Localizing Invariants and Algebraic K-Theory

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Abstract

These are my notes from Georg Tamme's lecture *Localizing Invariants and Algebraic K-theory* from the 2023 IHES Summer School — Recent Advances in Algebraic K-theory. The lectures can be found on Youtube.

Please contact me at qzhu@mpim-bonn.mpg.de (or over social media) for comments or suggestions.

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0 Intro

Let A be a ring. According to Quillen we define **connective K-theory** as the connective spectrum associated to the $\mathbb{E}_\infty$-group

$$K(A) = K\left(\mathbf{Proj}^{\mathrm{fg}}(A)\right) = \left(\mathbf{Proj}^{\mathrm{fg}}(A)^{\mathrm{core}}, \oplus\right)^{\mathrm{gp}}$$

where $\mathbf{Proj}^{\mathrm{fg}}(A)$ is an exact category[1] in the sense of Quillen.

Now you can try to compute it for which one wants some descent statements, i.e. maybe one wants some excision type statements for Zariski covers. But with the above definition this is not so easy to prove. Quillen's theorems mainly work for abelian categories, so it's hard to prove descent statements because $\mathbf{Proj}^{\mathrm{fg}}(A)$ is not an abelian category.

Thomason proposed that you should replace $\mathbf{Proj}^{\mathrm{fg}}(A)$ by the category of perfect complexes **Perf**(A), i.e. finite complexes of finitely generated projectives. This is a small stable ∞-category and $\mathbf{Perf}(A) \simeq D(A)^{\omega}$. The main point is: Waldhausen further developed K-theory in terms of Waldhausen categories and he has a fibration theorem, called *Waldhausen Fibration Theorem*, which sometimes gives you fiber sequences of K-theory spectra.

1 Localizing Invariants

You can define $K(A)$ for $\mathbf{Perf}(A)$. More generally, you can define $K(\mathscr{C}) \in \mathbf{Sp}$ for any small stable ∞-category $\mathscr{C}$. In this setting you can formulate the analog of Waldhausen's fibration theorem and the functors satisfying this are called *localizing invariants*.

1.1 Verdier Quotients

Definition 1.1. Let $\mathscr{C} \in \mathbf{Cat}_\infty^{\mathrm{st}}$ and $\mathscr{D} \subseteq \mathscr{C}$ be a stable subcategory. The **Verdier quotient** is $\mathscr{C}/\mathscr{D} = \mathscr{C}[W^{-1}]$ with $W = \{f : c \to c' : \mathrm{cofib}\, f \in \mathscr{D}\}$.

Remark 1.2. So $\mathscr{C} \to \mathscr{C}/\mathscr{D}$ induces a functor $\mathrm{Fun}(\mathscr{C}/\mathscr{D}, \mathscr{E}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{E})$ is an equivalence onto $\mathrm{Fun}^W(\mathscr{C}, \mathscr{E})$, the full subcategory spanned by those functors sending $f \in W$ to equivalences.

Fact 1.3.

(i) The Verdier quotient $\mathscr{C}/\mathscr{D}$ is stable.

(ii) Let $\mathscr{E} \in \mathbf{Cat}_\infty^{\mathrm{st}}$. Then, $\mathrm{Fun}^{\mathrm{ex}}(\mathscr{C}/\mathscr{D}, \mathscr{E}) \simeq \mathrm{Fun}^{\mathscr{D}\mapsto 0}(\mathscr{C}, \mathscr{E})$.

(iii) Let $X, Y \in \mathscr{C}$ with $\overline{X}, \overline{Y} \in \mathscr{C}/\mathscr{D}$. Then,

$$\mathrm{Map}_{\mathscr{C}/\mathscr{D}}(\overline{X}, \overline{Y}) \simeq \operatorname*{colim}_{Z \in \mathscr{D}_{/Y}} \mathrm{Map}_{\mathscr{C}}(X, \mathrm{cofib}(Z \to Y)).$$

Proof. See e.g. [NS18]. The classical statements in the language of triangulated categories is already in work of Neeman. □

Since $\mathscr{D}$ is stable, $\mathscr{D}_{/Y}$ is filtered.

Remark 1.4. Here is some intuition about 1.3(iii): Pictorially, Pictorially,

[1] I.e. there is some *exact sequence* notion satisfying some axioms.

$$\begin{array}{ccccc} & & & & Z \\ & & & \swarrow & \\ X & & Y & & \\ & \searrow \quad \swarrow & & & \\ & \mathrm{cofib}(Z \to Y) & & & \end{array}$$

and since $Z \in \mathscr{D}$, the map $Y \to \mathrm{cofib}(Z \to Y)$ lies in W, i.e. it gets inverted in $\mathscr{C}/\mathscr{D}$ where $X \to \mathrm{cofib}(Z \to Y)$ amounts to $\overline{X} \to \overline{\mathrm{cofib}(Z \to Y)} \xrightarrow{\simeq} \overline{Y}$.

1.2 Ind-Completion

We were looking at small categories but we can pass to large categories via ind-completions.

Definition 1.5. Let $\mathscr{C} \in \mathbf{Cat}_\infty^{\mathrm{st}}$. Then, $\mathrm{Ind}(\mathscr{C}) \subseteq \mathbf{PSh}(\mathscr{C})$ is the full subcategory spanned by filtered colimits of representables.

This is now presentable, so you can use the adjoint functor theorem which is not available for small categories.

Remark 1.6. The Yoneda embedding よ $: \mathscr{C} \to \mathbf{PSh}(\mathscr{C})$ factors over $\mathrm{Ind}(\mathscr{C})$.

Question 1.7. Can you recover $\mathscr{C}$ from $\mathrm{Ind}(\mathscr{C})$?

