.

In fact, K-theory was first presented on such an Arbeitstagung. The initial one was in 1957[2] – and Alexander Grothendieck spoke about his *hellish* discoveries. He spoke about the Grothendieck–Riemann–Roch theorem, a formula that he called *Hexenküche*, i.e. witches' kitchen.

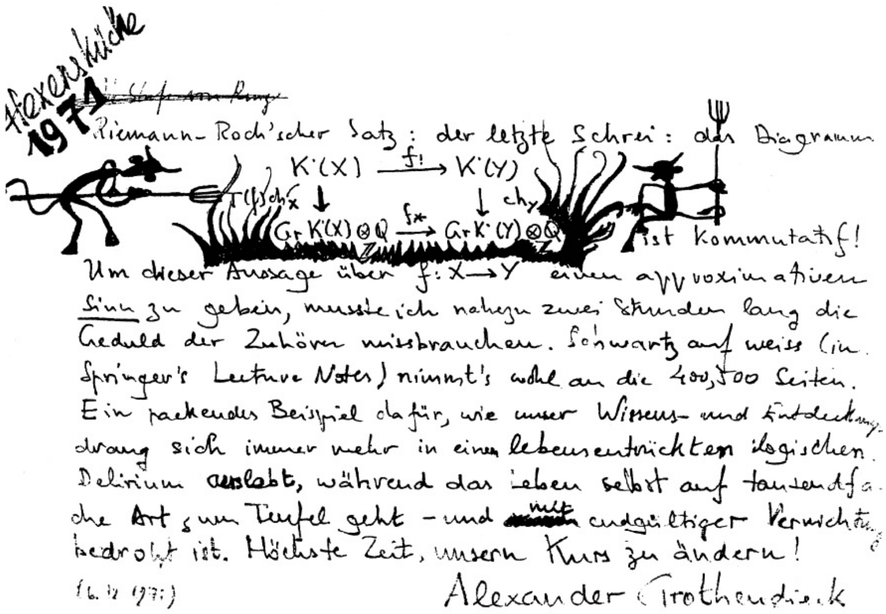

Figure 1: *To give this statement an approximate sense, I had to abuse the patience of my audience for almost two hours [...].*

The work of the witches is:

Theorem 1.1 (Grothendieck–Riemann–Roch)**.** Let $f\colon X \to Y$ be a proper morphism of complex manifolds. It induces a morphism $f_! = \sum_i (-1)^i \mathrm{R}^i f_* : K^0(X) \to K^0(Y)$ fitting into a commutative diagram:

$$\begin{array}{ccc} K^0(X) & \xrightarrow{f_!} & K^0(Y) \\ {\scriptstyle \mathrm{Td}(X)\,\mathrm{ch}(-)}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}(-)\,\mathrm{Td}(Y)} \\ H^*(X;\mathbb{Z}) & \xrightarrow[f_*]{} & H^*(Y;\mathbb{Z}) \end{array}$$

The bottom map comes from functoriality of homology and Poincaré duality.

[1] See https://www.mpim-bonn.mpg.de/node/11707.

[2] The famous prime 57... See also https://www.mpim-bonn.mpg.de/preblob/3347.

Ever since, we are left with this miraculous notion of algebraic K-theory that everyone should want to compute.

While Grothendieck required two hours of patience back in the days, I will make due with 2 minutes. Another testimony to the strength of higher category theory.

Definition 1.2. Let R be a classical ring and denote by $\mathbf{Proj}_R^{\mathrm{fg}}$ the 1-category of finitely generated projective R-modules.

(i) Applying ∞-categorical functors yields $K(R) = \left(\mathbf{Proj}_R^{\mathrm{fg,core}}\right)^{\mathrm{gp}}$, the **algebraic K-theory space** of R.

(ii) Let $n \in \mathbb{N}$. Then, $K_n(R) = \pi_n K(R)$.

By definition this is a grouplike $\mathbb{E}_\infty$-space and thus corresponds to a connective spectrum.

