Applications to Representation Theory

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Abstract

Condensed anima **Cond**(**An**) is nothing but a fancy word for stacks on (nice) topological spaces. As such, (locally) profinite groups G can be found in there and the homotopical direction allows us to define classifying spaces $* /\!\!/ G$ by tacking stacky quotients. This immediately attracts the desire to study higher representation theory from this viewpoint. Indeed, there is a six functor formalism on **Cond**(**An**) which restricts to a six functor formalism on higher representation theory through those classifying spaces.

Now, following Heyer–Mann [HM24], we can specialize all the words from abstract six functor formalism theory to this example and study their incarnations and consequences. In that regard, we will describe the six functors and study suave and prim objects in this example. This gives easier formal and more conceptual proofs of some classical results. We end by discussing an anti-involution on derived Hecke algebras through prim duality.

This is talk 5 given at the Six Functor Formalisms Seminar in WiSe 2025/26.

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1 Smooth Representation Theory through Condensed Anima

1.1 Condensed Anima & Classifying Stacks

Recall from last talk that $\mathbf{Cond(An)} = \mathbf{Sh}^{\mathrm{hyp}}(\mathbf{ProFin})$ and the six functor formalism of condensed anima.

Recollection 1.1. Let $\Lambda \in \mathbf{CAlg}$, then the universal property of Ind yields a diagram

$$\begin{tikzcd}
* \arrow[r, "\Lambda"] \arrow[d, hook, "\langle 1 \rangle"'] & \mathbf{CAlg} \\
\mathbb{F}^{\mathrm{op}} \arrow[ur, dashed] \arrow[d, hook] & \\
\mathbf{ProFin}^{\mathrm{op}} \arrow[uur, dotted, bend right, "(S_i)_i \mapsto \mathrm{colim}_i \prod_{x \in S_i} \Lambda"'] &
\end{tikzcd}$$

and postcomposing with $\mathbf{Mod}_{(-)}$ gives $D(-,\Lambda) : \mathbf{ProFin}^{\mathrm{op}} \to \mathbf{Cat}_\infty$. This can be extended to $\mathbf{Cond}(\mathbf{An})^{\mathrm{op}}$ and then to a six functor formalism $D(-,\Lambda) : \mathbf{Span}(\mathbf{Cond}(\mathbf{An}), \Lambda\text{-fine}) \to \mathbf{Cat}_\infty$ [HM24, Construction 3.5.16].

The ∞-topos $\mathbf{Cond}(\mathbf{An})$ contains $\mathbf{Top}$ but also a homotopical direction – in particular, it allows us to form classifying stacks of topological groups. We will use this observation to study smooth representation theory of locally profinite groups.

Definition 1.2. A **locally profinite group** is a Hausdorff, locally compact, totally disconnected topological group.

So compact locally profinite groups are precisely the profinite groups.

Example 1.3. This includes profinite groups like Galois groups of (infinite) field extensions $\mathrm{Gal}(L/K)$ or the Morava stabilizer group $\mathbb{G}$, but also discrete groups, $\mathbb{Q}_p$ and p-adic Lie groups such as $\mathrm{GL}_n(\mathbb{Q}_p)$.

Let G be a locally profinite group, then it is in particular a group object in $\mathbf{Cond}(\mathbf{An})$. If it acts on some $X \in \mathbf{Cond}(\mathbf{An})$, then we can form the stacky quotient

$$X /\!\!/ G = \operatorname*{colim}_{[n] \in \Delta^{\mathrm{op}}} G^{\times n} \times X \in \mathbf{Cond}(\mathbf{An}).$$

We will in particular care about the classifying stacks $* /\!\!/ G$. Indeed, it gives information about representation theory as follows!

1.2 Representation Theory

Let's define smooth representation theory!

Definition 1.4. Let G be a locally profinite group, $\Lambda \in \mathbf{CRing}$ and V be a continuous G-representation. It is **smooth** if $\mathrm{Stab}_G(v) \subseteq G$ is open for every $v \in V$. We write $\mathbf{Rep}_\Lambda(G)^\heartsuit$ for the 1-category of smooth G-representations and

$$\mathbf{Rep}_\Lambda(G) = \mathcal{D}\left(\mathbf{Rep}_\Lambda(G)^\heartsuit\right) \quad \text{and} \quad \widehat{\mathbf{Rep}}_\Lambda(G) = \widehat{\mathbf{Rep}_\Lambda(G)}$$

for its unbounded derived category and the left t-completion thereof.