Answer. Not quite! You can check with mapping space formulas that $\mathscr{C} \subseteq \mathrm{Ind}(\mathscr{C})^\omega$. In fact, $\mathrm{Ind}(\mathscr{C})^\omega$ is the **idempotent/Karoubi completion** of $\mathscr{C}$. In other words, it is closed under retracts and every object therein is a retract of an object in $\mathscr{C}$.

Indeed, let $X \in \mathrm{Ind}(\mathscr{C})^\omega$, so you can write it as a filtered colimit $X \xrightarrow{\simeq} \mathrm{colim}_i C_i$ but by compactness, this factors over some C_i, i.e.

$$\begin{array}{ccc} X & \xrightarrow{\quad\simeq\quad} & \mathrm{colim}_i C_i \\ & \searrow \quad \nearrow & \\ & C_i & \end{array}$$

showing the retract statement. □

Remark 1.8. One checks $\mathrm{Ind}(\mathscr{C}) \in \mathbf{Pr}_{\mathrm{st}}^{L,\omega}$, i.e. that it is a presentable stable compactly generated[2] ∞-category.

We saw that the passage $\mathscr{C} \mapsto \mathrm{Ind}(\mathscr{C})$ is almost an equivalence of categories, not quite, applying $(-)^\omega$ only returns the idempotent complete ones. These idempotent complete ∞-categories are the main players for localizing invariants.

Definition 1.9.

(i) We write $\mathbf{Cat}_\infty^{\mathrm{perf}}$ for the ∞-category of Karoubi complete small stable ∞-categories.

(ii) A sequence

$$\mathscr{D} \xrightarrow{i} \mathscr{C} \xrightarrow{p} \mathscr{E}$$

in $\mathbf{Cat}_\infty^{\mathrm{perf}}$ is called a **Karoubi sequence** if

[2]In particular, $(-)^\omega$ does not mean compact objects in $\mathbf{Pr}_{\mathrm{st}}^L$. We restrict to compactly generated categories and restrict to compact object-preserving functors. Equivalents, the right adjoint preserves filtered colimits.

- $p \circ i \simeq 0$,[3]
- i is fully faithful,
- the induced map[4] $\mathscr{C}/\mathscr{D} \to \mathscr{E}$ is an idempotent completion.[5]

(iii) A **(Karoubi) localizing invariant** is a functor $\mathbf{Cat}_\infty^{\mathrm{perf}} \to \mathbf{Sp}$ sending Karoubi sequences to fiber sequences.

Remark 1.10. Equivalently, a Karoubi sequence is a bifiber sequence in $\mathbf{Cat}_\infty^{\mathrm{perf}}$.

Remark 1.11. There is an equivalence $\mathrm{Ind} : \mathbf{Cat}_\infty^{\mathrm{perf}} \xrightarrow{\simeq} \mathbf{Pr}_{\mathrm{st}}^{L,\omega}$ with inverse $(-)^\omega$. So K-theory has as input objects in $\mathbf{Cat}_\infty^{\mathrm{perf}}$ but by this equivalence we can equivalently take objects in $\mathbf{Pr}_{\mathrm{st}}^{L,\omega}$. The point of *Efimov K-theory* is to extend the class $\mathbf{Pr}_{\mathrm{st}}^{L,\omega}$ to the so-called *dualizable categories*.

Theorem 1.12 (Waldhausen, Thomason, Schlichting, Blumberg-Gepner-Tabuada, Hebestreit-Lachmann-Steimle)**.** Non-connective K-theory $K(-)$ is Karoubi-localizing.

BGT uses Waldhausen and Schlichting's work while Hebestreit-Lachmann-Steimle is a more modern purely ∞-categorical approach.

Remark 1.13. A possible definition is the $K(-)$ is the universal localizing invariant. For rings you can define negative K-groups by forcing the fundamental theorem.

Let's just blackbox this and prove things about K-theories of schemes. There are some more localizing invariants, but not so many. Tamme only knows one, namely THH. Plus everything you can build out of these like $\mathrm{TC}, \mathrm{TP}, \cdots$. Plus you can restrict the domain $\mathbf{Cat}_\infty^{\mathrm{perf}}$.

1.3 Thomason-Neeman Localization Theorem

We know that Karoubi sequences gives us fiber sequences on K-theory, so that's of course nice. But we first need to produce some of these sequences. That's what Thomason-Neeman localization is about.

We start with some candidate sequence $\mathscr{D} \to \mathscr{C} \to \mathscr{E}$ and wish to check whether it is a Karoubi sequence. This might not be so easy on $\mathbf{Cat}_\infty$ but passing to the ind-completions we have presentable ∞-categories, so we have the adjoint functor theorems and so on. Such tools often make a check possible.

Theorem 1.14. Let $\mathscr{C} \in \mathbf{Cat}_\infty^{\mathrm{perf}}$ and $\mathscr{D} \subseteq \mathscr{C}$ be a stable subcategory with $p : \mathscr{C} \to \mathscr{C}/\mathscr{D}$.

(i) Then,
$$L = \mathrm{Ind}(p) : \mathrm{Ind}(\mathscr{C}) \to \mathrm{Ind}(\mathscr{C}/\mathscr{D})$$
is a Bousfield localization.

(ii) There is an equivalence $\ker L \simeq \mathrm{Ind}(\mathscr{D})$.

Proof.

(i) The functor L preserves filtered colimits by definition. Plus everything is stable and L is also exact[6], so L commutes with all colimits. By the adjoint functor theorem, it has a right adjoint R. On $\mathscr{C}/\mathscr{D}$ this R is given by
$$\mathscr{C}/\mathscr{D} \to \mathrm{Ind}(\mathscr{C}),\ \overline{Y} \mapsto \mathrm{Map}_{\mathscr{C}/\mathscr{D}}(p(-), \overline{Y}).$$

[3]You can check that this is a condition: the space of nullhomotopies of $p \circ i$ is contractible or empty!
[4]It exists because of $p \circ i \simeq 0$.
[5]The Verdier quotient of idempotent complete ∞-categories in general need not be idempotent complete.
[6]This comes from functoriality of Ind, as maps in $\mathbf{Pr}_{\mathrm{st}}^{L,\omega}$ are in particular exact. The difficulty would of course lie in proving that such a functoriality can be provided.