After Grothendieck's announcement, higher K-theory groups eventually emerged with deep connections to other fields. To name just a few: The Lichtenbaum–Quillen conjecture connected K to arithmetic information via étale cohomology. Waldhausen was interested in its rich information about manifolds. Homotopy theorists are curious e.g. due to the patterns observed with regards in chromatic homotopy theory.

An invariant this powerful necessarily is hard to compute. Very little is computed as of today, but one of the main tools is via trace methods: There is a map $\mathrm{tr}\colon K(R) \to \mathrm{THH}(R)$ which is close to an equivalence and we have more computational tools on the THH side.

1.2 Classical Trace Map

Before going into the definition of tr, let us first consider a classical version.

Construction 1.3. The trace from linear algebra can be categorified greatly. Using that language, the usual approach for a finite projective $P \in \mathbf{RMod}_R$ is the map

$$\mathrm{Hom}_R(P,P) \cong P \otimes_R \mathrm{Hom}_R(P,R) \xrightarrow{\mathrm{ev}} R,\ x \otimes \varphi \mapsto \varphi(x).$$

This is fine if R is a commutative ring, but doesn't quite work for non-commutative rings. Indeed, the relative tensor product imposes a condition coming from

$$r\varphi(x) \leftarrow x \otimes r\varphi = xr \otimes \varphi \mapsto \varphi(xr) = \varphi(x)r.$$

This can be salvaged: Replace the target by $R/[R,R]$. We have then defined a map

$$\mathrm{tr}\colon \mathrm{Hom}_R(P,P) \to R/[R,R].$$

The target is a familiar object!

Exercise 1.4. Check $R/[R,R] \cong R \otimes_{R \otimes R^{\mathrm{op}}} R \cong \mathrm{HH}_0(R)$.

Construction 1.5. The map extends:

$$\begin{array}{ccc} \pi_0\left(\mathbf{Proj}_R^{\mathrm{fg,core}}\right) & \xrightarrow{P \mapsto \mathrm{tr}(\mathrm{id}_P)} & \mathrm{HH}_0(R) \\ \downarrow & \overset{\exists ! \mathrm{tr}}{\nearrow} & \\ K_0(R) & & \end{array}$$

which is the **Hattori–Stallings trace**.

Remark 1.6. There is also a derived version $\mathrm{RHom}_R(M,R) \otimes_R^{\mathbb{L}} M \to R \otimes^{\mathbb{L}}_{R \otimes^{\mathbb{L}} R^{\mathrm{op}}} R$.[3]

The main first goal of this talk is a generalization of this map.

[3] See https://mathoverflow.net/questions/458447.

1.3 Homotopical Enhancement

One can define K is a much more general fashion, but group-completion K-theory (Definition 1.2) will no longer remain the correct definition. One typically adheres to Quillen's Q-construction or Waldhausen's S-construction. These constructions are a bit technical and the precise definition of K will not be relevant in this talk, so we will not recall them here. We will give an ad hoc definition of K-theory but to be on the same page, it will have the following form:

Assumption 1.7. There is an K-theory functor $K : \mathbf{Cat}_\infty^{\mathrm{st}} \to \mathbf{Sp}_{\geq 0}$.[4]

Example 1.8. Let $R \in \mathbf{CAlg}$, then $\mathbf{Perf}_R = \mathbf{Mod}_R^\omega \in \mathbf{Cat}_\infty^{\mathrm{st}}$. If R is an ordinary commutative ring, then it is represented by bounded chain complexes of finitely generated R-modules. In the latter case, we recover $K(R) \simeq K(\mathbf{Perf}_R)$.

We start by first defining the world in which K lives.

Definition 1.9.

(i) A small stable ∞-category $\mathscr{C}$ is **idempotent complete** if idempotents in $\mathscr{C}$ split. We write $\mathbf{Cat}_\infty^{\mathrm{perf}} \subseteq \mathbf{Cat}_\infty^{\mathrm{st}}$ for the full subcategory generated by idempotent complete ones.