Theorem 1.5 ([HM24, Proposition 5.1.12]). Let $\Lambda \in \mathbf{CRing}$ and G be a locally profinite group. There, there is a natural t-exact equivalence $D(* /\!\!/ G, \Lambda) \simeq \widehat{\mathbf{Rep}}_\Lambda(G)$.

Proof Idea. The proof strategy is by *derived descent from abelian descent*.

1. One first develops some general abstract nonsense to discuss the question for which $X \in \mathbf{Cond}(\mathbf{An})$ the ∞-category $D(X,\Lambda)$ is the (left t-completion[1] of the) derived category

[1]This part is automatic [HM24, Lemma 3.5.14].

of its heart. This turns out to be true for $* /\!\!/ G$, so $D(* /\!\!/ G, \Lambda) \simeq \widehat{\mathcal{D}}\left(D(* /\!\!/ G, \Lambda)^{\heartsuit}\right)$ [HM24, Example 5.1.2, Proposition 5.1.8].

Thus, it suffices to prove $D(* /\!\!/ G, \Lambda)^{\heartsuit} \simeq D(* /\!\!/ G, \Lambda)^{\heartsuit}$. In other words, it suffices to study the relevant abelian descent data to obtain derived descent.

2. To perform abelian descent one notices $D(G^n, \Lambda) \simeq \mathcal{D}\left(\mathbf{Mod}^{\heartsuit}_{\Lambda_c(G^n)}\right)$ where we denote by $\Lambda_c(G^n) \subseteq \Lambda_c(G^n)$ locally constant functions $G^n \to \Lambda$ with compact support [HM24, Lemma 5.1.9].[2] Writing out the descent diagram for $* /\!\!/ G$ and noting that we are working in 1-categories, we obtain that $D(* /\!\!/ G, \Lambda)^{\heartsuit}$ is the limit of

$$\mathbf{Mod}^{\heartsuit}_{\Lambda} \rightrightarrows \mathbf{Mod}^{\heartsuit}_{\Lambda_c(G)} \overset{\pi_2^*}{\underset{\pi_1^*}{\overset{m^*}{\rightrightarrows}}} \mathbf{Mod}^{\heartsuit}_{\Lambda_c(G\times G)},$$

i.e. abelian descent.

At this point, writing out an equivalence $\mathbf{Rep}_{\Lambda}(G)^{\heartsuit} \to D(* /\!\!/ G, \Lambda)^{\heartsuit}$ is a 1-categorical problem which can be handled by hand [HM24, Proposition 5.1.12].

□

Remark 1.6 ([HM24, Corollary 5.1.14, Remark 5.1.15])**.** Let $\varphi : H \to G$ be a map of locally profinite groups. This induces an adjunction

$$D(* /\!\!/ G, \Lambda) \underset{f_*}{\overset{f^*}{\rightleftarrows}} D(* /\!\!/ H, \Lambda)$$

which can be described in terms of smooth representations.

(i) The pullback f^* is the derived functor of taking a G-representation to its underlying H-representation. It is called **restriction/inflation** depending on whether f is injective or surjective.

(ii) If φ is the inclusion of a closed subgroup, then f_* is the right derived functor of smooth induction $R\,\mathrm{Ind}_H^G$. It φ is a topological quotient map with kernel U, then f_* is the right derived functor of taking U-fixed points $R(-)^U$, also denoted $(-)^U$.

(iii) The symmetric monoidal structure corresponds to the underlying tensor product of Λ-modules with diagonal G-action.

2 Six Functors in Representation Theory

We have already described some of the six operations. Now, we shall also describe the !-functor and discuss some of the six functor formulaic features.

2.1 !

Let $\Lambda \in \mathbf{CRing}$ and G be a locally profinite group, then natural maps such as $* /\!\!/ G \to *$ need not be Λ-fine, but we want shriekability to study six functor phenomena like being suave/prim. We fix this by posing mild conditions.

[2] This result is stated for disjoint unions of profinite sets. To apply it to the locally profinite G we note that by van Dantzig's theorem there exists a compact open subgroup $K \leq G$, so we obtain a disjoint union decomposition $G = \bigsqcup_{[g]\in G/K} gK$.