A priori, this formula on the right is an object in $\mathbf{PSh}(\mathscr{C})$ but by 1.3(iii) it lies in $\operatorname{Ind}(\mathscr{C})$.

About fully faithfulness. Via 1.3(iii) you can check that $LR \simeq \mathrm{id}$ on $\mathscr{C}/\mathscr{D}$ and now we want to extend it to all of $\operatorname{Ind}(\mathscr{C}/\mathscr{D})$. We do this by observing that R commutes with filtered colimits which follows from the fact that L preserves compact objects.[7] Thus, L and R both commute with filtered colimits, so $LR \simeq \mathrm{id}_{\operatorname{Ind}(\mathscr{C}/\mathscr{D})}$, i.e. R is fully faithful.

(ii) Let $X \in \ker L \subseteq \operatorname{Ind}(\mathscr{C})$, so we can write it as a filtered colimit $X \simeq \operatorname{colim}_i C_i$. Consider the fiber sequence

$$D_i \longrightarrow C_i \xrightarrow{\eta_{C_i}} RL(C_i)$$

From the explicit formula you can check $D_i \in \operatorname{Ind}(\mathscr{D})$. Passing to colimits yields

$$\operatorname{colim}_i D_i \longrightarrow X \longrightarrow RL(X)$$

but $RL(X) \simeq 0$, so $X \simeq \operatorname{colim}_i D_i \in \operatorname{Ind}(\mathscr{D})$ using $D_i \in \operatorname{Ind}(\mathscr{D})$. The other inclusion is clear.

□

The point is to show that everything is completely formal!

Corollary 1.15. There is an equivalence $\operatorname{Ind}(\mathscr{C}/\mathscr{D}) \simeq \operatorname{Ind}(\mathscr{C})/\operatorname{Ind}(\mathscr{D})$.

Proof. This is a general fact about these sequences. Anyway, since L is a Bousfield localization, it inverts a class of morphisms W. A map $\varphi : x \to y$ in $\operatorname{Ind}(\mathscr{C})$ is such that $L\varphi$ is an equivalence if and only if

$$\operatorname{cofib}(L\varphi) \simeq L(\operatorname{cofib}\varphi) \simeq 0.$$

In other words, $\operatorname{cofib}\varphi \in \ker L$. So

$$\operatorname{Ind}(\mathscr{C}/\mathscr{D}) \simeq \operatorname{Ind}(\mathscr{C})[\{\varphi : \varphi \in \ker L = \operatorname{Ind}(\mathscr{D})\}^{-1}] \simeq \operatorname{Ind}(\mathscr{C})/\operatorname{Ind}(\mathscr{D}).$$

Yeah! □

You usually apply this theorem in the other direction. You're given a sequence of large (presentable) ∞-categories where it's often easy to check that in fact you have some Bousfield localization sequence and then you want apply $(-)^\omega$ to get a Karoubi sequence of small stable ∞-categories.

Corollary 1.16 (Thomason-Neeman Localization)**.** A sequence $\mathscr{D} \to \mathscr{C} \to \mathscr{E}$ is a Karoubi sequence if and only if

$$\operatorname{Ind}(\mathscr{D}) \longrightarrow \operatorname{Ind}(\mathscr{C}) \longrightarrow \operatorname{Ind}(\mathscr{E})$$

is a Bousfield localization sequence.[8]

Proof. For $\Longleftarrow$ consider the comparison map

$$\begin{array}{ccccc} \mathscr{D} & \longrightarrow & \mathscr{C} & \longrightarrow & \mathscr{C}/\mathscr{D} \\ \| & & \| & & \downarrow \\ \mathscr{D} & \longrightarrow & \mathscr{C} & \longrightarrow & \mathscr{E} \end{array}$$

[7] This uses that $\operatorname{Ind}(\mathscr{C})$ is compactly generated so that we may perform the Yoneda argument on compact objects where we can pull out filtered colimits.

[8] It's the same thing as saying that this is a Karoubi sequence in $\mathbf{Pr}^L_{\mathrm{st}}$.

induced by the universal property of the Verdier quotient. By the same argument as in 1.15 and using it, we have

$$\mathrm{Ind}(\mathscr{E}) \simeq \mathrm{Ind}(\mathscr{C})/\,\mathrm{Ind}(\mathscr{D}) \simeq \mathrm{Ind}(\mathscr{C}/\mathscr{D}).$$

Pass back by applying $(-)^{\omega}$.

I believe some steps are missing to make this into a full proof of the result with our technology. □

This is the usual way the theorem is applied: In practice it is often feasible to prove that $\mathrm{Ind}(\mathscr{D}) \to \mathrm{Ind}(\mathscr{C}) \to \mathrm{Ind}(\mathscr{E})$ is a Bousfield localization sequence because it is often possible to explicitly write down the right adjoint of $\mathrm{Ind}(\mathscr{C}) \to \mathrm{Ind}(\mathscr{E})$.

Remark 1.17 (Warning)**.** You could start with (large) presentable ∞-categories: Let

$$\widehat{\mathscr{D}} \longrightarrow \widehat{\mathscr{C}} \longrightarrow \widehat{\mathscr{E}}$$

be a Bousfield localization sequence in $\mathbf{Pr}_L^{\mathrm{st}}$. Let's even assume $\widehat{\mathscr{C}}, \widehat{\mathscr{E}} \in \mathbf{Pr}_L^{\mathrm{st},\omega}$ and that $\widehat{\mathscr{C}} \to \widehat{\mathscr{E}}$ preserves compact objects. This does not imply that

$$\widehat{\mathscr{D}}^{\omega} \longrightarrow \widehat{\mathscr{C}}^{\omega} \longrightarrow \widehat{\mathscr{E}}^{\omega}$$

is a Karoubi sequence.