(ii) A sequence $\mathscr{C} \xrightarrow{i} \mathscr{D} \xrightarrow{p} \mathscr{E}$ in $\mathbf{Cat}_\infty^{\mathrm{perf}}$ is a **Verdier sequence** if it is a bifiber sequence, i.e. if it is a cofiber sequence and a fiber sequence. A Verdier sequence is a **split Verdier sequence** if p has both adjoints.

(iii) A functor $F : \mathbf{Cat}_\infty^{\mathrm{perf}} \to \mathbf{Sp}$ that takes split Verdier sequences to fiber sequences and preserves filtered colimits is called **additive invariant**.

Example 1.10. Let $R \in \mathbf{CAlg}$, then $\mathbf{Perf}_R \in \mathbf{Cat}_\infty^{\mathrm{perf}}$. This is the example leading to the 'perfect' terminology. If F is an additive invariant, then we write $F(R) := F(\mathbf{Perf}_R)$.

Fact 1.11 ([BGT13])**.**

(i) The K-theory functor is an additive invariant.

(ii) One can extend THH to $\mathbf{Cat}_\infty^{\mathrm{perf}}$, see [HNS24, Definition 4.3], and it becomes an additive invariant then. On the other hand, TC is not an additive invariant.

Theorem 1.12 ([BGT13])**.** Let $F\colon \mathbf{Cat}_\infty^{\mathrm{perf}} \to \mathbf{Sp}$ be an additive invariant. Then,

$$\mathrm{map}_{\mathrm{add}}(K, F) \simeq F(\mathbb{S}).$$

Construction 1.13. We compute

$$\mathrm{map}_{\mathrm{add}}(K, \mathrm{THH}) \simeq \mathrm{THH}(\mathbb{S}) \simeq \mathbb{S} \otimes_{\mathbb{S}\otimes\mathbb{S}^{\mathrm{op}}} \mathbb{S} \simeq \mathbb{S}$$

whose π_0 is $\mathbb{Z}$. The **Dennis trace** is the map $\mathrm{tr}\colon K \to \mathrm{THH}$ corresponding to $1 \in \mathbb{Z}$.

Remark 1.14. It turns out that this enhances the Hattori–Stallings trace (Construction 1.5).

Construction 1.15. The same strategy does not immediately apply to TC. However, one can approximate TC by additive invariants

$$\mathrm{TC}^n = \mathrm{fib}\left(\mathrm{incl} - \varphi\colon\ \mathrm{TR}^n \to \mathrm{TR}^{n-1}\right)$$

[4] In fact, this works even more generally.

where $\mathrm{TR}^n(-) = \mathrm{THH}(-)^{C_{p^n}}$ with inclusion map incl and[5]

$$\varphi\colon \mathrm{THH}(\mathscr{C})^{C_{p^n}} \simeq (\mathrm{THH}(\mathscr{C})^{C_p})^{C_{p^{n-1}}} \to (\Phi^{C_p}\,\mathrm{THH}(\mathscr{C}))^{C_{p^{n-1}}} \simeq \mathrm{THH}(\mathscr{C})^{C_{p^{n-1}}}.$$

These satisfy $\mathrm{TC} \simeq \lim_n \mathrm{TC}^n$. We may then compute

$$\mathrm{map}(K, \mathrm{TC}) \simeq \lim_n \mathrm{map}(K, \mathrm{TC}^n) \simeq \lim_n \mathrm{TC}^n(\mathbb{S}) \simeq \mathrm{TC}(\mathbb{S}) \simeq \mathbb{S} \oplus \Sigma\,\mathrm{fib}(\mathbb{S}_{hS^1} \to \Sigma^{-1}\mathbb{S}).$$

The **cyclotomic trace**

$$\mathrm{cyctr}\colon K \to \mathrm{TC}$$

corresponds to $1 \in \pi_0\mathbb{S}$.

It is the ultimate computational tool for K-theory, as the famous theorem for Dundas–Goodwillie–McCarthy says that TC is a good approximation of K.

