Evenness 1

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Abstract

This is a first talk to Hahn-Raksit-Wilson's *A Motivic Filtration on the Topological Cyclic Homology of Commutative Ring Spectra* [HRW24].

After a motivational overview about K-theory we recall/learn about (even) $\mathbb{E}_\infty$-rings. Then, we will construct the even filtration and discuss descent type statements through the flat topology. We make an effort to fill in some details of the proofs which are of course known to the experts.

This is a talk given at the Kleine AT 2025.

Contents

0 Overview: The Even Filtration

Let me recount Jeremy's very nice motivation from his talk *Prismatic and syntomic cohomology of ring spectra.*[1]

Let R be a nice discrete commutative ring. A major problem in topology and algebra is to compute the K-theory groups $K(R)$ of R. It is an incredibly hard problem, for example $K(\mathbb{Z}/4)$ is not even known.

The state of the art is a combination of trace methods and compatible motivic filtrations

$$\begin{tikzcd}
K(R) \arrow[r, "\text{cyctr}"] & \mathrm{TC}(R) \arrow[r] \arrow[rr, bend left] & \mathrm{TP}(R) & \mathrm{THH}(R) = R \otimes_{R \otimes R} R \\
H_{\mathrm{mot}} \arrow[u, Rightarrow, "\text{slice SS}"] & H_{\mathrm{syn}} \arrow[u, Rightarrow] & H_{\mathrm{pris}} \arrow[u, Rightarrow] & H_{\mathrm{HT}} \arrow[u, Rightarrow]
\end{tikzcd}$$

One tries to understand $K(R)$ by these trace maps which are close to equivalences by a famous result from Dundas-Goodwillie-McCarthy [DGM13]. Even though these approximations are said to be easier to compute, they are still really hard. Bhatt-Morrow-Scholze [BMS19] recently constructed compatible motivic filtrations on $\mathrm{TC}(R), \mathrm{TP}(R), \mathrm{THH}(R)$ by qrsp (quasiregular semiperfectoid) descent. These induce compatible spectral sequences that try to compute those desired groups.

These words are scary to homotopy theorists[2] and the amazing observation of Hahn-Raksit-Wilson [HRW24] is that one can recover those filtrations by pure abstract non-sense! In fact, their construction extends to a more general setting and for example recovers the Adams-Novikov filtration.

That's the *even filtration*.

1 Higher Algebra Recap

1.1 $\mathbb{E}_\infty$-Rings

Commutativity is more subtle in higher algebra which is now structure instead of a property ensured via certain homotopies realizing commutativity relations. One way to encode such multiplicative structure is through so-called ∞-operads, for example there is a sequence of ∞-operads

$$\mathbb{E}_1 \longrightarrow \mathbb{E}_2 \longrightarrow \cdots \longrightarrow \mathbb{E}_\infty$$

ranging from the associative operad to the commutative operad filtering a 'level' of commutativity. Here is one down-to-earth way of phrasing the definition:

Definition 1.1. An **$\mathbb{E}_\infty$-ring spectrum** is a spectrum $A \in \mathbf{Sp}$ together with structure maps

$$* \times_{\Sigma^n} A^{\otimes n} \simeq A^{\otimes n}_{h\Sigma_n} \to A.$$

There is an associated ∞-category $\mathbf{CAlg} = \mathbf{CAlg}(\mathbf{Sp}) = \mathbf{Alg}_{\mathbb{E}_\infty}(\mathbf{Sp})$.

Observation 1.2. Let $A \in \mathbf{CAlg}$.

[1] See https://www.youtube.com/watch?v=y92tka3bwLc.
[2] At least they scare me.

(i) For $n = 2$ this is a map $(A \otimes A)_{h\Sigma_2} \to A$ which by adjunction corresponds to a map $A \otimes A \to A$ of $B\Sigma_2$-spectra where the right side has the trivial Σ_2-action. In other words, the diagram

$$\begin{tikzcd} A \otimes A \arrow[rr, "\tau"] \arrow[dr] & & A \otimes A \arrow[dl] \\ & A & \end{tikzcd}$$

commutes.