The point is that you have to be a bit careful applying the Thomason-Neeman localization theorem (1.16). Even if you have a Bousfield localization sequence, it doesn't have to be in this Ind-shape. One still needs that $\widehat{\mathscr{D}}$ is compactly generated (1.11). This is precisely what sometimes goes wrong.[9]

Example 1.18. There is a famous counterexample by Keller which uses precisely this situation. He constructs a ring R and an ideal $I \trianglelefteq R$ and considers $\mathcal{D}(R) \to \mathcal{D}(R/I)$ and shows that this is a Bousfield localization but its kernel has no non-trivial compact objects.

Remark 1.19. But: $\widehat{\mathscr{D}}$ is still dualizable, so you can apply Efimov K-theory!

Example 1.20. Let $A \in \mathbf{CRing}$ and $f \in A$. Consider

$$\mathcal{D}(A) \xrightarrow{-\otimes_A A[f^{-1}]} \mathcal{D}(A[f^{-1}])$$

which is a Bousfield localization with kernel $\mathcal{D}(A \text{ on } (f)) \subseteq \mathcal{D}(A)$. This is compactly generated! Indeed, consider the Koszul complex

$$K(f) = \left[A \xrightarrow{f} A\right] \simeq \mathrm{cofib}(f : A \to A)$$

which is compact since A is compact which also lives in $\mathcal{D}(A \text{ on } f)$, as inverting f has the effect $\mathrm{cofib}\, f \simeq 0$.

It generates: Let $M \in \mathcal{D}(A \text{ on } (f))$. Suppose

$$\mathrm{map}_{\mathcal{D}(A \text{ on } (f))}(K(f), M) \simeq 0,$$

i.e. $\mathrm{fib}(f : M \to M) \simeq 0$, i.e. $f : M \to M$ is an equivalence. So $0 \simeq M \otimes_A A[f^{-1}] \simeq M$ since $M \in \mathcal{D}(A \text{ on } (f))$.

This is the base case of the following theorem which nowadays has been generalized to qcqs spectral algebraic spaces.

Theorem 1.21 (Neeman, Bondal-van den Bergh, Thomason)**.** Let X be a qcqs scheme and $U \subseteq X$ be a qc open subspace with complement Z. Then,

$$\mathcal{D}_{\mathrm{qc}}(X \text{ on } Z) = \ker(\mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{D}_{\mathrm{qc}}(U))$$

is compactly generated.[10]

[9] If $\widehat{\mathscr{D}}$ and $\widehat{\mathscr{C}}$ are compactly generated, then so is $\widehat{\mathscr{E}}$.

[10] Here, $\mathcal{D}_{\mathrm{qc}}$ is the ∞-category of quasicoherent sheaves.

Comment. There is an inductive argument coming from Thomason to do this. That's why Thomason's name is included. □

Corollary 1.22. There is a Karoubi sequence

$$\mathbf{Perf}(X \text{ on } Z) \longrightarrow \mathbf{Perf}(X) \longrightarrow \mathbf{Perf}(U).$$

Proof. Apply Ind to 1.21 via the Thomason-Neeman Localization (1.16). □

This is really the important thing that will give Nisnevich/Zariski descent. The easiest case is an open cover $X = U \cup V$. Build a square coming from $\mathbf{Perf}(X), \mathbf{Perf}(U), \mathbf{Perf}(V), \mathbf{Perf}(U \cap V)$ and one can check that the kernels of those localization sequences are equivalent! Thus, we get a pullback square, so you get a Mayer-Vietoris sequence on K-theory. The same argument works for Nisnevich descent.

This also allows you to reduce to local rings.

So all these descent statements are (up to 1.21) formal to obtain!

2 Excision

If you want to study singularities, then it's usually not enough to consider Zariski or Nisnevich coverings. You need something more general, which are the cdh-covering – or in the simplest case: closed coverings.

2.1 Milnor Squares

Example 2.1. Let $\operatorname{char} k \neq 2$ and $A = k[x,y]/(y^2 - x^3 - x^2)$ be the algebra representing the nodal curve. Consider its normalization $\mathbb{A}^1 \to \operatorname{Spec} A$ whose singularity has two preimages. Gluing them yields the nodal curve:

$$\begin{array}{ccc} * \amalg * & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ * & \longrightarrow & \operatorname{Spec} A \end{array}$$

On rings this corresponds to a cartesian square

$$\begin{array}{ccc} A & \longrightarrow & k[t] \\ \downarrow & & \downarrow {\scriptstyle \mathrm{ev}_{(-1,1)}} \\ k & \xrightarrow[\Delta]{} & k \times k \end{array}$$

in **CRing** where the right map is surjective since $\operatorname{char} k \neq 2$. Clearly, those rings besides A are much simpler than A. So you can ask if you can say something about $K(\mathbf{Perf}_A)$ if you know something about the other corners. This is a typical situation of a Milnor square.

Definition 2.2. A **Milnor square** is a cartesian square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

in **CRing** with surjective vertical arrows.

Theorem 2.3 (Bass, Milnor, Murthy)**.** For every Milnor square there is a LES

$$K_1(A) \longrightarrow K_1(A') \oplus K_1(B) \longrightarrow K_1(B') \xrightarrow{\partial} K_0(A) \longrightarrow \cdots$$

extending infinitely to the right.

This theorem is also called excision. Using this result, it is easy to compute the non-positive K-groups of the nodal curve (2.1).

This statement was known before K_2 and higher K-groups were even found. People tried to extend this sequence to the left.