Theorem 1.16 ([DGM13])**.** Let $f : B \to A$ be a map in $\mathbf{CAlg}_{\geq 0}$ with $\pi_0 f$ being surjective with a nilpotent kernel. Then,

$$\begin{array}{ccc} K(B) & \xrightarrow{\mathrm{cyctr}} & \mathrm{TC}(B) \\ \downarrow & & \downarrow \\ K(A) & \xrightarrow[\mathrm{cyctr}]{} & \mathrm{TC}(A) \end{array}$$

is a pullback square.

So one can start to compute via Mayer–Vietoris arguments. It for example featured in Antieau–Krause–Nikolaus' [AKN24] work on $K(\mathbb{Z}/p^n\mathbb{Z})$ and in the celebrated disproof of the telescope conjecture by Burklund–Hahn–Levy–Schlank [BHLS23].

Nowadays, one should also cite results from Clausen–Mathew–Morrow [CMM21] in this group of theorems.

2 Application: Localization Sequences

2.1 Chromatic Homotopy Theory & K-Theory

A classical technique in classical algebra is to localize at prime numbers, to study the problem a prime at a time and then to glue this information back. Let p be a prime number. In higher algebra, there are more intermediate prime fields[6] larger than $\mathbb{F}_p$, namely one *Morava K-theory* $K(n)$ for each height $n \in \mathbb{N} \cup \{\infty\}$. Here, $K(0) = H\mathbb{Q}$ and $K(\infty) = H\mathbb{F}_p$. The program of chromatic homotopy theory is to study stable homotopy theory one height at a time.

These $K(n)$ have homotopy groups $\pi_\bullet K(n) \cong \mathbb{F}_p[v_n^{\pm 1}]$ with $|v_n| = 2(p^n - 1)$. There are related theories which control all the lower heights. Namely, there is the so-called *truncated Brown–Peterson theory* $\mathrm{BP}\langle n\rangle$ with $\pi_* \mathrm{BP}\langle n\rangle \cong \mathbb{Z}_{(p)}[v_1, v_2, \cdots, v_n]$ and *Johnson–Wilson theory* $E(n)$ with $\pi_* E(n) = \mathbb{Z}_{(p)}[v_1, v_2, \cdots, v_n^{\pm 1}]$. Let $\mathrm{BP}\langle -1\rangle = E(-1) = \mathbb{F}_p$.

The K-theory groups $K(\mathrm{BP}\langle n\rangle)$ are already interesting in their own rights, as it exhibits so-called *redshift* as shown by Hahn–Wilson [HW22]. Roughly it says that K-theory shifts chromatic height. This is a higher version of the now resolved Lichtenbaum–Quillen conjecture which connects K-theory to arithmetic via étale cohomology. All of this stems from the Ausoni–Rognes program to compute $K(\mathbb{S})$. An approximation is to first compute the K-theory of chromatic

[5] The comparison map comes from $\Phi^{C_p}\,\mathrm{THH}(\mathscr{C}) \simeq (\widetilde{E\mathcal{P}} \otimes \mathrm{THH}(\mathscr{C}))^{C_p}$.

[6] A spectrum X is a field if $\pi_* X$ is a field. Any field is a direct sum of shifts of Morava K-theories.

layers.

Here are beautiful and classical observations from Quillen and Blumberg–Mandell using a devissage argument. These more generally follow from Barwick's theorem of the heart.

Theorem 2.1 ([Qui73, BM08]). There are fiber sequences

$$K(\mathbb{F}_p) \longrightarrow K(\mathbb{Z}_p) \longrightarrow K(\mathbb{Q}_p) \qquad \text{and} \qquad K(\mathrm{H}\mathbb{Z}) \longrightarrow K(\mathrm{ku}) \longrightarrow K(\mathrm{KU}).$$

More generally, there are maps

$$\mathrm{BP}\langle n-1\rangle \simeq \mathrm{BP}\langle n\rangle/v_n \longleftarrow \mathrm{BP}\langle n\rangle \longrightarrow \mathrm{BP}\langle n\rangle[v_n^{-1}] \simeq E(n).$$

Apply covariant and contravariant functorialities of $K(\mathbf{Perf}_{-})$.[7] Now, we are set to ask a higher height analog question of Theorem 2.1.