Definition 2.1. Let $\Lambda \in \mathbf{CRing}$.

(i) Let G be a profinite group. We call

$$\operatorname{cd}_\Lambda G = \sup\left\{n : H^n(G, V) \neq 0 \text{ for some } V \in \mathbf{Rep}_\Lambda(G)^\heartsuit\right\} \in \mathbb{N} \cup \{\infty\}$$

the **Λ-cohomological dimension** of G.

(ii) We say that a locally profinite group G has **locally finite Λ-cohomological dimension** if there exists an open profinite subgroup $K \leq G$ such that $\operatorname{cd}_\Lambda K < \infty$.

Many p-adic Lie groups satisfy this condition [HM24, Example 5.2.2].

Lemma 2.2. Let $\Lambda \in \mathbf{CRing}$.

(i) Let G be a locally profinite group and $H \leq K \leq G$ be compact subgroups with open K and (closed) H. The map $* /\!\!/ K \to * /\!\!/ G$ is Λ-étale and $* /\!\!/ H \to * /\!\!/ K$ is Λ-proper.

(ii) Let G be a profinite group with $\operatorname{cd}_\Lambda G < \infty$. Then, $* /\!\!/ G \to *$ is Λ-proper.

(iii) Let $H \to G$ be a map of locally profinite groups with locally finite Λ-cohomological dimension. Then, $* /\!\!/ H \to * /\!\!/ G$ is Λ-fine.

Proof.

(i) First note that $* \to * /\!\!/ G$ is a $*$-cover since $* \to * /\!\!/ G$ is an effective epimorphism[3] and D is sheafy. Thus, we need to check that the pullback[4] $G/K \to *$ is Λ-étale [HM24, Lemma 4.6.3(ii)].[5] This can be checked on open covers [HM24, Corollary 4.8.4(i)] but G/K is discrete, so it reduces to $* \to *$ being Λ-étale.

Similarly, for Λ-properness, we need to check that $K/H \to *$ is Λ-proper. This is true because K/H is a profinite set [HM24, Lemma 4.8.2(ii)].

(ii) We apply backwards 2-out-of-3 [HM24, Corollary 4.7.5] on

$$* \xrightarrow{g} * /\!\!/ G \xrightarrow{f} *$$

so we need to show that g is Λ-prim, $fg = \mathrm{id}_*$, that f is truncated, Λ-proper and that $g_*\mathbb{1} \in D(* /\!\!/ G, \Lambda)$ is descendable. The first part follows from (i), the second part is clear. Truncatedness follows from $\Omega B \simeq \mathrm{id}$ [Lur09, Lemma 7.2.2.1]. and that $g_*\mathbb{1}$ is descendable requires the finite cohomological dimension [HM24, Proposition 5.2.5].

(iii) Since the shriekable maps are right cancellative (by definition of geometric setups), it suffices to check that $* /\!\!/ G \to *$ (and $* /\!\!/ H \to *$) is Λ-fine. This can be checked after restriction to $* /\!\!/ K$ for some compact open subgroup $K \leq G$ with $\operatorname{cd}_\Lambda K < \infty$.

Indeed, such $K \leq G$ exists by locally finite Λ-cohomological dimension and (i) shows that $* /\!\!/ K \to * /\!\!/ G$ is Λ-suave. It is furthermore $*$-conservative since this is just the restriction of a representation. Thus, the map is a universal !-cover and Λ-fine maps can be tested !-locally on the source. This then follows from (ii).

□

[3] This means that it is equivalent to its Čech nerve, which can be checked by hand.

[4] To compute the pullback we use the delooping $\Omega(* /\!\!/ G) \simeq G$, some pullback pastings and the LES associated to fiber sequences [NSS15, Definition 2.26].

[5] This pullback is truncated, so in particular, the map $* /\!\!/ K \to * /\!\!/ G$ is truncated.

In particular, those maps $* /\!/ H \to * /\!/ G$ are shriekable, so we should describe the shrieks.

Construction 2.3. Let $\Lambda \in \mathbf{CRing}$ and let $H \leq G$ be a closed subgroup of a locally profinite group.