Proposition 2.4 (Swan)**.** This sequence does not extend (functorially) to the left with

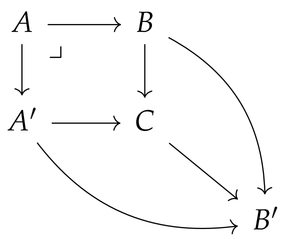
$$\cdots \longrightarrow K_2(B') \xrightarrow{\partial} K_1(A).$$

So Milnor squares will not give rise to a pullback of K-theory spectra.

Theorem 2.5 (Land-T.)**.** Let

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

be a cartesian square in $\mathbf{Alg}_{\mathbb{E}_1}(\mathbf{Sp})$.[11] Then, there is a naturally associated ring spectrum $C = A' \odot_A^{B'} B$ and a commutative diagram



such that any localizing invariant takes $\mathbf{Perf}(-)$ of this inner square to a cartesian square in $\mathbf{Sp}$. Moreover, the underlying spectrum of C is $A' \otimes_A B$.

Example 2.6. If the square we start with is a Milnor square, then $\pi_\bullet(C) \cong \mathrm{Tor}_\bullet^A(A', B)$ and $\pi_0(C) \cong A' \otimes_A B \cong B'$ using the surjectivity of those maps in the Milnor square.[12] Recall that K-theory increases connectivity by 1. Relatedly, $K_n(R)$ only depends on $\tau_{\leq n-1}R$. So computing $K_1(C)$ only depends on $\pi_0 C$. Applying $K(-)$ gives BMM (2.3). This also shows why the sequence cannot extend.

Remark 2.7. Because we also have Zariski descent, we can reduce to the affine setting, so these results give something for schemes. For truncating localizing invariants (2.8) we will get cdh descent but this will need some geometric input.

There was a lot of work on specific Milnor squares. This uses completely different techniques than those from the first lecture. One of the motivating questions before the Land-Tamme theorem was why you cannot use the machinery of localizing invariants to treat these kinds of problems. But in fact, you can.

[11] Every Milnor square is an example.
[12] You should think of $C \to B'$ as modding out a nilpotent ideal.

2.2 Applications

In the example 2.6 we recover B' from $\pi_0 C$. There are localizing invariants which don't see this difference, so then we get excision for free. Such localizing invariants deserve a name.

Definition 2.8. A localizing invariant E is called **truncating** if $E(C) \to E(\pi_0 C)$ is an equivalence for every $C \in \mathbf{Sp}_{\geq 0}$.[13]

Example 2.9. The following are truncating.

(i) $K^{\text{inv}} = \text{fib}(\text{tr}^{\text{cyctr}} : K \to \text{TC})$ by the famous theorem of Dundas-Goodwillie-McCarthy.[14] So K does not satisfy excision (i.e. it does not send Milnor square to pullback squares) but the failure of doing so is the same for K and for TC. Because sometimes you can effectively compute TC, this lets you compute something about the K-theory of Milnor squares.

(ii) HP over $\mathbb{Q}$[15] by Goodwillie.

(iii) KH over $\mathbb{Z}$-algebras, Weibel's homotopy K-theory. So K is not $\mathbb{A}^1$-invariant but you can force it to be and that's the result. If you do this you also kill all the higher homotopy information, i.e. it is truncating.

(iv) $K(-)[1/p]$ is truncating on $\mathbb{Z}/p^n$-algebras by Weibel.

Corollary 2.10. Let E be a truncating localizing invariant.

(i) Then, E satisfies Milnor excision, i.e. E sends Milnor squares to cartesian squares.

(ii) If $I \trianglelefteq A$ is nilpotent, then $E(A) \to E(A/I)$ is an equivalence.

(iii) Then, E satisfies *cdh-descent*, i.e. if

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & X \end{array}$$

is an abstract blow-up, i.e. the horizontal maps are closed immersions, $X' \to X$ is proper and an isomorphism outside Y and potentially some finiteness conditions, then E sends this square to a cartesian square.

Proof.

(i) This is because $B' \simeq \pi_0 C$.

(ii) The main idea is that we can assume $I^2 = 0$ since I is nilpotent and are in the situation of a square-zero extension which can be written as a pullback. In such a situation we can compute everything showing up in the Land-Tamme square.

(iii) This needs some geometric input and essentially relies on the fact that abstract blow-up squares can be built from blow-ups.

□

[13] Here, $E(C) = E(\mathbf{Perf}_C)$.

[14] McCarthy proved it for simplicial rings and Dundas found a way to make this work for connective ring spectra. They prove it after p-completion and Goodwillie added the rational case (which was proven earlier).

[15] Restrict all algebras to $\mathbb{Q}$-algebras and categories to $\mathbb{Q}$-linear ones.

2.3 Schwede-Shipley Theorem

Proof Idea of 2.5. We want to have a cartesian square

$$\begin{array}{ccc} E(A) & \longrightarrow & E(B) \\ \downarrow & & \downarrow \\ E(A') & \longrightarrow & E(C) \end{array}$$

which equivalently means that

$$E(A) \longrightarrow E(A') \oplus E(B) \longrightarrow E(C)$$

is a fiber sequence. The idea is to construct some category $\mathscr{C}$ such that $E(\mathscr{C}) \simeq E(A') \oplus E(B)$ and hope that $\mathbf{Perf}_A \hookrightarrow \mathscr{C}$. Then, we can take the Verdier quotient and will at least get a fiber sequence. □

The Schwede-Shipley theorem is some sort of Morita theory for ring spectra and is a recognition theorem about when some stable presentable category is the category of modules over some ring spectrum.

Theorem 2.11 (Schwede-Shipley, 2003)**.** Let $\mathscr{C} \in \mathbf{Pr}^L_{\mathrm{st}}$ and $c \in \mathscr{C}^\omega$ generating $\mathscr{C}$. Then, there is an equivalence $\mathscr{C} \simeq \mathbf{RMod}_{\mathrm{End}_{\mathscr{C}}(c)}$.