Problem 2.2 (Rognes). Let $n \in \mathbb{N}$. Is the sequence

$$K(\mathrm{BP}\langle n-1\rangle) \longrightarrow K(\mathrm{BP}\langle n\rangle) \longrightarrow K(E(n))$$

is a fiber sequence of connective spectra?

Quillen and Blumberg–Mandell's theorems answer Rognes' question (Problem 2.2) affirmatively for chromatic heights $n = 0, 1$. These are and would be great results since they relate the K-theory of nonconnective ring spectra to the K-theory of connective ring spectra. There are basically no tools to compute the K-theory of nonconnective ring spectra, so this would be useful!

However, Antieau–Barthel–Gepner [ABG18] show that Rognes' hope was too optimistic:

Theorem 2.3 ([ABG18, Theorem 6.5]). Let $n > 1$. Then, the sequence

$$K(\mathrm{BP}\langle n-1\rangle) \longrightarrow K(\mathrm{BP}\langle n\rangle) \longrightarrow K(E(n))$$

is not a fiber sequence.

2.2 Sketch of Proof

It turns out that one can give one description of the fiber by formal means.

Theorem 2.4 ([ABG18, Theorem 6.2]). Let $n > 0$ and $A\langle n-1\rangle = \mathrm{end}_{\mathrm{BP}\langle n\rangle}(\mathrm{BP}\langle n-1\rangle)^{\mathrm{op}}$. Then, there is a fiber sequence

$$K(A\langle n-1\rangle) \longrightarrow K(\mathrm{BP}\langle n\rangle) \longrightarrow K(E(n))$$

of K-theory spectra.

Proof. This follows from a general result on fibers of $K(A) \to K(A[r^{-1}])$ under suitable conditions [ABG18, Theorem 1.11].

Here is the vibe. Let X be a qcqs scheme and $U \subseteq X$ be a qc Zariski open with complement Z. Then, there is a fiber sequence

$$K(X \text{ on } Z) \longrightarrow K(X) \longrightarrow K(U)$$

[7] There is maybe a slight inaccuracy here and one should really look at nonconnective K-theory.

where $K(X \text{ on } Z)$ is the K-theory of perfect complexes on X that are acyclic on U. It is in general not $K(Z)$. However, it is still the K-theory of a ring spectrum. In our setting, it is the cited theorem. □

So we need to compare $K(\mathrm{BP}\langle n-1\rangle)$ to $K(A\langle n-1\rangle)$.

Remark 2.5. In fact, if the sequence in Theorem 2.3 was a fiber sequence, then it turns out that the comparison map

$$K(\mathrm{BP}\langle n-1\rangle) \to K(A\langle n-1\rangle)$$

induced by $\mathrm{BP}\langle n-1\rangle \to A\langle n-1\rangle$ is an equivalence [ABG18, Lemma 6.4].[8] The point of this remark is that it requires an argument to see that this is the comparison map.

Proof of Theorem 2.3. Let $A \in \mathbf{Alg}_{\mathbb{E}_1}(\mathbf{Sp})$, then there is a natural map

$$\mathrm{THH}(A) \to \mathrm{THH}(A) \otimes \mathrm{H}\mathbb{Q} \simeq \mathrm{THH}(A \otimes \mathrm{H}\mathbb{Q}/\mathrm{H}\mathbb{Q})$$

via THH base change formulas. Naturality of the Dennis trace yields a commutative square

$$\begin{array}{ccc} K(\mathrm{BP}\langle n-1\rangle) & \longrightarrow & \mathrm{THH}(\mathrm{BP}\langle n-1\rangle_{\mathbb{Q}}/\mathrm{H}\mathbb{Q}) \\ \downarrow & & \downarrow \\ K(A\langle n-1\rangle) & \longrightarrow & \mathrm{THH}(A\langle n-1\rangle_{\mathbb{Q}}/\mathrm{H}\mathbb{Q}) \end{array}$$

We want to show that the left vertical arrow is not an equivalence. So it suffices to show on homotopy groups that there exists a class in the image of the bottom map which does not lie in the image of the right map. The upshot is that rational computations are doable. In particular, we have the following Hochschild–Kostant–Rosenberg-type isomorphism.