(ii) The structure leads to so-called power operations. Let $X \in \mathbf{Sp}$, then we can form the map

$$A^0(X) \to A^0(X^{\otimes n}_{h\Sigma_n}),\ [\alpha : X \to A] \mapsto \left[X^{\otimes n}_{h\Sigma_n} \xrightarrow{\alpha} A^{\otimes n}_{h\Sigma_n} \to A\right].$$

(iii) Passing to $\pi_\bullet$ in particular yields a graded-commutative ring $\pi_\bullet A$ – so $\pi_0 A$ is a commutative ring and $\pi_n A$ is a $\pi_0 R$-module with graded-commutative multiplication. The graded-commutativity comes from the swap map $\mathbb{S}^m \otimes \mathbb{S}^n \simeq \mathbb{S}^n \otimes \mathbb{S}^m$.

Remark 1.3. In Lurie's language [Lur17] this is a lax symmetric monoidal functor $\mathbb{F}_* \to \mathbf{Sp}^{\otimes}$. If $\langle 1 \rangle \mapsto X$, then the lax symmetric monoidality can be used to show $\langle n \rangle \mapsto A^{\otimes n}$. One needs a little argument to recover those maps $A^{\otimes n}_{h\Sigma_n} \to A$ though. Funnily, this is essentially the same technique as in 2.2(iv).

Proof. Indeed, by adjunction we want to equivalently construct a natural transformation $X^{\otimes n} \Rightarrow \operatorname{const} X$ as functors $B\Sigma_n \to \mathbf{Sp}$. So we need an element in

$$\mathrm{Fun}([1], \mathrm{Fun}(B\Sigma_n, \mathbf{Sp})) \simeq \mathrm{Fun}(B\Sigma_n, \mathrm{Ar}(\mathbf{Sp}))$$

which we obtain as

$$B\Sigma_n \longrightarrow \mathbb{F}_* \simeq (\mathbb{F}_*)_{/\langle 1 \rangle} \longrightarrow \mathscr{C}^{\otimes}_{/c} \longrightarrow \mathscr{C}_{/c} \longrightarrow \mathrm{Ar}(\mathscr{C})$$

where the second-to-last functor is the unique factorization through coCartesian lifts

$$\begin{tikzcd} & c^{\otimes n} \arrow[dr, dashed, "\exists!"] & \\ (c, \cdots, c) \arrow[ur] \arrow[rr] & & c \end{tikzcd}$$

which can be made functorial. □

The theory of $\mathbb{E}_\infty$-rings is nice because it gives rise to nice structure. For example, there is an ∞-category of modules $\mathbf{Mod}_R(\mathbf{Sp})$ over some $R \in \mathbf{CAlg}(\mathbf{Sp})$ which can be endowed with a symmetric monoidal structure, pushouts are relatively computable, etc.

Example 1.4. Many everyday examples of spectra are $\mathbb{E}_\infty$. Common ones are

$$\mathbb{S}, H\mathbb{Z}, \mathrm{ku}, \mathrm{ko}, \mathrm{KU}, \mathrm{KO}, \mathrm{MU}, \mathrm{MO}, \mathrm{tmf}$$

and so on.

Example 1.5. Not every example is $\mathbb{E}_\infty$.

(i) Let p be a prime. Then, $\mathbb{S}/p = \mathrm{cofib}(p : \mathbb{S} \to \mathbb{S})$ is never $\mathbb{E}_\infty$. It is not even $\mathbb{E}_1$ by results of Toda, Kraines, Kochmann. For the longest time, this was conjectured to be true for all Moore spectra until Robert Burklund recently miraculously proved that for any $m \in \mathbb{N}$ the Moore spectrum $\mathbb{S}/p^n$ is $\mathbb{E}_m$ for $m \gg 0$ using technology very related to filtered spectra [Bur22].

(ii) The p-local complex bordism spectrum splits as a direct sum $\mathrm{MU}_{(p)} \simeq \bigoplus \Sigma^{?}\,\mathrm{BP}$ of equivalent spectra called *Brown-Peterson spectrum*. It is one of the main players in chromatic homotopy theory, only known to be $\mathbb{E}_4$ [BM13] and provably known not to be $\mathbb{E}_{12}$ [Law18, Sen24].

(iii) Morava K-theory $K(n)$ can be made $\mathbb{E}_1$ but not $\mathbb{E}_2$ [ACB19, Corollary 5.4]. These are basic designer spectra from chromatic homotopy theory.