Proof. Let us write $R = \mathrm{End}_{\mathscr{C}}(c) \in \mathbf{Alg}_{\mathbb{E}_1}(\mathbf{Sp})$. There is a functor

$$G : \mathscr{C} \to \mathbf{RMod}_R,\ d \mapsto \mathrm{map}_{\mathscr{C}}(c, d).$$

We observe:

(1) Then, G preserves all limits by construction.

(2) Moreover, G preserves filtered colimits since c is compact. Since we are in a stable setting, G thus preserves all colimits.

(3) The generating condition implies that G is conservative.

By (1), the adjoint functor theorem thus provides a left adjoint F. The module category $\mathbf{RMod}_R$ has a compact generator R and we claim $F(R) \simeq c$. Let us compute this:

$$\mathrm{map}_{\mathscr{C}}(F(R), d) \simeq \mathrm{map}_R(R, G(d)) \simeq \mathrm{map}_{\mathscr{C}}(c, d),$$

so $F(R) \simeq c$ by Yoneda.

So $G(F(R)) \simeq G(c) \simeq R$. So, $\eta : \mathrm{id}_{\mathbf{RMod}_R} \Rightarrow G \circ F$ is an equivalence on the compact generator R but G, F preserve colimits by (2), so η is an equivalence.

Now about the counit $\varepsilon : F \circ G \Rightarrow \mathrm{id}_{\mathscr{C}}$. By the triangle identities we obtain that $G\varepsilon$ is an equivalence, so conservativity of G by (3) implies that ε is an equivalence. □

2.4 Proof of Theorem

The idea that we already mentioned in the previous subsection comes from the proof of this BMM theorem (2.3). They work with finitely projective A-modules and shows that $\mathbf{Proj}^{\mathrm{fg}}_A$ is the pullback of the other categories involved in a Milnor square. But in general, K is not well-behaved with pullbacks of categories. A pullback

$$\begin{array}{ccc} \mathbf{Proj}_A^{\mathrm{fg}} & \longrightarrow & \mathbf{Proj}_B^{\mathrm{fg}} \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{Proj}_{A'}^{\mathrm{fg}} & \longrightarrow & \mathbf{Proj}_{B'}^{\mathrm{fg}} \end{array}$$

is a pair of modules together with an equivalence after base changing to B'. We will relax this equivalence condition.

Definition 2.12. Consider a diagram

$$\begin{array}{ccc} & & \mathscr{B} \\ & & \downarrow{\scriptstyle q} \\ \mathscr{A}' & \xrightarrow[p]{} & \mathscr{B}' \end{array}$$

in $\mathbf{Cat}_\infty^{\mathrm{perf}}$ (or just $\mathbf{Cat}_\infty$). The **lax/oriented pullback** is an ∞-category $\mathscr{A}' \overset{\rightarrow}{\times}_{\mathscr{B}'} \mathscr{B}$ with objects $(X \in \mathscr{A}', Y \in \mathscr{B}, pX \to qY \text{ in } \mathscr{B}')$.

There is a full subcategory in $\mathscr{A}' \overset{\rightarrow}{\times}_{\mathscr{B}'} \mathscr{B}$ where you require the map in $\mathscr{B}'$ to be an equivalence. In other words, this is the usual pullback $\mathscr{A}' \times_{\mathscr{B}'} \mathscr{B}$.

Observation 2.13. There exists a split Karoubi sequence

$$\mathscr{B} \xrightarrow{(0,-,0)} \mathscr{A}' \overset{\rightarrow}{\times}_{\mathscr{B}'} \mathscr{B} \xrightarrow{\mathrm{pr}_1} \mathscr{A}'$$

So $E(\mathscr{A}' \overset{\rightarrow}{\times}_{\mathscr{B}'} \mathscr{B}) \simeq E(\mathscr{A}') \oplus E(\mathscr{B})$ since we get fiber sequences from the localizing invariant condition and these split.

Observation 2.14. There is a preferred functor

$$i : \mathbf{Perf}_A \to \mathbf{Perf}_{A'} \overset{\rightarrow}{\times}_{\mathbf{Perf}_{B'}} \mathbf{Perf}_B$$

which already exists for every commutative square of rings.[16] This is fully faithful if that original square is a pullback square. Define $\mathscr{C}$ as the *Karoubi quotient* of i, i.e. take the Verdier quotient and then idempotent complete.

We stated Schwede-Shipley for presentable ∞-categories but now we have small ∞-categories but that's okay, we can just ind-complete. It turns out that $\mathrm{Ind}(-)$ commutes with lax pullbacks (while it does not for strict pullbacks), so this is possible. Then, we can pass back to compact objects.

Proof of 2.5. Note that the oriented pullback is generated by $(A', 0, 0)$ and $(0, B, 0)$. On the other hand, we have the fiber sequence

$$(0, B, 0) \longrightarrow (A', B', B' \xrightarrow{\simeq} B') \longrightarrow (A', 0, 0).$$

In $\mathscr{C}$ we modded out $\mathbf{Perf}_A$, i.e. the middle term, so $(A', 0, 0)$ and $(0, B, 0)$ only differ up to a shift. This implies that $\mathscr{C}$ is generated by the image $\overline{B}$ of $(0, B, 0)$. By Schwede-Shipley we deduce $\mathscr{C} \simeq \mathbf{Perf}_{\mathrm{End}_{\mathscr{C}}(\overline{B})}$ and so it remains to compute the underlying spectrum of $C = \mathrm{End}_{\mathscr{C}}(\overline{B})$. All the other desired properties are already provided (as can be checked)! After ind-completion we have the adjoint functor theorem and get a sequence of adjunctions

[16] It really already maps to the strict pullback.

$$\mathbf{RMod}_A \underset{s}{\overset{i}{\rightleftarrows}} (\cdots) \overset{\rightarrow}{\times}_{(\dots)} (\cdots) \underset{r}{\overset{p}{\rightleftarrows}} \mathrm{Ind}(\mathscr{C})$$

It's formal that $r \circ p \simeq \mathrm{cofib}(\varepsilon : i \circ s \Rightarrow \mathrm{id}_{\overset{\rightarrow}{\times}})$ but all these functors are explicit! For instance, $s(X, Y, g)$ is the pullback

$$\begin{array}{ccc} s(X,Y,g) & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow[g]{} & Y \otimes_B B' \end{array}$$

and now it's a doable computation by computing this on the generator. □

3 Pro-cdh Descent

This is on work from Morrow – no:[17] Krishna-Weibel, Srinivas.