(i) For $V \in \mathbf{Rep}_\Lambda(H)^\heartsuit$ we set $\text{c-Ind}_H^G(V)$ as the set of elements $f : G \to V$ such that

   (a) f is locally constant,

   (b) $f(hg) = hf(g)$ for all $h \in H, g \in G$,

   (c) the image of $\operatorname{supp} f$ in $H \setminus G$ is compact.

   It becomes a smooth G-representation via the right translation action on the domain.

(ii) The functor c-Ind_H^G is exact, so we denote its derived functor by

$$\text{c-Ind}_H^G : \widehat{\mathbf{Rep}}_\Lambda(H) \to \widehat{\mathbf{Rep}}_\Lambda(G).$$

This is the **compact induction functor**.

Proposition 2.4 ([HM24, Lemma 5.4.2, Proposition 5.4.4])**.** Let $\Lambda \in \mathbf{CRing}$ and $H \leq G$ be a closed subgroup in a locally profinite group with locally finite Λ-cohomological dimension.

(i) Then, $f_! : D(* /\!/ H, \Lambda) \to D(* /\!/ G, \Lambda)$ is t-exact.

(ii) The diagram

$$\begin{tikzcd} \widehat{\mathbf{Rep}}_\Lambda(H) \arrow[r, "\text{c-Ind}_H^G"] \arrow[d, "\simeq"'] & \widehat{\mathbf{Rep}}_\Lambda(G) \arrow[d, "\simeq"] \\ D(* /\!/ H, \Lambda) \arrow[r, "f_!"'] & D(* /\!/ G, \Lambda) \end{tikzcd}$$

commutes.

Remark 2.5. In fact, $\widehat{\mathbf{Rep}} \simeq \mathbf{Rep}$ in this setting [HM24, Proposition 5.3.10].

2.2 Suave & Prim in Representation Theory

Let us describe suave and prim objects and hence recover notions of duality.

Definition 2.6. Let $\Lambda \in \mathbf{CRing}$ and let G be a locally profinite group with $f : * /\!/ G \to *$.

(i) Let $V \in D(* /\!/ G, \Lambda)$. We write $V^G = \Gamma(* /\!/ G, V) = f_* V$ for the **derived invariants** of V.

(ii) Suppose that G has locally finite Λ-cohomological dimension. An object $V \in D(* /\!/ G, \Lambda)$ is called **admissible** if $V^K \in \mathbf{Mod}_\Lambda$ is dualizable for all compact open $K \leq G$ with $\operatorname{cd}_\Lambda K < \infty$.

(iii) Suppose that G is a profinite group with $d = \operatorname{cd}_\Lambda G < \infty$. We say that it is **$\Lambda$-Poincaré** (of dimension d) if $f_* : D(* /\!/ G, \Lambda) \to \mathbf{Mod}_\Lambda$ preserves dualizable objects.

(iv) A locally profinite group is **locally Λ-Poincaré** (of dimension d) if it admits an open profinite subgroup which is Λ-Poincaré (of dimension d).

Lemma 2.7 ([HM24, Lemma 5.3.11])**.** Let $\Lambda \in \mathbf{CRing}$ and G be a locally profinite group with $i_K : K \hookrightarrow G$ a compact open subgroup with $\operatorname{cd}_\Lambda K < \infty$. Let $V \in D(* /\!/ G, \Lambda)$. The following are equivalent:

(i) V is dualizable,

(ii) $i_K^* V$ is dualizable in $D(* /\!\!/ K, \Lambda)$,

(iii) the underlying Λ-module of V is dualizable.

Proof.

(i) $\implies$ (iii): The implication (i) $\implies$ (iii) is because $D(* /\!\!/ G, \Lambda) \to \mathbf{Mod}_\Lambda$ is symmetric monoidal.

(ii) $\implies$ (i): Let $V^\vee = \underline{\mathrm{Map}}_{D(* /\!\!/ G,\Lambda)}(V, \mathbb{1})$. It suffices to check that $V \otimes V^\vee \to \underline{\mathrm{Map}}_G(V, V)$ is a G-equivariant equivalence. To do so, consider the following commutative diagram:

$$\begin{tikzcd} i_K^*(V \otimes V^\vee) \arrow[rr] \arrow[d, "\simeq"'] & & i_K^*\underline{\mathrm{Map}}_G(V,V) \arrow[d, "\simeq"] \\ i_K^* V \otimes i_K^*(V^\vee) \arrow[r, "\simeq"'] & i_K^* V \otimes (i_K^* V)^\vee \arrow[r, "\simeq"'] & \underline{\mathrm{Map}}_K(i_K^* V, i_K^* V) \end{tikzcd}$$