1.2 Evenness

As so often in mathematics, 2 is a special number.

Definition 1.6.

(i) A spectrum $X \in \mathbf{Sp}$ is **even** if $\pi_{2i-1}X = 0$ for all $i \in \mathbb{Z}$.

(ii) An **even $\mathbb{E}_\infty$-ring** is an $\mathbb{E}_\infty$-ring whose underlying spectrum is even. We denote by $\mathbf{CAlg}^{\mathrm{ev}}$ the full subcategory of $\mathbf{CAlg}$ spanned by even $\mathbb{E}_\infty$-rings.

There are many nice computational features of even spectra. For example, it can be responsible for degenerate behaviours on spectral sequences. With the Atiyah-Hirzebruch spectral sequence one consequence is the following:

Remark 1.7. Any even homotopy ring spectrum is complex orientable.

Example 1.8.

(i) Examples include $H\mathbb{Z}, \mathrm{KU}, \mathrm{MU}, \mathrm{THH}(\mathbb{F}_p)$.

(ii) Non-examples include $\mathbb{S}, \mathrm{KO}, \mathrm{MO}, \mathrm{tmf}$.

It seems desirable to approximate non-even spectra by even spectra to profit from the computational tools.

2 The Even Filtration

2.1 The Construction

In recent years, people have tried computing $\mathrm{THH}(R)$ by realizing it as a totalization of even ring spectra which has inspired the following construction [HRW24, p. 2].

Construction 2.1 (Hahn-Raksit-Wilson, 2022)**.** There is a lax symmetric monoidal[3] double-speed Whitehead tower functor $\tau_{\geq 2\star} : \mathbf{Sp} \to \mathbf{Fil}(\mathbf{Sp})$ sending $X \in \mathbf{Sp}$ to

$$\cdots \longrightarrow \tau_{\geq 4}X \longrightarrow \tau_{\geq 2}X \longrightarrow \tau_{\geq 0}X \longrightarrow \tau_{\geq -2}X \longrightarrow \tau_{\geq -4}X \longrightarrow \cdots$$

By lax symmetric monoidality it induces a functor $\tau_{\geq 2\star} : \mathbf{CAlg} \to \mathbf{CAlg}(\mathbf{Fil}(\mathbf{Sp}))$. We restrict this to $\mathbf{CAlg}^{\mathrm{ev}}$ and right Kan extend along $\mathbf{CAlg}^{\mathrm{ev}} \hookrightarrow \mathbf{CAlg}$ to obtain

$$\begin{tikzcd} \mathbf{CAlg}^{\mathrm{ev}} \arrow[r, "\tau_{\geq 2\star}"] \arrow[d, hook] & \mathbf{CAlg}(\mathbf{Fil}(\mathbf{Sp})) \\ \mathbf{CAlg} \arrow[ur, dashed, "\mathrm{fil}^\star_{\mathrm{ev}}"'] & \end{tikzcd}$$

also called the **even filtration**.

[3] The target is endowed with the Day convolution symmetric monoidal structure.

Proof. This right Kan extension exists if the limit

$$\mathrm{fil}^{\star}_{\mathrm{ev}}(A) \overset{!}{\simeq} \lim_{\substack{A\to B\\ B\in \mathbf{CAlg}^{\mathrm{ev}}}} \tau_{\geq 2\star} B$$

exists. A priori, there are some size issues: The indexing category need not be small but we only know that small limits exist. To get around this, we realize that there is a pullback square

$$\begin{tikzcd} \mathbf{CAlg}^{\mathrm{ev}} \arrow[r] \arrow[d, "\pi_\bullet"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{CAlg} \arrow[d, "\pi_\bullet"] \\ \mathbf{GrAb}^{\mathrm{ev}} \arrow[r] & \mathbf{GrAb} \end{tikzcd}$$

and it is well-known/checkable that $\mathbf{CAlg}, \mathbf{GrAb}, \mathbf{GrAb}^{\mathrm{ev}}$ are accessible and that the functors $\pi_\bullet : \mathbf{CAlg} \to \mathbf{GrAb}$ and $\mathbf{GrAb}^{\mathrm{ev}} \to \mathbf{GrAb}$ preserve filtered colimits, hence are accessible. This implies that $\mathbf{CAlg}^{\mathrm{ev}} \to \mathbf{CAlg}$ is an accessible functor [Lur09, Proposition 5.4.6.6]. In particular, this functor