3.1 Results

Consider an abstract blow-up square

$$\begin{array}{ccc} Z' & \hookrightarrow & X' \\ \downarrow & \lrcorner & \downarrow p \\ Z & \hookrightarrow & X \end{array}$$

of schemes, i.e. it is cartesian, the horizontal arrows are closed immersions, p is proper and an isomorphism outside of Z plus potential finiteness conditions if we are not in the Noetherian setting.

Example 3.1. A typical example is the normalization of the nodal curve (2.1):

$$\begin{array}{ccc} * \amalg * & \hookrightarrow & \mathbb{A}^1 \\ \downarrow & \lrcorner & \downarrow \\ * & \hookrightarrow & \mathrm{Spec}\, A \end{array}$$

In general, K of abstract blow-up squares will not be cartesian.

Observation 3.2. Now cdh-descent basically means that you send abstract blow-up squares to cartesian squares. The idea of pro cdh descent is that you place the closed subscheme Z by an infinitesimal tubular neighbourhood. As such, let $Z(n)$ be the n-th infinitesimal thickening of Z in X. So we get a sequence

$$Z = Z(0) \hookrightarrow Z(1) \hookrightarrow \cdots \hookrightarrow X$$

leading to an ind-scheme $\{Z(n)\}_n$. This looks a bit more like something open, so maybe there is a better chance to get some descent property.

Theorem 3.3 (Krishna-Srinivas, Krishna, Morrow, Kerz-Strunk-T., Bachmann-Khan-Ravi-Sosnilo)**.** Assume that X is a Noetherian[18] ANS stack. Then, the square

[17] Correction by Morrow, I think.

[18] It's special to the Noetherian setting but the scheme structure for Z didn't really play a role since we pass to infinitesimal thickenings anyway.

$$\begin{array}{ccc} K(X) & \longrightarrow & \{K(Z(n))\}_n \\ \downarrow & & \downarrow \\ K(X') & \longrightarrow & \{K(Z'(n))\}_n \end{array}$$

in Pro(**Sp**) is (weakly) cartesian, i.e. $\tau_{\leq k}(K(X) \to \text{pullback})$ is an equivalence[19] in Pro(**Sp**) for every k.

Here, Krishna-Srinivas and Krishna consider special cases. Morrow does it more generally assuming resolution of singularity using trace methods. Finally, KST proved it in the Noetherian setting. Then, BKRS generalized it to a stacky setup and removed the word 'weakly'.

Example 3.4 (Dahlhausen-T.)**.** What about the non-Noetherian setting? Let

$$A = F\left[x, y, \frac{x}{y}, \frac{x}{y^2}, \cdots\right]/(xy),\ I = (y),\ J = (x).$$

Then,

$$J^2 = 0 \subseteq J \subseteq \cdots \subseteq I^3 \subseteq I^2 \subseteq I.$$

With these we can construct an abstract blow-up square

$$\begin{array}{ccc} \operatorname{Spec}(A/I) & \hookrightarrow & \operatorname{Spec}(A/J) \\ \| & \lrcorner & \downarrow \\ \operatorname{Spec}(A/I) & \hookrightarrow & \operatorname{Spec}(A) \end{array}$$

If we had pro-cdh descent, then

$$\begin{array}{ccc} K(A) & \longrightarrow & \{K(A/I^n)\}_n \\ \downarrow & & \downarrow \\ K(A/J) & \longrightarrow & \{K(A/(I^n + J = I^n)\}_n \end{array}$$

and now the right side is an equivalence which implies that the left side is a weak equivalence, so in particular isomorphisms on pro-homotopy groups – which are just groups in this case. We deduce $K_1(A) \xrightarrow{\sim} K_1(A/J)$. Since A is commutative, the units sit inside:

$$\begin{array}{ccc} K_1(A) & \xrightarrow{\sim} & K_1(A/J) \\ \uparrow & & \uparrow \\ A^\times & \longrightarrow & (A/J)^\times \end{array}$$

and now $1 + x \mapsto 1$ but $x \neq 0$, so this contradicts injectivity since also $1 \mapsto 1$.

The following is formulated for derived schemes because derived schemes show up anyway.

Theorem 3.5 (Kelly-Saito-T., in progress)**.** Let $p : X' \to X$ be proper and locally almost finitely presented morphism of derived schemes which is an isomorphism outside a closed $Z \subseteq |X|$ with $|X| \setminus |Z|$ qc and X qcqs. Then, the square

$$\begin{array}{ccc} K(X) & \longrightarrow & K(X^\wedge_Z) \\ \downarrow & & \downarrow \\ K(X') & \longrightarrow & K(X'^\wedge_Z) \end{array}$$

[19] So we get equivalences of pro-homotopy groups.

is weakly cartesian.[20]

More localizing invariants hold are THH, TC, $\cdots$.

Example 3.6. Let $X = \operatorname{Spec} A$ and Z be cut out by $f_1, \cdots, f_m \in \pi_0(A)$. Then, $X_Z^\wedge$ is the pro-derived scheme $\{X /\!/ (f_1^n, \cdots, f_m^n)\}_n$.

This result (3.5) implies a vanishing conjecture from Weibel.

Remark 3.7. Let A be a discrete Noetherian ring. Then, $\{A /\!/ f^n\}_n \xrightarrow{\simeq} \{A/f^n\}_n$.