The left map is an equivalence since i_K^* is symmetric monoidal. The lower right map is an equivalence by assumption (ii). For the remaining equivalences, we consider the projection formula

$$i_{K!} \circ (i_K^* V \otimes -) \overset{\simeq}{\Rightarrow} V \otimes i_{K!}(-)$$

whose two-fold right adjoints form an equivalence

$$i_K^* \underline{\mathrm{Map}}_G(V, -) \overset{\simeq}{\Rightarrow} \underline{\mathrm{Map}}_K(i_K^* V, i_K^* -).$$

This explains the bottom left and the right equivalence. In particular, the top arrow must be an equivalence. We conclude with conservativity of i_K^*.[6]

(iii) $\implies$ (ii): Since $f_K : * /\!\!/ K \to *$ is Λ-proper (2.2(ii)) we conclude that the f_K-prim and dualizable objects in $D(* /\!\!/ K, \Lambda)$ agree [HM24, Lemma 4.6.3(iii)]. So (iii) means that $q^* V$ is prim where $q : * \to * /\!\!/ K$ and we need to show that V is f_K-prim. But q is Λ-prim (2.2(i)) and $q_* \mathbb{1}$ is descendable [HM24, Proposition 5.2.5].[7] So V is prim [HM24, Corollary 4.7.5].

□

In special settings there are more checkable conditions for admissibility [HM24, Remark 5.3.13]. Another finiteness condition is compactness which will thus naturally show up in our arguments below. Let us briefly state it here.

Lemma 2.8 ([HM24, Corollary 5.3.4])**.** Consider $\Lambda \in \mathbf{CRing}$ and a profinite group G with $\mathrm{cd}_\Lambda G < \infty$. Then, $\mathbb{1} \in D(* /\!\!/ G, \Lambda)$ is compact.

Proof. By 2.2(ii) the map $f : * /\!\!/ G \to *$ is Λ-proper, so we can compute

$$\mathrm{RHom}_{D(* /\!\!/ G,\Lambda)}(\mathbb{1}, -) \simeq f_* \underline{\mathrm{Map}}_G(\mathbb{1}, -) \simeq f_* \simeq f_!$$

which commutes with colimits as a left adjoint. Here, the first equivalence follows by passing to left adjoints. Now we can pass to the underlying spectrum and then apply Ω^∞ to obtain the underlying space and both of these passages commute with filtered colimits. □

[6] On models, we are just forgetting an action but the map being an equivalence can be tested underlying.
[7] This uses $\mathrm{cd}_\Lambda K < \infty$.

Proposition 2.9 ([HM24, Proposition 5.3.14, 5.3.19])**.** Let $\Lambda \in \mathbf{CRing}$ and G be a profinite group with locally finite Λ-cohomological dimension. Let $V \in D(* /\!\!/ G, \Lambda)$ and $f : * /\!\!/ G \to *$.

(i) The object V is f-prim if and only if it is compact.

(a) In this case, $\mathrm{PD}_f(V) \simeq \underline{\mathrm{Map}}_G(V, \Lambda_c(G))$.

(b) If $K \leq G$ is a compact open subgroup with $\mathrm{cd}_\Lambda K < \infty$ and $V \in D(* /\!\!/ K, \Lambda)$ is dualizable, then $\mathrm{PD}_f(\text{c-Ind}_K^G V) \simeq \text{c-Ind}_K^G V^\vee$.

(ii) The object V is f-suave if and only if it is admissible. In this case, $\mathrm{SD}_f(V) \simeq \underline{\mathrm{Map}}_G(V, f^! \mathbb{1})$.

(iii) The map $* /\!\!/ G \to *$ is Λ-suave if and only if G is locally Λ-Poincaré.

Proof. Let's start by recalling a classical result from smooth representation theory that we will use.

Lemma [HM24, Lemma 5.3.7]**.** For $V \in D(* /\!\!/ G, \Lambda)$ we have $\operatorname{colim}_{\substack{K \leq G \text{ open} \\ \mathrm{cd}_\Lambda K < \infty}} V^K \simeq V$.

The fun thing is that you can also recover this result via a 6FF argument [HM24, Lemma 5.3.7].