Proposition 2.6 ([ABG18, Corollary 2.3])**.** Let M be a compact $\mathrm{H}\mathbb{Q}$-module with basis for π_{2*} given by homogeneous elements $x_1, \cdots, x_m$ and for π_{2*-1} by homogeneous elements $y_1, \cdots, y_n$. So $\pi_*(\mathrm{Sym}_{\mathrm{H}\mathbb{Q}} M) \cong \mathbb{Q}[x_1, \cdots, x_m] \otimes \Lambda_{\mathbb{Q}}\langle y_1, \cdots, y_n\rangle$. Then,

$$\mathrm{THH}_*(\mathrm{Sym}_{\mathrm{H}\mathbb{Q}} M/\mathrm{H}\mathbb{Q}) \cong \mathbb{Q}[x_1, \cdots, x_m, \delta(y_1), \cdots, \delta(y_n)] \otimes \Lambda_{\mathbb{Q}}\langle y_1, \cdots, y_n, \delta(x_1), \cdots, \delta(x_m)\rangle.$$

Here, $|\delta(y_i)| = |y_i| + 1$ and $|\delta(x_i)| = |x_i| + 1$.

Via cdgA computations, one can obtain $A\langle n-1\rangle_{\mathbb{Q}} \simeq \mathrm{Sym}_{\mathrm{H}\mathbb{Q}}\, \mathbb{Q}\{v_1, \cdots, v_{n-1}, \varepsilon_{1-2p^n}\}$ [ABG18, Lemma 5.4, Proposition 5.5]. So we may apply Proposition 2.6 to get

$$\mathrm{THH}_*(A\langle n-1\rangle_{\mathbb{Q}}/\mathrm{H}\mathbb{Q}) \cong \mathbb{Q}[v_1, \cdots, v_{n-1}, \delta_{2-2p^n}] \otimes \Lambda_{\mathbb{Q}}\langle \varepsilon_{1-2p^n}, \sigma_1, \cdots, \sigma_{n-1}\rangle$$

with $|\sigma_i| = 2p^i - 1$ and $|\varepsilon_{1-2p^n}| = 1 - 2p^n$.

Now consider the diagram

$$\begin{array}{ccccc} \Sigma^\infty_+ \mathrm{BGL}_1(\mathrm{BP}\langle n-1\rangle) & \longrightarrow & K(\mathrm{BP}\langle n-1\rangle) & \longrightarrow & \mathrm{THH}(\mathrm{BP}\langle n-1\rangle_{\mathbb{Q}}/\mathrm{H}\mathbb{Q}) \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma^\infty_+ \mathrm{BGL}_1(A\langle n-1\rangle) & \longrightarrow & K(A\langle n-1\rangle) & \longrightarrow & \mathrm{THH}(A\langle n-1\rangle_{\mathbb{Q}}/\mathrm{H}\mathbb{Q}) \end{array}$$

[8] The map $\mathrm{BP}\langle n\rangle \to \mathrm{BP}\langle n-1\rangle$ induces a map $\mathbf{RMod}_{\mathrm{BP}\langle n-1\rangle} \to \mathbf{RMod}_{\mathrm{BP}\langle n\rangle}$ and thus a map

$$\mathrm{BP}\langle n-1\rangle \simeq \mathrm{end}_{\mathrm{BP}\langle n-1\rangle}(\mathrm{BP}\langle n-1\rangle)^{\mathrm{op}} \to \mathrm{end}_{\mathrm{BP}\langle n\rangle}(\mathrm{BP}\langle n-1\rangle)^{\mathrm{op}}.$$

I'm a bit confused by the $(-)^{\mathrm{op}}$ on the left side.