The proofs of all these cdh statements (including 2.10(iii)) depends on some geometric reductions and two features about K-theory:

- Pro-excision,
- Derived blow-ups.

3.2 Pro-excision

Observation 3.8. Let

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

be a pullback of ring spectra which is Tor-independent, i.e. $A' \otimes_A B \xrightarrow{\simeq} B'$. Then, K of this square is a pullback by 2.5 since $A' \odot_A^{B'} B \simeq B'$ in **Sp** by this Tor-independence assumption.

Proofs of the KST Theorems for finite p. Reduce to affine X. Let $X = \operatorname{Spec} A$ and $X' = \operatorname{Spec} B$ and $\varphi : A \to B$ be a finite[21] map, then it is of almost finite presentation. For simplicity of notation let us say that $Z \subseteq |X|$ is defined by $f \in \pi_0(A)$. The condition that φ is a map outside Z is to say that it is an equivalence after inverting f. Let $J = \operatorname{fib} \varphi$ which is an almost perfect (pseudocoherent) A-module. So $J[f^{-1}] \simeq 0$.

Claim 3.9. The square

$$\begin{array}{ccc} A & \longrightarrow & \{A /\!/ f^n\}_n \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ B & \longrightarrow & \{B /\!/ f^n\} \end{array}$$

is weakly cartesian.

Proof. Equivalently, we need to check that the map of fibers

$$\begin{array}{ccc} J & \longrightarrow & \{J /\!/ f^n\}_n \\ \downarrow & & \downarrow \\ A & \longrightarrow & \{A /\!/ f^n\}_n \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ B & \longrightarrow & \{B /\!/ f^n\} \end{array}$$

[20] Here, we see $X_Z^\wedge$ as an ind-derived scheme.
[21] I.e. it is on π_0.

is a weak equivalence. The fiber of that map is the pro system $\{J, - \cdot f\}$ which we want to show to be 0. We have

$$0 \simeq J[f^{-1}] \simeq \operatorname{colim}\left(J \xrightarrow{f} J \xrightarrow{f} \cdots\right).$$

If J were compact, then this would already imply that a fixed power of f acts as 0. This is not quite the case but let us apply

$$0 \simeq \tau_{\leq k} J[f^{-1}] \simeq \operatorname{colim}\left(\tau_{\leq k} J \xrightarrow{f} \tau_{\leq k} J \xrightarrow{f} \cdots\right)$$

but $\tau_{\leq k} J$ is compact by the finiteness assumptions, so this argument holds now, i.e. f^N is nullhomotopy for some $N \in \mathbb{N}$ on $\tau_{\leq k} J$. This precisely says that the pro-system $\{J, - \cdot f\}$ is weakly contractible. □

Moreover, $B \otimes_A A /\!\!/ f^n \simeq B /\!\!/ f^n$. Tinkering a bit with pro-systems then lets us conclude the KST results with 3.8 for finite p. □

3.3 Derived Blow-Ups

We want a derived version of the following. The following works for all localizing invariants but Thomason phrased it for K.

Theorem 3.10 (Thomason)**.** Let R be a ring and $t_1, \cdots, t_m$ be a regular sequence and let

$$\begin{array}{ccc} E & \overset{j}{\hookrightarrow} & \widetilde{X} = \mathrm{Bl}_X Z \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ Z = V(t_1, \cdots, t_m) & \hookrightarrow & X = \operatorname{Spec} R \end{array}$$

be a (not necessarily derived) cartesian square of schemes. Then,

$$\begin{array}{ccc} K(X) & \longrightarrow & K(Z) \\ \downarrow & & \downarrow \\ K(\widetilde{X}) & \longrightarrow & K(E) \end{array}$$

is cartesian.

Proof. The main idea is that one basically fully understands $\mathbf{Perf}_{\widetilde{X}}$. Consider

$$0 \subseteq P_0 \subseteq \cdots \subseteq P_\ell = \langle \mathcal{O}_{\widetilde{X}}, j_* \mathcal{O}_E(-k) : k = 1, \cdots, \ell \rangle \subseteq \cdots \subseteq P_{m-1} = \mathbf{Perf}_{\widetilde{X}}.$$

Then, one computes $p^* : \mathbf{Perf}_X \xrightarrow{\simeq} P_0$ and $j_* q^*(-\ell) : \mathbf{Perf}_Z \xrightarrow{\simeq} P_\ell / P_{\ell-1}$ and similarly for E. Comparing the filtration on $\widetilde{X}$ and E one obtains the theorem. □

Now, the derived version.

Definition 3.11. Let X be a derived scheme and $f : X \to \mathbb{A}^n$. The **derived blow-up** of X in f is $\widetilde{X} = X \times^R_{\mathbb{A}^m} \mathrm{Bl}_{\mathbb{A}^m}(0)$.

Theorem 3.12 (Weak version: KST, Antieau)**.** Let $Z = V(f) = V(f_1, \cdots, f_m) = X \times_{\mathbb{A}^m} 0$[22] with thickenings $Z(n) = V(f_1^n, \cdots, f_m^n)$ and

[22] These are the derived versions, i.e. with $/\!\!/$.

$$\begin{array}{ccc} D(n) & \longrightarrow & \widetilde{X} \\ \downarrow & \lrcorner & \downarrow \\ Z(n) & \longrightarrow & X \end{array}$$

in derived schemes. Then,

$$\begin{array}{ccc} K(X) & \longrightarrow & \{K(Z(n))\}_n \\ \downarrow & \lrcorner & \downarrow \\ K(\widetilde{X}) & \longrightarrow & \{K(D(n))\}_n \end{array}$$

is cartesian.

References

[NS18] Thomas Nikolaus and Peter Scholze. On topological cyclic homology. *Acta Math.*, 221(2):203–409, 2018. (Cited on page 1.)