$$\mathbf{CAlg}^{\mathrm{ev}}_{A/} \longrightarrow \mathbf{CAlg}^{\mathrm{ev}} \xrightarrow{\tau_{\geq 2\star}} \mathbf{CAlg}(\mathbf{Fil}(\mathbf{Sp}))$$

is accessible. This implies that its limit is equivalent to a small limit.[4] □

In the paper, the notation **FilCAlg** is used for **CAlg**(**Fil**(**Sp**)) which I find a little misleading since it is not supposed to be filtered objects in **CAlg**.

Observation 2.2.

(i) Let $A \in \mathbf{CAlg}$. As seen in the proof, we explicitly have

$$\mathrm{fil}^{\star}_{\mathrm{ev}}(A) \simeq \lim_{\substack{A\to B\\ B\in \mathbf{CAlg}^{\mathrm{ev}}}} \tau_{\geq 2\star} B,$$

i.e. we approximate A by even $\mathbb{E}_\infty$ rings and apply the double-speed Whitehead filtration.

(ii) If $A \in \mathbf{CAlg}^{\mathrm{ev}}$, then $\mathrm{fil}^{\star}_{\mathrm{ev}}(A) \simeq \tau_{\geq 2\star} A$ since id_A is then initial in the system of maps we are taking a limit over.

(iii) Let $A \in \mathbf{CAlg}$. Then, the even filtration $\mathrm{fil}^{\star}_{\mathrm{ev}}(A)$ is complete since the $\tau_{\geq 2\star} B$ is complete for any $B \in \mathbf{CAlg}$ by a Milnor $\lim^1$ argument and complete filtered spectra are closed under limits.

(iv) Let $A \in \mathbf{CAlg}$, then there is always a comparison map $\tau_{\geq 2\star} A \to \mathrm{fil}^{\star}_{\mathrm{ev}} A$ in $\mathbf{CAlg}(\mathbf{Fil}(\mathbf{Sp}))$. Taking underlying objects yields a comparison map $A \to \operatorname{colim} \mathrm{fil}^{\star}_{\mathrm{ev}} A$. This is not always an equivalence but it often is [HRW24, Remark 2.1.6].

Proof of (iv). The construction of this comparison map was not as obvious as I had expected, so let me give a justification here. By adjunction, it suffices to construct a natural transformation $\mathrm{const}\, \tau_{\geq 2\star} A \to \tau_{\geq 2\star} B$ of functors $\mathbf{CAlg}^{\mathrm{ev}}_{A/} \to \mathbf{CAlg}(\mathbf{Fil}(\mathbf{Sp}))$, so by functoriality it suffices to construct a natural transformation $\mathrm{const}\, A \to B$ of functors $\mathbf{CAlg}^{\mathrm{ev}}_{A/} \to \mathbf{CAlg}$. Such a natural transformation is an object in

$$\mathrm{Fun}([1], \mathrm{Fun}(\mathbf{CAlg}^{\mathrm{ev}}_{A/}, \mathbf{CAlg})) \simeq \mathrm{Fun}(\mathbf{CAlg}^{\mathrm{ev}}_{A/}, \mathrm{Ar}(\mathbf{CAlg}))$$

out of which we can pick the preferred functor $\mathbf{CAlg}^{\mathrm{ev}}_{A/} \to \mathrm{Ar}(\mathbf{CAlg})$ which is the structure map in the pullback square that defines the comma category $\mathbf{CAlg}^{\mathrm{ev}}_{A/}$. We check by hand that this natural transformation is indeed of the desired signature. □

[4] Let $F : \mathscr{C} \to \mathscr{D}$ be an accessible functor with a small $\mathscr{C}^0$ such that $\mathscr{C} \simeq \mathrm{Ind}(\mathscr{C}^0)$. Then, we need that

$$\mathrm{Map}_{\mathrm{Fun}(\mathscr{C},\mathscr{D})}(\mathrm{const}\, d, F) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{Ind}\,\mathscr{C}^0,\mathscr{D})}(\mathrm{const}\, d, F) \simeq \mathrm{Map}_{\mathrm{Fun}^{\text{filt-colim}}(\mathscr{C}^0,\mathscr{D})}(\mathrm{const}\, d, F|_{\mathscr{C}^0})$$

is representable for all $d \in \mathscr{D}$. But this exists and is $\lim F|_{\mathscr{C}^0}$ over the small ∞-category $\mathscr{C}^0$.