where the right vertical map is the inclusion

$$\mathbb{Q}[v_1, \cdots, v_{n-1}] \otimes \Lambda_{\mathbb{Q}}\langle \sigma_1, \cdots, \sigma_{n-1}\rangle \to \mathbb{Q}[v_1, \cdots, v_{n-1}, \delta_{2-2p^n}] \otimes \Lambda_{\mathbb{Q}}\langle \sigma_1, \cdots, \sigma_{n-1}, \varepsilon_{1-2p^n}\rangle$$

by Proposition 2.6. Antieau–Barthel–Gepner prove $\pi_* A\langle n-1\rangle \cong \Lambda_{\pi_* \mathrm{BP}\langle n-1\rangle}\langle \varepsilon_{1-2p^n}\rangle$ in [ABG18, Lemma 5.4]. So we may consider the class

$$x = v_1^{a_1} \cdots v_{n-1}^{a_{n-1}} \varepsilon_{1-2p^n} \in \pi_d A\langle n-1\rangle$$

with exponents chosen such that $d > 0$. Let $y \in \pi_{d+1} \mathrm{BGL}_1(A\langle n-1\rangle) \cong \pi_d A\langle n-1\rangle$ be the associated class. Along the bottom composite it can be shown [ABG18, Corollary 3.2] that y maps to the non-zero element

$$v_1^{a_1} \cdots v_{n-1}^{a_{n-1}} \delta_{2-2p^n} + \sum_{i=1}^{n-1} a_i v_1^{a_1} \cdots v_{n-1}^{a_{n-1}} \sigma_i \varepsilon_{1-2p^n} \in \mathrm{THH}_{d+1}(A\langle n-1\rangle_{\mathbb{Q}}/\mathrm{H}\mathbb{Q}).$$

This term involves ε_{1-2p^n} and δ_{2-2p^n}, so it cannot come from $\mathrm{THH}_*(\mathrm{BP}\langle n-1\rangle_{\mathbb{Q}}/\mathrm{H}\mathbb{Q})$. □

When $n = 0, 1$, then v_1 does not exist in $A\langle n-1\rangle$, so the above argument does not go through. That's why this proof does not contradict Theorem 2.1.

The famous telescope conjecture in chromatic homotopy theory was only recently resolved [BHLS23]. The next big problems in this area should concern the study of various K-theory spectra of chromatic objects.[9] We are now at least equipped with the tool of trace methods to initiate this approach.

References

[ABG18] Benjamin Antieau, Tobias Barthel, and David Gepner. On localization sequences in the algebraic K-theory of ring spectra. *J. Eur. Math. Soc. (JEMS)*, 20(2):459–487, 2018. (Cited on pages 5, 6, and 7.)

[AKN24] Benjamin Antieau, Achim Krause, and Thomas Nikolaus. On the k-theory of $\mathbf{Z}/p^n$, 2024. (Cited on page 4.)

[BGT13] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada. A universal characterization of higher algebraic K-theory. *Geom. Topol.*, 17(2):733–838, 2013. (Cited on page 3.)

[BHLS23] Robert Burklund, Jeremy Hahn, Ishan Levy, and Tomer M. Schlank. K-theoretic counterexamples to Ravenel's telescope conjecture, 2023. arXiv:2310.17459. (Cited on pages 4 and 7.)

[BM08] Andrew J. Blumberg and Michael A. Mandell. The localization sequence for the algebraic K-theory of topological K-theory. *Acta Math.*, 200(2):155–179, 2008. (Cited on page 5.)

[CMM21] Dustin Clausen, Akhil Mathew, and Matthew Morrow. K-theory and topological cyclic homology of henselian pairs. *J. Amer. Math. Soc.*, 34(2):411–473, 2021. (Cited on page 4.)

[DGM13] Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy. *The local structure of algebraic K-theory*, volume 18 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2013. (Cited on page 4.)

[9] See `https://www.youtube.com/watch?v=bYiAAHYYI1U`.

[HNS24] Yonatan Harpaz, Thomas Nikolaus, and Victor Saunier. Trace methods for stable categories i: The linear approximation of algebraic k-theory, 2024. (Cited on page 3.)

[HW22] Jeremy Hahn and Dylan Wilson. Redshift and multiplication for truncated Brown-Peterson spectra. *Ann. of Math. (2)*, 196(3):1277–1351, 2022. (Cited on page 4.)

[Qui73] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341, 1973. (Cited on page 5.)