(i) Note that $\Lambda \in \mathbf{Mod}_\Lambda$ is compact. This implies that every f-prim object is compact in $D(* /\!\!/ G, \Lambda)$ [HM24, Lemma 4.4.18(ii)]. So onto the converse.

Claim. Let

$$\mathcal{G} = \{ i_{K!}\mathbb{1} : i_K : * /\!\!/ K \to * /\!\!/ G,\ K \leq G \text{ compact open with } \mathrm{cd}_\Lambda K < \infty \}.$$

Then, $\mathcal{G}$ consists of compact and f-prim objects and generates $D(* /\!\!/ G, \Lambda)$.

Proof. We have seen that $f_K : * /\!\!/ K \to *$ is Λ-prim (2.2(ii)), i.e. $\mathbb{1} \in D(* /\!\!/ K, \Lambda)$ is f_K-prim. Moreover, i_K is Λ-suave (2.2(i)), so $i_{K!}\mathbb{1}$ is f-prim [HM24, Lemma 4.4.9(ii)].

Furthermore, $\mathbb{1}$ is compact by 2.8. Since $i_{K!} \dashv i_K^! \simeq i_K^* \dashv i_{K*}$ by Λ-étaleness of i_K (see 2.2(i)), it admits a right adjoint who commutes with (filtered) colimits and hence preserves compact objects. So $i_{K!}\mathbb{1}$ is compact.

To see that $\mathcal{G}$ is generating we observe

$$P^K = f_{K*} i_K^* P \simeq f_* i_{K*} \underline{\mathrm{Map}}_K(\mathbb{1}, i_K^* P) \simeq f_* \underline{\mathrm{Map}}_K(i_{K!}\mathbb{1}, P) \simeq \mathrm{RHom}_{D(* /\!\!/ G, \Lambda)}(i_{K!}\mathbb{1}, P)$$

where the third equality is general 6FF nonsense [HM24, Proposition 3.2.2]. By the result discussed in the beginning of the proof, we conclude. ☐

Denote by $\langle \mathcal{G} \rangle \subseteq D(* /\!\!/ G, \Lambda)$ the full subcategory generated by $\mathcal{G}$ under (co-)fibers and retracts. Since primness is closed under these operations [HM24, Corollary 4.4.13] we get $\langle \mathcal{G} \rangle \subseteq \mathrm{Prim}(* /\!\!/ G)$. On the other hand, $\mathrm{Ind}(\langle \mathcal{G} \rangle) \simeq D(* /\!\!/ G, \Lambda)$ since $\mathcal{G}$ consists of compact generators [Lur09, Proposition 5.3.5.11]. Passing to compact objects yields $\langle \mathcal{G} \rangle \simeq D(* /\!\!/ G, \Lambda)^\omega$.

(a) This follows from the general prim dual formula [HM24, Lemma 4.4.6] while using the c-Ind to understand $\Delta_!$ from that formula.

(b) The map $f_K : * /\!\!/ K \to *$ is Λ-proper (2.2(ii)). So, the dualizables agree with the f_K-prims in $D(* /\!\!/ K, \Lambda)$ [HM24, Lemma 4.6.3(iii)] which in particular means that f_K-prim duality is the usual duality. Moreover, $h_! = \text{c-Ind}_K^G$ commutes with prim duality [HM24, Lemma 4.4.9]. So

$$\mathrm{PD}_f(\text{c-Ind}_K^G V) \simeq \text{c-Ind}_K^G \left(\mathrm{PD}_{f_K}(V) \right) \simeq \text{c-Ind}_K^G V^\vee$$

as desired.

(ii) We use

> **Lemma** [HM24, Corollary 4.4.15]**.** Let D be a 6FF on some geometric setup $(\mathscr{C}, \mathscr{E})$ and $f : X \to S$ be a map in $\mathscr{E}$. Let $(Q_i)_{i \in I}$ be a family of objects in $D(X)$. Assume that the Q_i are f-prim and $D(X \times_S X)$ is generated by $\pi_1^* Q_i \otimes \pi_2^* Q_j$. Then, $P \in D(X)$ is f-suave if and only if $f_* \underline{\mathrm{Map}}(Q_i, P)$ is dualizable for all Q_i.