We will see examples in the next talk. For example, $\mathrm{fil}^{\star}_{\mathrm{ev}}(\mathbb{S})$ will recover the Adams-Novikov filtration [HRW24, Corollary 2.2.21]. We also recover the BMS motivic filtration, the Hochschild-Kostant-Rosenberg filtration, the Morin/Bhatt-Lurie filtration and the synthetic analog functor ν into Morel-Voevodsky's $\mathbf{SH}(\mathbb{C})$ [HRW24, Section 1.1].

Remark 2.3.

(i) Pstrągowski constructed an alternative version of the even filtration [Pst24] which works for $\mathbb{E}_1$-algebras and mostly coincides with the Hahn-Raksit-Wilson even filtration.

(ii) There are variants and more structured versions of the concepts appearing in this talk, for example taking into account p-completeness, an S^1-action, cyclotomic structures and so on. These are made to understand all these approximations to K-theory.

2.2 The Flat Topology

The (faithful) flatness in Hahn-Raksit-Wilson differs from Lurie's [Lur17, Definition 7.2.2.10]. We want to define an even version of the fpqc topology for spectral affine schemes.

Definition 2.4.

(i) A map $f : A \to B$ in $\mathbf{CAlg}^{\mathrm{ev}}$ is **(faithfully) flat** if $\pi_{2\bullet}(f) : \pi_{2\bullet}(A) \to \pi_{2\bullet}(B)$ is (faithfully) flat.

(ii) A sieve on $A \in (\mathbf{CAlg}^{\mathrm{ev}})^{\mathrm{op}}$ is a **flat covering sieve** if it contains a finite collection of maps $\{A \to B_i\}_{1\leq i\leq n}$ such that $A \to \prod_{i=1}^{n} B_i$ is faithfully flat.

Proposition 2.5 ([HRW24, Proposition 2.2.10])**.**

(i) The flat covering sieves define a Grothendieck topology on $(\mathbf{CAlg}^{\mathrm{ev}})^{\mathrm{op}}$.

(ii) Let $\mathscr{C}$ be an ∞-category with limits. A functor $F : \mathbf{CAlg}^{\mathrm{ev}} \to \mathscr{C}$ is a sheaf for the flat topology if and only if the following are satisfied:

  (a) F preserves finite products.

  (b) For every faithfully flat map $A \to B$ in $\mathbf{CAlg}^{\mathrm{ev}}$ the map $F(A) \to \lim_{\Delta} F(B^{\otimes_A \bullet+1})$ is an equivalence.

Theorem 2.6. The functor $\tau_{\geq 2\star}(-) : \mathbf{CAlg}^{\mathrm{ev}} \to \mathbf{CAlg}(\mathbf{Fil}(\mathbf{Sp}))$ is a sheaf for the flat topology.

Proof. The product condition follows because $\tau_{\geq 2\star}(-)$ is a right adjoint. Consider the map of complete filtered spectra $\tau_{\geq 2\star}(A) \to \lim_{\Delta} \tau_{\geq 2\star}\left(B^{\otimes_A \bullet+1}\right)$. That this is an equivalence can be checked on associated graded [vN25, Proposition 3.30]. □

Lemma 2.7. The functor $\pi_{2\star}(-) : \mathbf{CAlg}^{\mathrm{ev}} \to \mathbf{GrSp}$ is a sheaf for the flat topology.[5]

The proof of this result essentially tries to reduce the HRW flat topology to Lurie's flat topology at which point we can cite his tomes.

Definition 2.8 (Lurie)**.**

(i) Let $R \in \mathbf{CAlg}$ and $M \in \mathbf{Mod}_R$. Then, M is **(Lurie) flat** if the following conditions are satisfied:

  (a) The homotopy group $\pi_0 M$ is flat over $\pi_0 R$ in the classical sense.

[5] The homotopy groups are viewed as Eilenberg-MacLane spectra.

(b) For every $n \in \mathbb{Z}$ the map $\pi_n R \otimes_{\pi_0 R} \pi_0 M \to \pi_n M$ is an isomorphism of abelian groups.

(ii) Let $A \to B$ be a map of connective $\mathbb{E}_\infty$-rings. Then, B is **(Lurie) faithfully flat** over A if it is flat and for every $0 \not\simeq M \in \mathbf{Mod}_A$ we have $M \otimes_A B \not\simeq 0$.

Example 2.9. Let $A \to B$ be a (faithfully) flat map of discrete commutative rings. Then, it is Lurie (faithfully) flat.

(i) Part (a) of flatness is by assumption and part (b) is clear since the only non-trivial homotopy group is π_0.

(ii) Let $0 \not\simeq M \in \mathbf{Mod}_A \simeq \mathcal{D}(A)$. Since $A \to B$ is flat, we can compute $M \otimes_A^{\mathbb{L}} B$ as the tensor product $M \otimes_A B$ of chain complexes. Moreover,

$$\pi_\bullet(M \otimes_A^{\mathbb{L}} B) \cong H_\bullet(M \otimes_A B) \cong H_\bullet(M) \otimes_A B$$

using flatness of $A \to B$. Now $M \not\simeq 0$ implies $H_\bullet(M) \not\cong 0$ and thus, the above term is non-zero by faithful flatness.

Similar to 2.5 this also yields a topology and descent type results can be found in Lurie's work.

Fact 2.10 ([Lur18, Theorem D.6.3.5])**.** The functor $\mathrm{id}_{\mathbf{CAlg}} : \mathbf{CAlg} \to \mathbf{CAlg}$ is a sheaf for the Lurie fpqc topology.

Lemma 2.11. Let $A \to B$ be a flat map in $\mathbf{CAlg}^{\mathrm{ev}}$ and $A \to C$ be another map in $\mathbf{CAlg}^{\mathrm{ev}}$. Then, $\pi_\bullet(B \otimes_A C) \cong \pi_\bullet(B) \otimes^{\mathbb{L}}_{\pi_\bullet(A)} \pi_\bullet(C)$.

Proof. Consider the filtered object $\tau_{\geq 2\star} B \otimes_{\tau_{\geq 2\star} A} \tau_{\geq 2\star} C$ where we mean the filtered relative tensor product. We note:

(1) It is complete for which we need to check that for fixed $n \in \mathbb{Z}$ we see

$$\pi_n(\tau_{\geq 2\star} B \otimes_{\tau_{\geq 2\star} A} \tau_{\geq 2\star} C)_s = 0$$

for $s \gg 0$ [vN25, Proposition 2.53]. Phrased differently, it suffices to check that the object $\tau_{\geq 2\star} B \otimes_{\tau_{\geq 2\star} A} \tau_{\geq 2\star} C$ is bounded below in the diagonal t-structure,[6] i.e. the t-structure coming from $\tau_{\geq 2\star}$. We check that $\tau_{\geq 2\star} B \otimes \tau_{\geq 2\star} C$ is connective, then it will also follow for the relative tensor product since $\mathbf{Fil}(\mathbf{Sp})_{\geq 0} \to \mathbf{Fil}(\mathbf{Sp})$ is a left adjoint and thus preserves geometric realizations.

But for connectivity we note that

$$(\tau_{\geq 2\star} B \otimes \tau_{\geq 2\star} C)_s \simeq \operatorname*{colim}_{i+j \geq s} (\tau_{\geq 2i} B \otimes \tau_{\geq 2j} C)$$

is a filtered colimit of terms which are s-connective, so it must itself be s-connective using that $\mathbf{Sp}_{\geq s} \to \mathbf{Sp}$ preserves colimits.