We take the family $(Q_i)_{i \in I} = \mathcal{G}$ from (i). We have seen there that its consists of Λ-prim objects and moreover,

$$\pi_1^* i_{K!} \mathbb{1} \otimes \pi_2^* i_{K'!} \mathbb{1} \simeq i_{(K \times K')!} \mathbb{1}$$

generates $D(* /\!\!/ (G \times G), \Lambda)$ by the same argument as in (i). We have also seen in the proof of (i) that $f_* \underline{\mathrm{Map}}(i_{K!} \mathbb{1}, V) \simeq V^K$, so the only if part of the statement translates to admissibility. The suave dual formula is an instance of the general formula [HM24, Lemma 4.4.5].

(iii) Suppose first that G is locally Λ-Poincaré. Let $H \leq G$ be a compact open Λ-Poincaré subgroup. As in the proof of 2.2(iii) we see that $* /\!\!/ H \to * /\!\!/ G$ is a universal !-cover, so it suffices to show that $* /\!\!/ H \to *$ is Λ-suave [HM24, Lemma 4.5.8(i)]. So WLOG G is Λ-Poincaré.

We need to show that $\mathbb{1} \in D(* /\!\!/ G, \Lambda)$ is Λ-suave, i.e. admissible by (ii). In other words, we need that $V^K = f_{K*} \mathbb{1}$ is dualizable for every compact open $K \leq G$ with $\mathrm{cd}_\Lambda K < \infty$. For this, we note $f_{K*} \mathbb{1} \simeq f_* i_{K*} \mathbb{1}$ and f_* preserves dualizables because G is Λ-Poincaré. On the other hand, i_K is Λ-proper (2.2(i)), so $i_{K*} \mathbb{1} \simeq i_{K!} \mathbb{1}$. Now $\mathbb{1}$ is compact by 2.8 and $i_{K!}$ preserves compacts as demonstrated in the proof of (i). On the other hand, compacts and dualizables agree (2.10).

Conversely, suppose that $* /\!\!/ G \to *$ is Λ-suave. Since G has locally finite Λ-cohomological dimension, it has a compact open subgroup K with $\mathrm{cd}_\Lambda K < \infty$. Moreover, being Λ-suave is the same as admissibility by (ii), so $* /\!\!/ K \to *$ is still Λ-suave. So WLOG G is profinite with $\mathrm{cd}_\Lambda G < \infty$. Since $\mathbb{1} \in D(* /\!\!/ G, \Lambda)$ is f-suave, i.e. admissible, the object $f_* i_{K*} \mathbb{1} \simeq f_{K*} \mathbb{1}$ is dualizable in $\mathbf{Mod}_\Lambda$ for every compact open $K \leq G$. On the other hand, $i_{K*} \mathbb{1} \simeq i_{K!} \mathbb{1}$ generate the dualizables in $D(* /\!\!/ G, \Lambda)$ under (co-)fibers and retractions as demonstrated in (i). So f_* preserves dualizables.

□

This prim duality is also called *Bernstein–Zelevinsky duality* and it is an example of a statement that is really terrible to prove by writing down formulas but follows formally from six functor nonsense! Just from the formulas, it's not clear that this formula for the prim duality is interesting and it's hard to get this explicit prim duality formula on compact inductions by only playing around with the formulas. With 6FF nonsense it's not that bad!

Corollary 2.10. Let $\Lambda \in \mathbf{CRing}$ and G be a profinite group with $\mathrm{cd}_\Lambda G < \infty$.

(i) Then, $D(* /\!\!/ G, \Lambda)$ is compactly generated.

(ii) An object is compact if and only if it is dualizable.

Proof.

(i) We have seen this in the proof of 2.9(i), it is compactly generated by what we called $\mathcal{G}$.

(ii) By 2.9(i) the compact objects agree with the f-prim objects where $f : * /\!\!/ G \to *$. So we need to show that f-primality agrees with dualizability. But $* /\!\!/ G \to *$ is Λ-proper (2.2(ii)) and in this setting we are done [HM24, Lemma 4.6.3(iii)].

□

Example 2.11 ([HM24, Example 5.3.21, 5.3.22])**.** Let p be a prime.

(i) Let Λ be a $\mathbb{Z}[1/p]$-algebra and G be locally pro-p. Then, G is locally Λ-Poincaré.