(2) It has underlying object $B \otimes_A C$. Let's first do the case $B \otimes C$. Recall that the tensor product on filtered objects comes from Day convolution, so we need to check

$$\operatorname*{colim}_{s \to -\infty} \operatorname*{colim}_{i+j \geq s} \tau_{\geq 2i} B \otimes \tau_{\geq 2j} C \simeq B \otimes C.$$

To do so we observe that the subposet $\{(i,j,s) \in \mathbb{N}^3 : i + j \geq s\} \to \mathbb{N}^3$ is colimit cofinal. Indeed, by Quillen's Theorem A we need to check that certain slices are (weakly)

[6] In particular, this is not the Beilinson t-structure which was the focus of the first half of this workshop.

contractible but these slices are posets where it's straightforward to check that it's filtered implying that it's contractible.[7] Thus,

$$\begin{aligned}\operatorname*{colim}_{s\to-\infty}\operatorname*{colim}_{i+j\geq s}\tau_{\geq 2i}B\otimes\tau_{\geq 2j}C &\simeq \operatorname*{colim}_{i\in\mathbb{N}}\operatorname*{colim}_{j\in\mathbb{N}}\operatorname*{colim}_{s\in\mathbb{N}}\tau_{\geq 2i}B\otimes\tau_{\geq 2j}C\\ &\simeq \operatorname*{colim}_{i\in\mathbb{N}}\operatorname*{colim}_{j\in\mathbb{N}}\tau_{\geq 2i}B\otimes\tau_{\geq 2j}C\\ &\simeq \left(\operatorname*{colim}_{i\in\mathbb{N}}\tau_{\geq 2i}B\right)\otimes\left(\operatorname*{colim}_{j\in\mathbb{N}}\tau_{\geq 2j}C\right)\\ &\simeq B\otimes C.\end{aligned}$$

To pass to relative tensor products we use that geometric realizations commute with $\mathrm{colim}_{s\to-\infty}$.

(3) Since taking the associated graded is strong monoidal [Lur15, Proposition 3.2.1], we deduce that it is $(\pi_{2\bullet}B\otimes^{\mathbb{L}}_{\pi_{2\bullet}A}\pi_{2\bullet}C)$ where we mean the graded relative tensor product.

Fix $n\in\mathbb{Z}$. Since $\pi_n(\tau_{\geq 2\star}B\otimes_{\tau_{\geq 2\star}A}\tau_{\geq 2\star}C)_s=0$ for $s\gg 0$, we deduce that the spectral sequence associated to this filtration converges [vN25, Proposition 2.53].

Moreover, $\pi_{2\star}B$ is flat over $\pi_{2\star}A$ by assumption, so the higher homotopy groups of the graded spectrum $\pi_{2\star}B\otimes^{\mathbb{L}}_{\pi_{2\star}A}\pi_{2\star}C$, i.e. (graded) Tor-groups, vanish. In other words, the spectral sequence is concentrated in homological degree 0 and thus collapses. This implies the desired result. □

The proof strategy essentially uses a spectral sequence which is similar to the Künneth spectral sequence (but which is not the same).[8] It seems to me like the Künneth spectral sequence would also work here but the phrasing in [HRW24, Proposition 2.2.8] adapts better to the p-complete setting. Now, we are all set to prove the main result.

Proof of Lemma 2.7. The product condition follows because homotopy groups commute with finite products. Let $A\to B$ be a faithfully flat map in $\mathbf{CAlg}^{\mathrm{ev}}$. We need to check that the map

$$\pi_{2\bullet}(A)\to\lim_{\Delta}\pi_{2\bullet}\left(B^{\otimes_A\bullet+1}\right)\simeq\lim_{\Delta}\pi_{2\bullet}(B)^{\otimes^{\mathbb{L}}_{\pi_{2\bullet}(A)}\bullet+1}$$

is an equivalence where the second equivalence follows from 2.11. But $A\to B$ is faithfully flat, so $\pi_{2\bullet}A\to\pi_{2\bullet}B$ is faithfully flat which implies Lurie faithful flatness (2.9). Thus, we conclude via faithfully flat descent (2.10). □

At the end of the day, everything follows from Lurie's tomes!

The flat descent will be used to control the even filtration which will be the content of the next talk. Examples should also appear in that talk which is why no examples are included in this talk.

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[7] More concretely, a subposet $B\subseteq A$ is cofinal if for every $a\in A$ there exists some $b\in B$ with $a\to b$.

[8] The Künneth spectral sequence would be $\mathrm{Tor}^{\pi_\bullet A}_{s,t}(\pi_\bullet B,\pi_\bullet C)\Rightarrow\pi_{s+t}(B\otimes_A C)$, so here one works with graded objects in the E_2-page.

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