(ii) Let Λ be a $\mathbb{Z}/p^n$-algebra and G be a p-adic Lie group. Then, G is locally Λ-Poincaré.

In each case one can give explicit descriptions of the dualizing complex and so suave duality (2.9(iii)) recovers Poincaré duality in these settings. This is not really a new proof of Poincaré duality because it relies on results from classical representation theory which are close to Poincaré duality.

3 What the Hecke?

What the heck is a Hecke algebra?

They show up in various areas of mathematics. Frankly, I know neither of the motivations but `https://www.math.columbia.edu/~martinez/Notes/introtohecke.pdf` seems useful.

Definition 3.1. Let Λ be a field with char $\Lambda = p > 0$ and $K \leq G$ be a compact open subgroup of a locally profinite group with $V \in \mathbf{Rep}_\Lambda(K)^\heartsuit$. Then, $\mathcal{H}(G, K, V) = \mathrm{End}_G(\text{c-Ind}_K^G V)$ is the associated **Hecke algebra**.

Fact 3.2 ([HM24, Remark 5.5.1])**.**

(i) There is an isomorphism

$$\mathcal{H}(G, K, V) \xrightarrow{\sim} \{f : G \to \mathrm{End}_\Lambda(V) : f \text{ is } K\text{-}K\text{-linear}, \operatorname{supp} f \text{ compact}\}.$$

(ii) Under this identification there is an involutive anti-isomorphism of algebras

$$\iota : \mathcal{H}(G, K, V) \xrightarrow{\sim} \mathcal{H}(G, K, V^*), \; \iota(T)(g) = (T(g^{-1}))^*.$$

There are more refined derived versions of this construction by taking derived endomorphisms instead of the underived version [HM24, Remark 5.5.1].

Definition 3.3. Let $\Lambda \in \mathbf{CRing}$ and G be a locally profinite group with a compact open subgroup $K \leq G$ with $\mathrm{cd}_\Lambda K < \infty$.

(i) We denote by $\mathcal{H}_K$ the $\mathbf{Mod}_\Lambda$-enriched ∞-category whose objects are the dualizable objects in $\mathbf{Rep}_\Lambda(K)$ and whose mapping objects are

$$\mathcal{H}_K(V, W) = \mathrm{RHom}_G(\text{c-Ind}_K^G V, \text{c-Ind}_K^G W) \in \mathbf{Mod}_\Lambda.$$

(ii) We denote by $\mathcal{H}_K^\bullet = \mathcal{H}_K(\mathbb{1}, \mathbb{1}) \in \mathbf{Alg}_{\mathbb{E}_1}(\mathbf{Mod}_\Lambda)$ and **derived Hecke algebra** of weight $\mathbb{1}$.

Theorem 3.4 ([HM24, Proposition 5.5.4, 5.5.6])**.** Let $\Lambda \in \mathbf{CRing}$ and G be a locally profinite group with a compact open subgroup $K \leq G$ with $\mathrm{cd}_\Lambda K < \infty$.

(i) Prim duality PD on $\mathrm{Prim}(* /\!\!/ G)$ induces an involutive equivalence

$$\mathcal{H}_K^{\mathrm{op}} \xrightarrow{\simeq} \mathcal{H}_K, \; V \mapsto V^\vee = \mathrm{RHom}_\Lambda(V, \Lambda)$$

of $\mathbf{Mod}_\Lambda$-enriched ∞-categories.

(ii) Let Λ be a field with char $\Lambda = p > 0$ and G be a p-adic Lie group with a p-torsionfree compact open subgroup $I \leq G$. Then, (i) induces an anti-involution $\mathrm{Inv} : (\mathcal{H}_I^\bullet)^{\mathrm{op}} \xrightarrow{\simeq} \mathcal{H}_I^\bullet$ which coincides with Schneider–Sorensen's anti-involution $\mathrm{Inv}_{\mathrm{SS}}$ [HM24, Remark 5.5.1].

It seems like previously this was only defined for fields of positive characteristic Λ and you need to work a little to write down these maps. Prim duality immediately yields a map and works for all $\Lambda \in \mathbf{CRing}$.

A fruitful plan of developing new mathematics seems to be: Find/Take any six functor formalism and try to specialize all of the general abstract 6FF notions that we have learned to the example. Anyhow, the next goal of the seminar will be to carry out this plan on the category of topological spaces.

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